SOAR2001 — GEOMETRY
SUMMER 2001

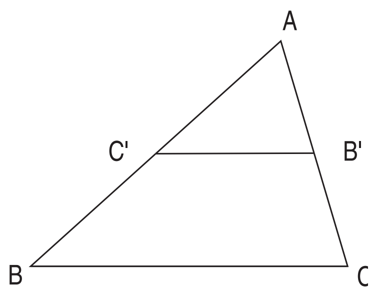
1. Introduction to plane geometry

In this Chapter we review some elementary plane geometry. We assume that the notions of isosceles triangles, parallel lines, similar triangles, area, etc. are already familiar. We will review the definitions of medians, angle bisectors, perpendicular bisectors and altitudes, and show that the three medians/ angle bisectors/ perpendicular bisectors/ altitudes are concurrent. We will also prove the sine law and some formulas for the area of a triangle. At the end of this chapter we prove the nine point circle theorem.

Try to solve each problem by yourself first. If you need help use the hints and the pictures in the solutions, but try not to read the solutions.

The line joining the midpoints of two sides in a triangle is called a *midline*.

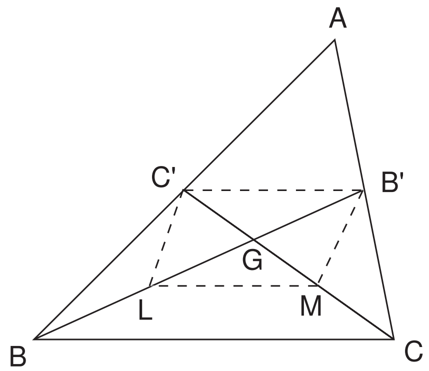
Problem 1. Show that a midline in a triangle is parallel to the base (the third side of the triangle), and is half as long.
Hint: Draw the picture and find similar triangles.



Solution: Triangles ABC and $AC'B'$ are similar, since they share angle A and $AC'/AB = AB'/AC = \frac{1}{2}$. Therefore $C'B'/BC = 1/2$, i.e. the midline is half as long as the base. Since $\angle AC'B' = \angle ABC$, the midline $C'B'$ is parallel to the base BC.

The line joining a vertex to the midpoint of the opposite side is called a *median*.

Problem 2. a) Given a triangle ABC, draw 2 of the three medians, say BB' and CC'. Let them meet at point G. Show that BB' and CC' trisect each other at G, i.e. $2C'G = GC$ and $2B'G = GB$.
Hint: Look at the picture below.
b) Conclude that all the three medians of any triangle pass through one point.



Solution: a) Since $C'B'$ is a midline of ABC, it is parallel to the base BC and is half as long. LM is a midline of GBC, therefore it is parallel to the base BC and is half as long. It follows that $C'B'$ and LM are parallel and equal, i.e. $B'C'LM$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, we have

$$B'G = GL = LB, \qquad C'G = GM = MC.$$

Thus the two medians BB', CC' trisect each other at G.
b)Draw one of the medians, say AA'. Let G be the point on this median, such that $AG = 2AG'$. Since each of the other two medians has to trisect the median AA' they both have to pass through the point G. We conclude that all the medians pass through one point.

The point of intersection of the medians G is called the *centroid* of a triangle.

Problem 3. Show that the three bisectors of the three angles of a triangle all pass through one point I.
Hint: Prove and use the fact that the bisector of an angle is the loci of points equidistant from the sides of the angle.

Solution: The point of intersection of the first two angle bisectors is equidistant from all three sides, therefore it has to belong to the the third angle bisector, i.e. all three bisectors pass through one point.

The point I of intersection of the three angle bisectors, is equidistant from all three sides of the triangle, and hence is the center of the inscribed circle(also called *incircle*). We denote the radius of this circle by r and call the point I the *incenter*.

Problem 4. In a triangle ABC let $a = BC$, $b = CA$, $c = AB$. Let $p = \frac{1}{2}(a + b + c)$ be the semiperimeter and S be the area of ABC. Show that $S = pr$.

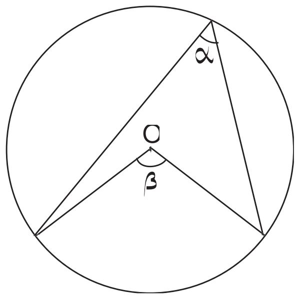
Solution: The area of ABC is equal to the sum of the areas of triangles BIC, AIC, AIB. Each of these three triangles has height r. Adding the areas of these three triangles we deduce

$$S = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = pr.$$

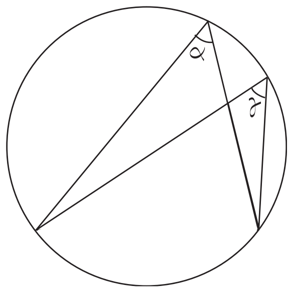
Problem 5. Show that the perpendicular bisectors of the three sides of a triangle all pass through one point O.
Hint: The perpendicular bisector of a line segment is the loci of the points equidistant from the end points.

This point O is equidistant from all three vertices of a triangle, and hence is the center of the circumscribed circle. This circle is called *circumcircle* and the center O is called the *circumcenter*. We denote the radius of the circumcircle by R.

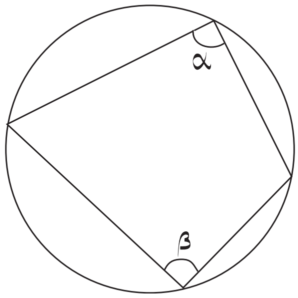
Problem 6. Prove the following statements:
a) In a circle the angle at the center is double the angle at the circumference, if these two angles are subtended by the same arc, i.e on the picture, $\beta = 2\alpha$.



b) In a circle, if two angles at the circumference are subtended by the same arc, these two angles are equal , i.e. in the picture, $\alpha = \gamma$.

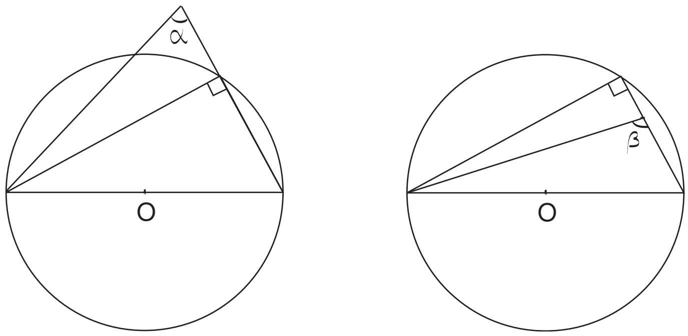


c) In a circle, the angle at the circumference that is subtended by a diameter is equal to $\pi/2$, and vice versa, if a right angle is subtended by a diameter, the vertex of the angle belongs to the circumference.
d) The opposite angles of any quadrangle inscribed in a circle are together equal to π, i.e. in the picture, $\alpha + \beta = \pi$
Solution: a) Since $AO = OB = OC = R$, triangles BOA, BOC and AOC are isosceles. Denote $\angle OBA = \angle BAO$ by α_1, $\angle OBC = \angle OCB$ by α_2, $\angle OCA = \angle OAC = \beta'$. Now, $\pi = \angle ABC + \angle BAC + \angle ACB = \alpha + \alpha_1 + \alpha_2 + 2\beta' = 2\alpha + 2\beta'$. On the other hand, from triangle AOC, $2\beta' = \pi - \beta$. It follows that $\beta = 2\alpha$.



b) Each of these two angles is equal to a half of the angle at the center, therefore these angles are equal.

c) The sum of these two angles is equal to half the sum of the angles at the center, i.e π.



d)The first part follows from a), since the corresponding angle at the center is π. Second part: if the vertex were not on the circumference, it would be either outside or inside of the circle. If the vertex is outside, the angle α is one of the acute angles of a right-angled triangle. If the vertex is inside, the angle β is obtuse, as a supplement to one of the acute angles of a right-angled triangle. It follows that the vertex is on the circumference.

Problem 7. Let a,b, c be the sides of triangle ABC, α, β, γ be the opposite angles, R be the circumradius, and S be the area of the triangle. Then

a)

$$S = \frac{1}{2} ab \sin\gamma$$

b)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \text{(Sine Law)}$$

Solution: a) Draw the height h from vertex A. Since $h = b\sin\gamma$, we get $S = \frac{1}{2}ab = \frac{1}{2}ab\sin\gamma$.

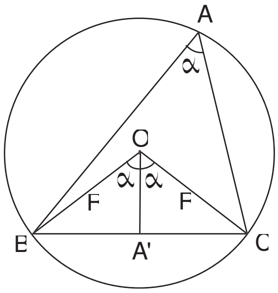
b) We use the formula proved in a):

$$S = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ac\sin\beta = \frac{1}{2}bc\sin\alpha.$$

From $ab\sin\gamma = ac\sin\beta$ we deduce $\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$. Similarly, $ac\sin\beta = bc\sin\alpha$ which implies $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$, and this completes the proof.

The line through a vertex perpendicular to the opposite side is called an *altitude*.

Problem 8. In the notations of the previous problem, $S = \frac{abc}{4R}$.



Solution: Draw the circumcircle, connect vertices B and C with the center O and draw the altitude OA' from the center O to the side BC. By part a), $\angle BOC = 2\alpha$. Since the triangle BOC is isosceles, the altitude OA' is an angle bisector and a median. Therefore, $\angle BOA' = \alpha$ and $BA' = \frac{1}{2}a$. From triangle BOA' we get $\frac{1}{2}a = R\sin\alpha$, i.e. $sin\alpha = \frac{a}{2R}$. Now, $S = \frac{1}{2}bc\sin\alpha = \frac{abc}{4R}$.

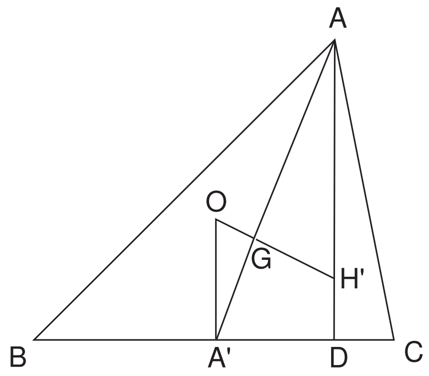
Problem 9. The three altitudes of any triangle ABC all pass through one point H.
Hint: Construct the triangle whose sides are the three lines through the three vertices of ABC, each of them parallel to the opposite side of ABC.

Solution: In this new triangle, the altitudes of ABC are the perpendicular bisectors of the sides, and we have already proved that they intersect at one point.

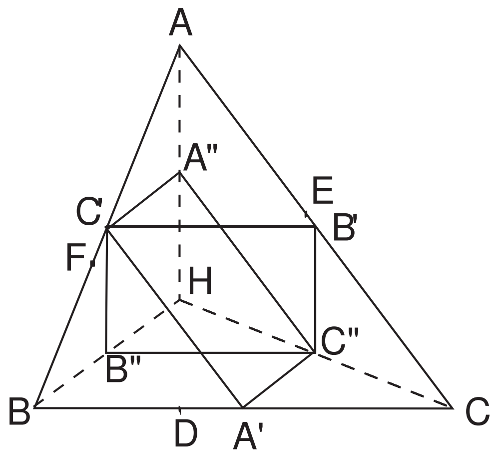
The point of intersection of the three altitudes H is called the *orthocenter* of the triangle.

Problem 10. The circumcenter O, the centroid G, and the orthocenter H, all lie on the same line and $HG = 2GO$. This line is called the *Euler line.*

Solution: Draw a line through the circumcenter O and centroid G. On this line, consider a point H' such that $GH' = 2OG$. Let A' be the midpoint of BC. Then G lies on the median AA' and OA' is a perpendicular bisector (see the picture). Triangles OGA' and AGH' are similar, since $AG = 2GA'$, $GH' = 2OG$ and $\angle AGH' = \angle OGA'$. Therefore OA' is parallel to AH', i.e. AH' is perpendicular to BC, and we deduce that AD is an altitude. Similarly BH' is perpendicular to CA, and CH' to AB, i.e. all the three altitudes pass through H'. We conclude that $H' = H$, i.e. the orthocenter H lies on the same line as the circumcenter O and the centroid G.



Problem 11. The midpoints of the three sides of a triangle, the midpoints of the lines joining the orthocenter to the three vertices, and the feet of the three altitudes, all lie on a circle. This circle is called the *nine-point circle*.



Solution: Let A',B',C',A'',B'',C'' be the midpoints of BC,CA,AB,HA,HB,HC respectively, and let D,E,F be the feet of the altitudes, as in the picture. Since $C'B'$ and $B''C''$ are midlines of triangles BAC, BHC, both $C'B'$ and $B''C''$ are parallel to the base BC, and are half as long. It follows that $B'C'B''C''$ is a parallelogram. On the other hand, $B''C'$ and $C''B'$ are both parallel to AH as midlines of triangles HBA and HCA. Since AH is an altitude it is perpendicular to BC. From this we conclude that $C'B'$ is perpendicular to $B''C'$, and $B'C'B''C''$ is a rectangle. Similarly $C'A'C''A''$ is a rectangle.

The three diagonals $A'A''$, $B'B''$, $C'C''$ bisect each other at the point of intersection. Hence $A'A''$, $B'B''$, $C'C''$ are three diameters of a circle. It remains to show that the points D,E,F belong to the same circle. The diameters $A'A$, $B'B''$, $C'C''$ subtend the right angles $A''DA'$, $B''EB'$, $C''FC'$ correspondingly, therefore the vertices D, E, F belong to the circle.