SOAR Math Course — Contest! — Winter, 2002

These are "solutions" for my own reference. Feel free to send me questions about them if you like at `garfield@math.toronto.edu`.

1. $G_1 = \langle a, b, c, d \mid a^2 = d,\ b^2 = d^{-2},\ c^2 = ba^2b,\ d^3ab = c,\ a^2b^2 = d^3cb^{-1}a^{-1} \rangle$

   Vishu: This is D_2, or $C_2 \times C_2$ if you like.

   Claim 1. $a^4 = b^{-2}$

   Proof. The first relations implies that $a^4 = d^2$ and the second that $d^2 = b^{-2}$. Thus $a^4 = b^{-2}$. □

   Let's now get rid of d for now. That is, let's consider the group

   $$G = \left\langle a, b, c \;\middle|\; a^4 = b^{-2},\ c^2 = ba^2b,\ a^7b = c,\ a^2b^2 = a^2cb^{-1}a^{-1} \right\rangle;$$

   so I've replaced the first two relations with one not involving d, and in the other relations I've replaced d with a^2.

   Claim 2. $a^{14} = 1$

   Proof. The last two relations imply that $a^2b^2 = a^6(a^7b)b^{-1}a^{-1}$. Simplifying this, we get $b^2 = a^{10}$. Since $b^2 = a^{-4}$ (from the first relation), we get $a^{-4} = a^{10}$, or $a^{14} = 1$. □

   Claim 3. $b^{14} = 1$

   Proof. First notice that $b^6 = a^2$. This follows from the first relation and the fact that a has order (that divides) 14:

   $$b^6 = (b^2)^3 = (a^{-4})^3 = a^{-12} = a^2.$$

   (Here we use the fact that $a^{14} = 1$ to see that $a^2 = a^{-12}$.) Then, using the first relation again,

   $$b^{14} = b^6 \cdot b^6 \cdot b^2 = a^2 \cdot a^2 \cdot a^{-4} = 1.$$

   □

   Claim 4. $b = a^7ba^5$

   Proof. Using the relations for c^2 and c, we see that $c^2 = ba^2b$ and $c = a^7ba^7b$. Hence $ba^2b = a^7ba^7b$. Simplifying, we get $b = a^7ba^5$, as claimed. □

   Claim 5. $a^2 = 1$.

   Proof. From the previous claim, we get two facts: first,

   $$a^7b = a^7a^7ba^5 = a^{14}ba^5 = ba^5$$

   and, second,

   $$a^7b = a^7ba^5a^{-5} = ba^{-5} = ba^9.$$

   Thus $ba^5 = ba^9$, or $a^4 = 1$. But $a^{14} = 1$, so $a^2 = 1$. □

Now everything falls into place: $d = a^2 = 1$, $b^2 = d^{-2} = 1$, and $c = d^3ab = ab$. Notice that $c^2 = ba^2b = b^2 = 1$, so we must have $ab = ba$. Thus $G_1 \cong D_2$.

2. $G_2 = \langle a, b \mid a^3 = 1,\ b^7 = 1,\ a^2b = b^3a^2 \rangle$

   Peter claims that this is C_3 (or $\mathbf{Z}_3$ if you prefer).

   Claim 6. $ab = b^2a$

   Proof. Multiply the last relation by a^2 and (using $a^4 = a$) to get $ab = a^2b^3a^2$. Now we commute a^2 past each of the three b's as follows:

$$\begin{aligned} ab &= a^2b^3a^2 \\ &= (a^2b)b^2a^2 \\ &= (b^3a^2)b^2a^2 \\ &= b^3(a^2b)ba^2 \\ &= b^3(b^3a^2)ba^2 \\ &= b^6(a^2b)a^2 \\ &= b^6(b^3a^2)a^2 \\ &= b^9a^4 \\ &= b^2a. \end{aligned}$$

   Claim 7. $b^2 = 1$

   Proof. Use the relations $a^2b = b^3a^2$ and $ab = b^2a$ to simplify $b = a^3b$ as follows:

$$\begin{aligned} b = a^3b &= a(a^2b) \\ &= a(b^3a^2) \\ &= (ab)b^2a^2 \\ &= (b^2a)b^2a^2 \\ &= b^2(ab)ba^2 \\ &= b^2(b^2a)ba^2 \\ &= b^4(ab)a^2 \\ &= b^4(b^2a)a^2 \\ &= b^6a^3 = b^6. \end{aligned}$$

   □

   Multiply both sides by b and we get $b^2 = 1$, as claimed. □

   Finally, we notice that $b^2 = 1$ and $b^7 = 1$ together imply that $b = 1$: $1 = b^7 = b^2 \cdot b^2 \cdot b^2 \cdot b = 1 \cdot 1 \cdot 1 \cdot b = b$. Thus G_2 simplifies to

$$G_2 = \langle a \mid a^3 = 1 \rangle,$$

   which is C_3 or $\mathbf{Z}_3$.

3. $G_3 = \langle a, b, c \mid a^5 = 1,\ b^{11} = 1,\ c^3 = 1,\ a^4 b = b^2 a^4,\ b^{10} c = c^2 b^{10},\ ac = ca^4 \rangle$

(David Bland) This is the trivial group $\{1\}$.

Claim 8. $a = 1$

Proof. Consider $a = ac^3$. Using the relation $ac = ca^4$, we move the a past each of the three c's to get (we claim) $a = c^3 a^{4^3} = c^3 a^{(-1)^3} = c^3 a^{-1} = a^{-1}$. (Since $a^5 = 1$, the fact that $a^2 = 1$ means that a must be 1.) Let's prove this:

$$\begin{aligned}
ac^3 &= (ac)c^2 \\
&= (ca^4)c^2 \\
&= ca^3(ac)c \\
&= ca^3(ca^4)c \\
&= ca^3ca^3(ac) \\
&= ca^3ca^3(ca^4) \\
&= ca^3ca^2(ac)a^4 \\
&= ca^3ca^2(ca^4)a^4 \\
&\vdots \\
&= ca^3c^2a^{4^2} \\
&\vdots \\
&= c^3a^{4^3}.
\end{aligned}$$

Thus $a = ac^3 = c^3 a^{4^3} = a^{64} = a^4 = a^{-1}$, or $a^2 = 1$. Hence $a = 1$. □

Now the relation $a^4 b = b^2 a^4$ is simply $b = b^2$, or $b = 1$ as well. Similarly, $b^{10} c = c^2 b^{10}$ means that $c = c^2$, so $c = 1$ too. Hence $G_3 \cong \{1\}$.

4. $$G_4 = \left\langle a, b, c, d, e, f, g, h \;\middle|\; \begin{array}{l} a^3 = c^6 = b,\ ca = be^2,\ d^2a^3 = h,\ h^2 = b = f^2, \\ da = gb, da^2da = c,\ e^2 f = d^2,\ ca^2 = h, \\ c^a = f,\ da^2 = b^2 e,\ d^2 a = c, da = g \end{array} \right\rangle$$

Ti writes:

> I've tried to shorten down the number of generators, but things get really confusing after a while... this is not as easy as i thought ;) anyway, it's the group A_4, with $b = 1$, $a^3 = c^3 = d^3 = e^3 = 1$, $f^2 = g^2 = h^2 = 1$, and $a = (12)(23)$, $c = (13)(34)$, $d = (23)(34)$, $e = (12)(24)$, $f = (12)(34)$, $g = (13)(24)$, and $h = (14)(23)$.

Unfortunately, there was a typo in this problem: $c^a = f$ should have been $c^2 a = f$, and this should have thrown everyone off as c^a doesn't make a whole lot of sense...

5. $G_5 = \langle x, y, z \mid x^2 = 1,\ y^4 = 1,\ z^2 = 1,\ y^2 z = 1,\ xyxy = 1, xy^{-1}zxy = 1\rangle$

Hamoon writes: "oh and that looks like a D_4 to me....What do you think??"

I think Hamoon is correct. The last two relations show that $xyxy = xy^{-1}zxy$. Cancelling the leading x and trailing xy, we get $y = y^{-1}z$, or $z = y^2$. The relation $xyxy = 1$ means that $xyx = y^{-1}$. But $y^4 = 1$, so $y^{-1} = y^3$ (similarly $x^{-1} = x$). This relation thus says $xyx = y^3$, or $yx = xy^3$. Thus G_5 may be written as

$$\begin{aligned} G_5 &= \left\langle x, y \mid x^2 = 1,\ y^4 = 1,\ yx = xy^3 \right\rangle \\ &\cong \left\langle m, r \mid m^2 = 1,\ r^4 = 1,\ rm = mr^3 \right\rangle \\ &= D_4. \end{aligned}$$

It's been pointed out to me that the last relation is really superfluous. Here's why: the relation $y^2 z = 1$ implies that $z^{-1} = y^2$. But $z^2 = 1$, so $z^{-1} = z$. Thus $z = y^2$. Now continue as above.