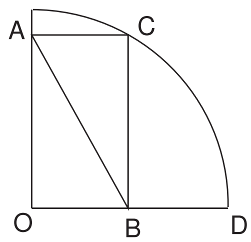
SOAR2001 — GEOMETRY
SUMMER 2001
PROBLEMS

Problem 1. A right-angled triangle is inscribed in a quarter circle, as in the picture. It is given that $OB = BD = 5$. Find the hypotenuse AB.



Problem 2. In a quadrangle $ABCD$, $AB = 4$, $CD = 5$. Let K, H be the midpoints of sides BC and AD, and let P, M be the midpoints of the diagonals AC, BD respectively. Find the perimeter of $PKMH$.

Problem 3. Show that any triangle having two equal medians is isosceles.

Problem 4. a)Where is the orthocenter of a right-angled triangle?
b)Where is the circumcenter of a right-angled triangle? Show that for a right-angled triangle $2R + r = p$, where R is the circumradius, r is the inradius and p is the semiperimeter of the triangle.

Problem 5. Show that in a right-angled triangle the bisector of the right angle cuts the angle between the median and the altitude in half.

Problem 6. Show that the sum of the lengths of the medians lies between $\frac{3}{4}P$ and P, where P is the perimeter of the triangle.

Problem 7. Show that the altitude AD of any triangle ABC is of length $2R\sin\beta\sin\gamma$, where $\beta = \angle B$ and $\gamma = \angle C$.

Problem 8. In a triangle ABC the angle bisector AD is drawn. Show that $AB \cdot CD = AC \cdot BD$.

Problem 9. Using the pictures below give two proofs of the Pythagorean Theorem.

Problem 10. a) Given an arbitrary triangle, show that it can be cut up into finitely many pieces that can be rearranged to get a rectangle.
b) Given an arbitrary rectangle, show that it can be cut up into finitely many pieces that can be rearranged to get a square.

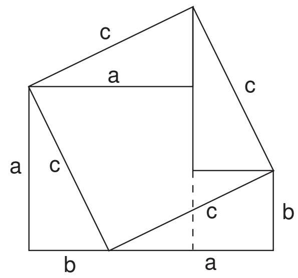


FIGURE 1. Proof 1

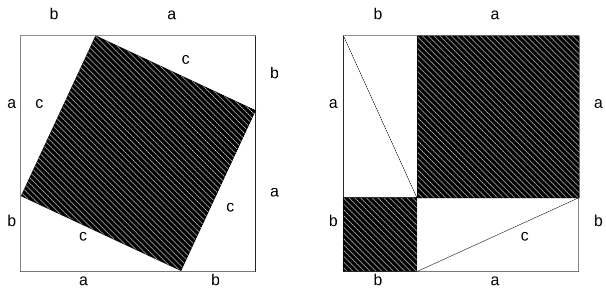
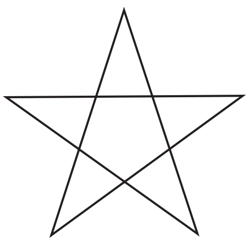


FIGURE 2. Proof 2

Problem 11. a) The plane is colored in two colors. Show that one can always find 2 points at unit distance that have the same color. Is it always possible to find two points at unit distance that have different color?
b) The plane is colored in three colors. Show that there are 2 points at unit distance that have the same color.

Problem 12. Draw a closed six-edge polygonal graph such that each edge intersects exactly one other edge. Can a polygonal graph with this property have 7 or 8 edges? (The star in the picture represents a closed five-edge polygonal graph such that each edge intersects exactly two other edges.)



Problem 13. How many diagonals does a 2001-gon have?

Problem 14. How can you connect three loops in such a way that they do not fall apart but when any of them is cut the other two fall apart? (The solution can be found in front of the Fields Institute.)