Proving the Quadratic Formula

I. Quadratic Forms

Recall that a quadratic polynomial is usually expressed in one of these popular forms:

1. **Standard Form**: $\boldsymbol{a\,x^2 + b\,x + c}$ where a, b, c are *constants* with $a \neq 0$.
2. **Factored Form**: $\boldsymbol{d\,(m\,x+s)(n\,x+t)}$ where d, m, n, s, t are *constants* with $d, m, n \neq 0$.
3. **Vertex Form**: $\boldsymbol{a\,(x-h)^2 + k}$ where a, h, k are *constants* with $a \neq 0$.

Suppose we were asked to solve a quadratic equation, $p(x) = 0$. Recall that the **ultimate goal** when solving any equation is to *isolate the variable*. Notice that with vertex form: $\boldsymbol{a\,(x-h)^2 + k}$, the variable x is already in *one* place, unlike standard and factored form. So, for the equation:

$$\boldsymbol{a\,(x-h)^2 + k} = 0$$

, we will isolate x by simply *doing the opposite*! The steps are as follows:

$$\begin{aligned}
a(x-h)^2 + k = 0 \quad &\Rightarrow \quad a(x-h)^2 = -k \\
&\Rightarrow \quad (x-h)^2 = -\frac{k}{a} \\
&\Rightarrow \quad \sqrt{(x-h)^2} = \sqrt{-\frac{k}{a}} \\
&\Rightarrow \quad |x-h| = \sqrt{-\frac{k}{a}} && \text{by property of } n\text{th roots} \\
&\Rightarrow \quad x-h = \pm\sqrt{-\frac{k}{a}} && \text{by definition of } \textit{absolute value} \\
&\Rightarrow \quad x = h \pm \sqrt{-\frac{k}{a}}
\end{aligned}$$

II. The Proof

Unfortunately, we rarely get quadratic equations, where the quadratic polynomial is already in vertex form. However, we know that we can always transform a quadratic from standard form to vertex form by **completing the square**.

So, to find the general solutions to the equation: $ax^2 + bx + c = 0$, we will complete the square on the quadratic (which is in standard form). Then we will isolate x, like we did above!

$$\begin{aligned}
& ax^2 + bx + c = 0 \\
\Longleftrightarrow \quad & (ax^2 + bx \qquad\quad) + c = 0 && \text{removing the } c \\
\Longleftrightarrow \quad & a\,(x^2 + \frac{b}{a}x \qquad\quad) + c = 0 && \text{factoring out } a
\end{aligned}$$

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$$\iff \quad a\left(\boxed{x^2+\frac{b}{a}x+\boldsymbol{\frac{b^2}{4a^2}}}\underbrace{-\boldsymbol{\frac{b^2}{4a^2}}}_{=0}\right)+c=0$$

for $m=\dfrac{b}{2a}$, we add and subtract $m^2=\dfrac{b^2}{4a^2}$

$$\iff \quad a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-a\left(\frac{b^2}{4a^2}\right)+c=0$$

distributing a to the *perfect square* and $-\dfrac{b^2}{4a^2}$

$$\iff \quad a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c=0$$

perfect square is $(x+m)^2$; $\dfrac{a}{a}=1$ since $a\neq 0$

$$\iff \quad a\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a}-c\right)=0$$

factoring out -1

$$\iff \quad a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}=0$$

getting an LCD

$$\iff \quad a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a}$$

adding $\dfrac{b^2-4ac}{4a}$ to both sides

$$\iff \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4\boldsymbol{a^2}}$$

multiply both sides by $\dfrac{1}{a}$ (i.e. divide both sides by a)

$$\iff \quad \sqrt{\left(x+\frac{b}{2a}\right)^2}=\sqrt{\frac{b^2-4ac}{4\boldsymbol{a^2}}}$$

take the *square root* of both sides

$$\iff \quad \left|x+\frac{b}{2a}\right|=\frac{\sqrt{b^2-4ac}}{\sqrt{4\boldsymbol{a^2}}}$$

by properties of nth roots

$$\iff \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{\mathbf{2}\sqrt{\boldsymbol{a^2}}}$$

by def'n of *absolute value* and property of nth roots

$$\iff \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2\,|\boldsymbol{a}|}$$

by property of nth roots

$$\iff \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2\,(\pm\boldsymbol{a})}$$

by def'n of *absolute value*

$$\iff \quad x+\frac{b}{2a}=\pm\left(\pm\frac{\sqrt{b^2-4ac}}{2a}\right)$$

rewriting

$$\iff \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

since $(\pm1)(\pm1)=\pm1$

$$\iff \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

subtracting $\dfrac{b}{2a}$ from both sides

$$\iff \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

getting an LCD

