

These homework problems are meant to expand your understanding of what goes on during class. Any you turn in will be graded and returned to you. Answers may or may not be posted on the web, depending on demand.

1. Construct a tetrahedron (if you haven't already) and find the number of order-preserving symmetries of this polyhedron. In class it was suggested that the order of this *tetrahedral group*  $T$  (that is, the number of elements of  $T$ ) is 12. Prove this by
  - (a) showing that there are at least 12 symmetries, and
  - (b) showing that there are at most 12 symmetries.

This is called a *combinatorial* (or counting) argument.

2. In class we showed that the cube and octahedron shared an orientation-preserving symmetry group (called  $O$ , the *octahedral group*) as did the dodecahedron and the icosahedron (this is  $I$ , the *icosahedral group*). People made the following conjectures:

**Hamoon's second conjecture:** *The order of each of these symmetry groups is twice the number of edges.*

**Martin's conjecture:** *The order of each of these symmetry groups is the number of faces times the number of edges per face.*

Both of these conjectures result in the following orders:  $|T| = 12$ ,  $|O| = 24$ , and  $|I| = 60$ . Prove either of these conjectures, or at least explain why you believe one of them.

3. A group  $G$  is called *abelian* (or *commutative*) if  $ab = ba$  for all elements  $a, b \in G$ . Prove that  $G$  is abelian if and only if
  - (a)  $(ab)^n = a^n b^n$  for all elements  $a, b \in G$  and for all positive integers  $n$
  - (b)  $(ab)^2 = a^2 b^2$  for all elements  $a, b \in G$
  - (c)  $a^{-1} b^{-1} a b = 1$  for all elements  $a, b \in G$
  - (d)  $aba^{-1} = b$  for all elements  $a, b \in G$
  - (e)  $(ab)^{-1} = a^{-1} b^{-1}$  for all elements  $a, b \in G$
  - (f)  $(ab)^n = a^n b^n$  for all elements  $a, b \in G$  and for all negative integers  $n$
  - (g) (Harder!)  $(ab)^n = a^n b^n$  for all elements  $a, b \in G$  for 3 consecutive integers  $n$ .
4. (Still difficult!) Give an example to show that part (g) of the previous problem would not be true if we only had the property for 2 consecutive integers  $n$ .

A group  $G$  is called *cyclic* if there is an element  $g \in G$  so that any element  $h \in G$  may be written as  $h = g^k$  for some positive integer  $k$ . The element  $g$  is called the *generator* of  $G$ .

5. Prove that every cyclic group is abelian.

6. Let  $G$  be a group of order 3. Prove that  $G$  is cyclic (and therefore abelian). Does your proof still work if  $|G| = 5$ ? Can you make a generalization of this statement (and proof)?
7. Let  $G$  be any finite group (this means that  $|G|$  is finite), and let  $A = \{x \in G : x^3 = 1\}$ . Prove that  $|A|$  is odd.
8. This one gets a little challenging towards the end.
  - (a) Prove that if  $G$  is a group of even order then  $G$  has an element of order 2.
  - (b) We now know that if 2 divides  $|G|$ , then  $G$  has an element of order 2. Prove that if 3 divides  $|G|$ , then  $G$  has an element of order 3.
  - (c) Can you state a generalization of this result?
  - (d) Prove your generalization.