A Solution of Rubik's Cube

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The basic outline of our solution is as follows. (This is adapted from the solution that I was taught over 20 years ago, and I do not claim it as original. My guess is that it was from an early book by Singmaster, although I'm not certain. After writing a draft of this solution, I've found a wonderful book by Frey and Singmaster [1] that has improved this version. Another good source (although a bit over our level at the moment) is the book by Joyner [2], which has the advantage of being freely available on-line. A more extensive bibliography will be included in later versions.)

To continue, our outline is as follows:

1. **Solve one side.** This means that one side is all one color, plus the first layer of small cubes also match the respective centers of the sides. This side will be the bottom of our solved cube.

2. **Solve the middle layer.** That is, put in place the four missing edges of the small cubes that are not on the top layer.

3. **Put a plus sign in place on the top side.** That is, orient the edge (not corner) small cubes so that the four edge cubes match the center color.

4. **Put the top edges in the proper spots.** This is simply moving around the cubes in the plus sign of the previous step so that they are in their final places. Now every small cube is in its proper location except possibly for the four corner cubes on the top side.

5. **Put the top corner pieces in place.** Here we move the corners into the proper spot, but they still may be twisted out of alignment.

6. **Twist the top corners into place.** Before this step, the corners were only in the right spot, but not necessarily oriented (twisted) correctly. In this step we twist the corners to finish solving the cube.

Notation For Cubes

We begin with some notation, so that we may write down descriptions of turning the sides of a cube. The notation we will use also, I believe, originated with Singmaster; in any case it is now fairly standard.

We describe the six sides of the cube as the front, back, right, left, up, and down. Unless otherwise stated, we will assume that the centers of each side are fixed in space. The six basic moves are each simply a clockwise quarter turn of own of these sides, and we write these turns as F (a clockwise quarter turn of the front side), B (back), R (right), L (left), U (up), and D (down). You should see now the reason for the slightly odd choice of names (up?): these names give us six distinct letters to use as abbreviations.

A *move* is some combination of our six basic moves. We write a combination as a string (a *word* in group theory terms) of basic moves, for example $\mathsf{FL}^2\mathsf{B}^{-1}$ is first a clockwise quarter turn of the front side, then a half turn of the left side, and a *counter-clockwise* quarter turn of the back side. Thus L^2 is simply L down twice, and B^{-1} is B done "backwards" (that is, in the opposite, counter-clockwise, direction). You should be very comfortable with the fact that $\mathsf{F}^4 = 1$ (where 1 is the identity, or move that does nothing) and so $\mathsf{F}^{-1} = \mathsf{F}^3$.

We will describe the effect of a move by what it does to the small cubes that make up the larger cube. The clever design of Rubik's cube gives the impression that the large cube is in fact made of $3 \times 3 \times 3 = 27$ smaller cubes. In fact, there are only 20 small "cubes" that move about: 8 corners and 12 edge pieces. The 6 centers of each side merely rotate, and the twenty-seventh small cube, the center, does not exist at all. (These remarks are of course specific to the $3 \times 3 \times 3$ cube. The $2 \times 2 \times 2$ cube has only the 8 corners, and the $4 \times 4 \times 4$ cube has 24 edge pieces and 24 center pieces to go with the 8 corners – the other $64 - 24 - 24 - 8 = 8$ small cubes are missing "center" cubes.)

The small cubes have acquired silly names like *cubies*, *cublets*, *facets*, or *cubicles*. We'll stick with *cubies* (as used by Frey and Singmaster [1]) for the small cubes and we'll call the cube as a whole simply *the cube*. As alluded to previously, there are three distinct classes of cubies on a $3 \times 3 \times 3$ cube: the edges, the corners, and the centers. We'll identify them at times by their position on the cube. For example, the edge piece between the front and up sides is called fu or uf. It is bordered by the corner pieces urf (or rfu or fur) and ufl (or flu or luf). (Notice that the corners are described clockwise.) The different descriptions allow us to be precise when describing moves. For example, if we say that uf moves to ru, this means that the face of the cubie uf that was on the up side is now on the right side (that is, u moves to r) and the face on the front side moves to the up side.

Step 1: Solving One Side

Here you're on your own. Think of it as acquiring familiarity with the puzzle.

I do have some hints, so you're not *completely* on your own. I usually start with putting the edge pieces in properly. Here we're not simply trying to put the four (say) blue edges next to the blue center, we're trying to put the proper edges aligned with the proper adjacent faces as well. Do be careful.

Next, some hints on the corners. Suppose you're working on a side that you put on top (the up side). Here's a simple move to put frd to urf: FDF^{-1}. What if you really wanted to move frd to fur (that is, a different orientation of the movement): try $\mathsf{R}^{-1}\mathsf{D}^{-1}\mathsf{R}$. Finally, to move frd to rfu, try $\mathsf{FD}^{-1}\mathsf{F}^{-1}\mathsf{R}^{-1}\mathsf{D}^2\mathsf{R}$. Let's summarize:

Effect	Move
$\mathsf{frd} \mapsto \mathsf{urf}$	FDF^{-1}
$\mathsf{frd} \mapsto \mathsf{fur}$	$\mathsf{R}^{-1}\mathsf{D}^{-1}\mathsf{R}$
$\mathsf{frd} \mapsto \mathsf{rfu}$	$\mathsf{FD}^{-1}\mathsf{F}^{-1}\mathsf{R}^{-1}\mathsf{D}^2\mathsf{R}$

Two remarks about these moves: first, they will mess with other parts of the cube, so you shouldn't assume that these moves will be useful in later steps. Second, we have always started with frd in this example, but perhaps you find it easier to always end with urf. (This is understandable: if you're working on the blue side, you should be thinking: "I want the blue side on top.") The above table can be re-written as simply:

Effect	Move
$\mathsf{frd} \mapsto \mathsf{urf}$	FDF^{-1}
$\mathsf{rdf} \mapsto \mathsf{urf}$	$\mathsf{R}^{-1}\mathsf{D}^{-1}\mathsf{R}$
$\mathsf{dfr} \mapsto \mathsf{urf}$	$\mathsf{FD}^{-1}\mathsf{F}^{-1}\mathsf{R}^{-1}\mathsf{D}^2\mathsf{R}$

Now you can look and see which side of frd needs to go to the up side.

I know this looks like I've given away the whole game here, but there is still a fair amount of puzzling about here to put the corners in properly.

Step 2: The Middle Layer

Now we put our completed side on the bottom and look to put the four edges that are not on the top layer in place. We'll call this the middle layer. Now is when difficulties arise, but really needlessly so.

The middle layer can be solved with one simple move. This is, of course, hyperbole. We will use two variations (a right-handed and a left-handed) of a move to solve the middle layer. These variations are themselves slight complications of a quite simple move. Let's begin with the simplest move: $(\mathsf{R}^2\mathsf{U}^2)^3$, or $\mathsf{R}^2\mathsf{U}^2\mathsf{R}^2\mathsf{U}^2\mathsf{R}^2\mathsf{U}^2$. The entire effect of this move is to switch to pairs of edges: it switches uf and ub and it switches rf and rb. (A simple corollary of this fact is that the move is its own inverse.)

Our goal is to move an edge cubie from the top layer to the middle layer. We do this as follows:

$$\mathsf{FUR}^2\mathsf{U}^2\mathsf{R}^2\mathsf{U}^2\mathsf{R}^2\mathsf{UF}^{-1}$$

switches uf and rb. (It also switches ur and ul – see Figure 1) How does this relate to our earlier move $(\mathsf{R}^2\mathsf{U}^2)^3$? This move is simply $\mathsf{FU}(\mathsf{R}^2\mathsf{U}^2)^3\mathsf{U}^{-1}\mathsf{F}^{-1}$! That is, it is simply $(\mathsf{R}^2\mathsf{U}^2)^3$ *preceded* by FU, then *followed* by $(\mathsf{FU})^{-1} = \mathsf{U}^{-1}\mathsf{F}^{-1}$. Do you see the logic in this?

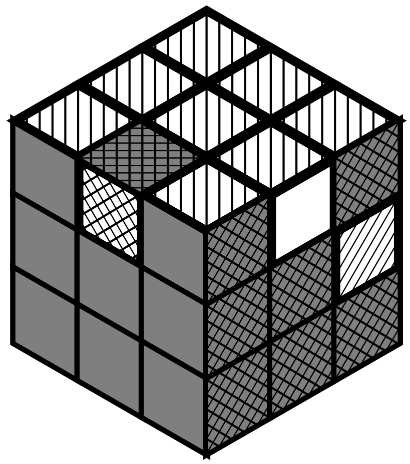

Figure 1: The Move $\mathsf{FUR^2U^2R^2U^2R^2UF^{-1}}$

The left-handed version of this is based on $\mathsf{(L^2U^2)^3}$, and is simply $\mathsf{F^{-1}U^{-1}(L^2U^2)^3UF = F^{-1}U^{-1}L^2U^2L^2U^2L^2U^{-1}F}$. We'll compile all these moves and their effects in a new chart:

Move	Effect
$\mathsf{(R^2U^2)^3}$	uf $\longleftrightarrow$ ub, rf $\longleftrightarrow$ rb
$\mathsf{FUR^2U^2R^2U^2R^2UF^{-1}}$	uf $\longleftrightarrow$ rb, ur $\longleftrightarrow$ ul
$\mathsf{(L^2U^2)^3}$	uf $\longleftrightarrow$ ub, lf $\longleftrightarrow$ lb
$\mathsf{F^{-1}U^{-1}L^2U^2L^2U^2L^2U^{-1}F}$	uf $\longleftrightarrow$ lb, ur $\longleftrightarrow$ ul

With these moves, you should be able to complete the middle layer.

Step 3: A Plus Sign On Top

Now we turn to the edge cubies on the top (up side) of the cube. If they're all one color (namely, the color of the center cubie), then we move on to the next step. If not, then we need a way of flipping edges.

We'll begin with simple edge flips that also move some of the corner cubies. A move that "flips" bu and lu is $\mathsf{BULU^{-1}L^{-1}B^{-1}}$. Another one, that "flips" fu and bu is $\mathsf{BLUL^{-1}U^{-1}B^{-1}}$. In fact, there's something to think about here, as these are inverses of each other! I'll leave it to you to think about, but I'll (as usual) leave you with a table of moves and effects (and detailed effects):

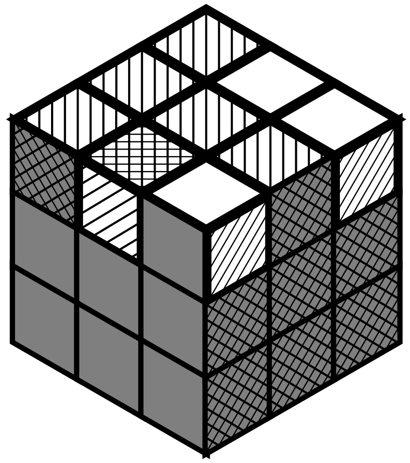

Figure 2: Effect Of The Move $\mathsf{BULU^{-1}L^{-1}B^{-1}}$.
(But We Want To Use $\mathsf{BLUL^{-1}U^{-1}B^{-1}}$ Here)

Move	Effect	Edge Details
$\mathsf{BULU^{-1}L^{-1}B^{-1}}$	"Flips" bu and lu	uf → ul → bu → uf
$\mathsf{BLUL^{-1}U^{-1}B^{-1}}$	"Flips" bu and fu	uf → bu → ul → uf

The down side to these short moves is that there are side-effects: they'll mess up the top corner cubies, as you can see in Figure 2. This isn't really an issue here, as we haven't dealt with the top corner cubies yet.

But suppose the corners were somehow miraculously in their proper places? All that needed to be done was to flip some edges, but we have to mess up corners in order to do so! Not really, as there are other moves to flip edges, and they illustrate a useful point.

To flip, say, fu and bu, we use $\mathsf{F^{-1}U^{-1}BR^2U^2B^2L^{-1}U^2LB^2U^2R^2B^{-1}UFU^2}$. This looks *awful*, but really it's $\mathsf{XYX^{-1}Y^{-1}}$, where $\mathsf{X = F^{-1}U^{-1}BR^2U^2B^2L^{-1}}$ and $\mathsf{Y = U^2}$. (You can check this easily.) What's going on here is actually pretty easy: we're flipping fu using X, then switching fu and bu using Y, then *un-flipping* bu (since it's where fu was) using $\mathsf{X^{-1}}$. Finally $\mathsf{Y^{-1}}$ switches fu and bu again. If you don't get it the first time, that's fine – this sort of thing will show up again. (Group theoretic aside: "this sort of thing" – group elements of the form $\mathsf{XYX^{-1}Y^{-1}}$ – is called a *commutator*. Notice that if X and Y commute with each other's inverses, then the commutator is $\mathsf{XX^{-1}YY^{-1}} = 1$.) If we'd used $\mathsf{Y = U}$ or $\mathsf{U^{-1}}$, we would have flipped edges on adjacent faces rather than opposite faces.

We thus have another set of flipping rules that we summarize in the following table. These moves *do not disturb corners*; that is, there are no unwanted side effects.

Move	Effect
$\mathsf{XU^2X^{-1}U^2}$	$\mathsf{fu} \longleftrightarrow \mathsf{uf}$, $\mathsf{bu} \longleftrightarrow \mathsf{ub}$
$\mathsf{XUX^{-1}U}$	$\mathsf{fu} \longleftrightarrow \mathsf{uf}$, $\mathsf{ru} \longleftrightarrow \mathsf{ur}$

where $\mathsf{X} = \mathsf{F^{-1}U^{-1}BR^2U^2B^2L^{-1}}$ and $\mathsf{X^{-1}} = \mathsf{LB^2U^2R^2B^{-1}UF}$.

Step 4: The Top Edges

Now we move the top edge cubies so that they are each in their proper place. Up to now, we have been concerned only with their alignment (getting the "plus sign" on the top face).

There is one move for this: $\mathsf{A} = \mathsf{R^2U^{-1}FB^{-1}R^2F^{-1}BU^{-1}R^2}$. This move cycles three of the top edge pieces: from the front, to the right, to the back, to the front. The inverse $\mathsf{A^{-1}}$ must therefore cycle in the opposite direction. We again have a table:

Move	Effect
$\mathsf{A} = \mathsf{R^2U^{-1}FB^{-1}R^2BF^{-1}U^{-1}R^2}$	$\mathsf{fu} \rightarrow \mathsf{ru} \rightarrow \mathsf{bu} \rightarrow \mathsf{fu}$
$\mathsf{A^{-1}} = \mathsf{R^2UFB^{-1}R^2BF^{-1}UR^2}$	$\mathsf{fu} \rightarrow \mathsf{bu} \rightarrow \mathsf{ru} \rightarrow \mathsf{fu}$

See also Figure 3.

I've called the move A a *cycle* because of an obvious consequence of the effect of this move: $\mathsf{A}^3 = 1$. Thus $\mathsf{A}^2 = \mathsf{A}^{-1}$ (which isn't *immediately* clear from the expressions written in the table, but should it should be clear from $\mathsf{A}^3 = 1$ and also the effect column).

It is possible that you will need to use this move twice in order to move the top edges into their proper positions.

Step 5: Top Corners In Place

There are more cycles that move three corners. The simpler moves rotate the corners as they are moved, but it is possible to cycle three

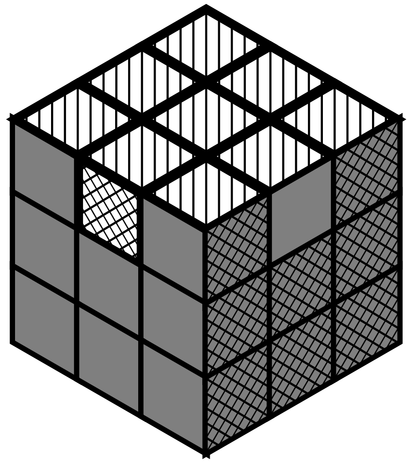

Figure 3: Effect Of $\mathsf{A}^{-1} = \mathsf{R}^2\mathsf{UFB}^{-1}\mathsf{R}^2\mathsf{BF}^{-1}\mathsf{UR}^2$

corners without rotating them. Here is a table of corner-switching moves, with one illustrated in Figure 4.

Move	Effect
$\mathsf{F}^{-1}\mathsf{L}^{-1}\mathsf{FR}^{-1}\mathsf{F}^{-1}\mathsf{LFR}$	$\mathsf{urf} \to \mathsf{bru} \to \mathsf{luf} \to \mathsf{fur}$
$\mathsf{ULU}^{-1}\mathsf{R}^{-1}\mathsf{UL}^{-1}\mathsf{U}^{-1}\mathsf{R}$	$\mathsf{urf} \to \mathsf{bru} \to \mathsf{flu} \to \mathsf{fur}$
$\mathsf{R}^{-1}\mathsf{B}^{-1}\mathsf{RUR}^{-1}\mathsf{BRU}^2\mathsf{R}^{-1}\mathsf{B}^{-1}\mathsf{RUR}^{-1}\mathsf{BR}$	$\mathsf{urf} \to \mathsf{ubr} \to \mathsf{ufl} \to \mathsf{urf}$

Notice that the first two moves listed are commutators. For example, the first one is

$$\mathsf{F}^{-1}\mathsf{L}^{-1}\mathsf{FR}^{-1}\mathsf{F}^{-1}\mathsf{LFR} = \left(\mathsf{F}^{-1}\mathsf{L}^{-1}\mathsf{F}\right)\mathsf{R}^{-1}\left(\mathsf{F}^{-1}\mathsf{L}^{-1}\mathsf{F}\right)^{-1}\left(\mathsf{R}^{-1}\right)^{-1},$$

which is $\mathsf{XYX}^{-1}\mathsf{Y}^{-1}$ with $\mathsf{X} = \mathsf{F}^{-1}\mathsf{L}^{-1}\mathsf{F}$ and $\mathsf{Y} = \mathsf{R}^{-1}$.

The inverses of these moves are also useful. I'll leave you to derive them (and their effects). You may need to use two of these moves to move the corner cubies into position.

Step 6: Twist The Top Corners

Finally, we twist the corner cubies so that they are properly aligned. All the twisting moves here will twist one corner by $+1/3$ of a rotation

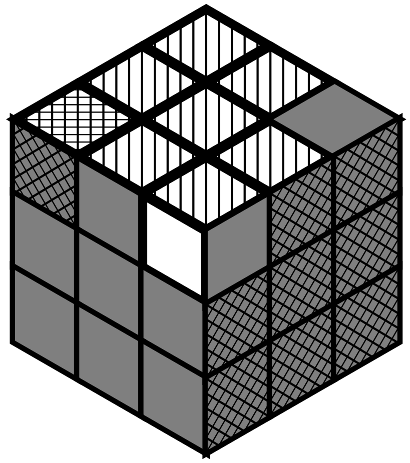

Figure 4: Effect Of The Move $\mathsf{F^{-1}L^{-1}FR^{-1}F^{-1}LFR}$
We Can Undo This With $\mathsf{R^{-1}F^{-1}L^{-1}FRF^{-1}LF}$

and twist another corner $-1/3$ rotation. I'll leave you to determine which way is positive.

These twists are all commutators, and they're all really just one move. Let $\mathsf{X} = \mathsf{R^{-1}DRFDF^{-1}}$. This twists urf to rfu and messes up the bottom two layers. All our moves in this step are $\mathsf{XU}^k\mathsf{X}^{-1}\mathsf{U}^{-k}$ for $k = 1,2,3$. The value of k simply determines which other corner gets twisted.

Move	Effect
$\mathsf{R^{-1}DRFDF^{-1}UFD^{-1}F^{-1}R^{-1}D^{-1}RU^{-1}}$	urf → rfu, ubr → rub
$\mathsf{R^{-1}DRFDF^{-1}U^2FD^{-1}F^{-1}R^{-1}D^{-1}RU^2}$	urf → rfu, ulb → bul
$\mathsf{R^{-1}DRFDF^{-1}U^{-1}FD^{-1}F^{-1}R^{-1}D^{-1}RU}$	urf → rfu, ufl → luf

(We show the last of these in Figure 5.) Each of these moves is a cycle: if repeated three times it is the identity. You may need to use two twisting moves while finishing your cube.

Congratulations! You should now be able to solve your cube!

References

[1] Alexander H. Frey, Jr. and David Singmaster. *Handbook of cubik math.* Enslow Publishers, Hillside, N.J., 1982.

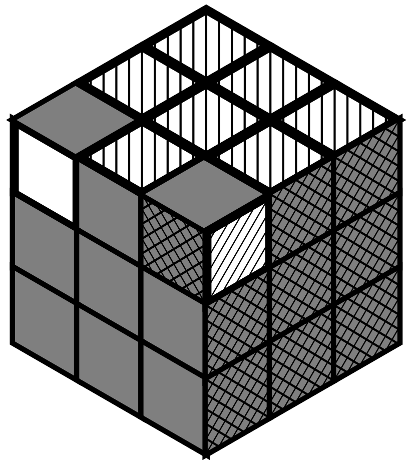

Figure 5: Effect Of $\mathsf{R}^{-1}\mathsf{DRFDF}^{-1}\mathsf{U}^{-1}\mathsf{FD}^{-1}\mathsf{F}^{-1}\mathsf{R}^{-1}\mathsf{D}^{-1}\mathsf{RU}$

[2] David Joyner. *Adventures in group theory: Rubik's cube, Merlin's machine and other mathematical toys.* Johns Hopkins University Press, Baltimore, MD, 2002. Early version available at `http://web.usna.navy.mil/~wdj/books.html`.