

COHOMOLOGY RINGS OF TORIC BUNDLES AND THE RING OF CONDITIONS

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To the memory of Ernest Borisovich Vinberg

ABSTRACT. The celebrated BKK Theorem expresses the number of roots of a system of generic Laurent polynomials in terms of the mixed volume of the corresponding system of Newton polytopes. In [PK92b], Pukhlikov and the second author noticed that the cohomology ring of smooth projective toric varieties over \mathbb{C} can be computed via the BKK Theorem. This complemented the known descriptions of the cohomology ring of toric varieties, like the one in terms of Stanley-Reisner algebras.

In [SU03], Sankaran and Uma generalized the “Stanley-Reisner description” to the case of toric bundles, i.e. equivariant compactifications of (not necessarily algebraic) torus principal bundles. We provide a description of the cohomology ring of toric bundles which is based on a generalization of the BKK Theorem, and thus extends the approach by Pukhlikov and the second author. Indeed, for every cohomology class of the base of the toric bundle, we obtain a BKK-type theorem. Furthermore, our proof relies on a description of graded-commutative algebras which satisfy Poincaré duality.

From this computation of the cohomology ring of toric bundles, we obtain a description of the ring of conditions of horospherical homogeneous spaces. We conclude the manuscript with a number of examples. In particular, we apply our results to toric bundles over a full flag variety G/B . The description that we get generalizes the corresponding description of the cohomology ring of toric varieties as well as the one of full flag varieties G/B previously obtained by Kaveh in [Kav11].

1. INTRODUCTION

In this paper we discuss descriptions of the cohomology ring of toric bundles and how to use them to compute the ring of conditions of horospherical homogeneous spaces. Our main contributions are:

- (1) A generalization of the celebrated BKK Theorem (Theorem 4.1). We also provide an alternative interpretation (Theorem 4.3).
- (2) A description of finite-dimensional graded commutative algebras with Poincaré duality (Theorem 7.16).
- (3) A description of the cohomology ring of toric bundles (Theorem 8.1).
- (4) A description of the ring of conditions of horospherical homogeneous spaces (Corollary 8.4).
- (5) We illustrate our results by computing the cohomology ring of some toric bundles over full flag varieties and the corresponding ring of conditions (Theorem 9.10).

Let us start with a recollection of the classical toric case. Let $T \simeq (\mathbb{C}^*)^n$ be an algebraic torus with character lattice M and lattice of one-parameter subgroups N . Further, let X_Σ be a smooth projective T -toric variety given by a fan $\Sigma \subseteq N_\mathbb{R} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Denote the rays of Σ by $\Sigma(1) = \{\rho_1, \dots, \rho_s\}$ and their primitive generators in N by e_1, \dots, e_s . Recall the following well-known description of the cohomology ring of X_Σ (see, for instance, [CLS11, Theorem 12.4.1]):

$$H^*(X_\Sigma, \mathbb{R}) \simeq \mathbb{R}[x_1, \dots, x_s]/(I + J) =: R_\Sigma,$$

where I is generated by monomials $x_{i_1} \cdots x_{i_t}$ such that $\rho_{i_1}, \dots, \rho_{i_t} \in \Sigma(1)$ are distinct and do not form a cone in Σ and $J = \langle \sum_{i=1}^s \chi(e_i) x_i : \chi \in M \rangle$. Note that I depicts the Stanley-Reisner ideal of the fan Σ and therefore we refer to this description as the *Stanley-Reisner description* of $H^*(X_\Sigma, \mathbb{R})$. This description yields an algorithm to evaluate products of cohomology classes in the top degree of R_Σ . In Section 2.1, we provide further details.

On the other hand, given line bundles L_1, \dots, L_t on X_Σ , one can directly compute a top degree intersection product $c_1(L_1)^{k_1} \cdots c_1(L_t)^{k_t}$ in $H^*(X_\Sigma, \mathbb{R})$ by using the BKK theorem [Kou76] (see also [Ber75, BKH76]). Here, $c_1(L_i) \in H^2(X_\Sigma, \mathbb{R})$ denotes the first Chern class of L_i . More precisely, as any line bundle on X_Σ is the difference of two ample line bundles, it suffices to evaluate products $c_1(L_1)^{k_1} \cdots c_1(L_t)^{k_t}$ with all L_i ample. By the toric dictionary, the ample line bundles L_i correspond to polytopes P_i whose normal fan coarsens the fan Σ . Using, the BKK Theorem, we obtain

$$c_1(L_1)^{k_1} \cdots c_1(L_t)^{k_t} = n! \cdot V(\underbrace{P_1, \dots, P_1}_{k_1 \text{ times}}, \dots, \underbrace{P_t, \dots, P_t}_{k_t \text{ times}})$$

where $V(P_1, \dots, P_1, \dots, P_t, \dots, P_t)$ denotes the mixed volume of the n -tuple $(P_1, \dots, P_1, \dots, P_t, \dots, P_t)$. In [PK92b, Section 1.4], Pukhlikov and the second author observed that the information on these intersection

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products suffices to regain a description of the cohomology ring $H^*(X_\Sigma, \mathbb{R})$. As virtual polytopes play a crucial role in this description, we refer to it as the *virtual polytope description* of the cohomology ring $H^*(X_\Sigma, \mathbb{R})$. In Section 2.2, we provide further details.

In [Kav11], Kaveh used a similar approach to study the cohomology ring of spherical varieties provided that it is generated in degree 2. In particular, he obtained a “volume-polynomial-description” for the cohomology ring of full flag varieties G/B by using the relation between the volume of string polytopes and the degree of the corresponding line bundle from [AB05]. Recently, in [KK20], Kaveh and the second author introduced the *ring of complete intersections* for an (arbitrary) algebraic variety. For a smooth complete algebraic variety X whose cohomology ring $H^*(X, \mathbb{R})$ is generated in degree 2, the ring of complete intersections coincides with $H^*(X, \mathbb{R})$. They also provide a description of the ring of complete intersections which generalizes the results of [Kav11].

Certainly, the cohomology ring of toric varieties accepts many more descriptions. In particular, we want to emphasize the description by Brion [Bri96]. Brion’s description is a byproduct of the computation of the *equivariant* cohomology ring. In [Bri96], Brion proves that $H^*(X_\Sigma, \mathbb{R})$ is isomorphic to the ring of piecewise polynomial functions on Σ modulo the ideal generated by global linear functions.

Our main contribution is a generalization of this picture to the case of toric bundles. For a principal T -torus bundle $p: E \rightarrow B$ over a closed orientable real manifold B and a T -toric variety X_Σ , let $E_\Sigma := (E \times X_\Sigma)/T$ be the associated toric bundle (see Section 3 for details). A generalization of the Stanley-Reisner description for the cohomology ring $H^*(E_\Sigma, \mathbb{R})$ was obtained by Sankaran and Uma [SU03]. In [Hof19], the first author noticed that Brion’s description also generalizes to the case of toric bundles (for more details see Section 8.2). Like in the toric case, the description by Sankaran and Uma implicitly contains an algorithm to compute products of cohomology classes in the top degree. In Section 3.1, we provide further details.

We complement the descriptions of the cohomology ring of toric bundles with a generalization of the virtual polytope description. Like in the toric case our description relies on (a generalization of) the BKK theorem (see Theorem 4.1). More precisely, we reduce the computation of intersection numbers on the toric bundle to intersection numbers on the base. This provides a “BKK-type theorem” for any choice of a cohomology class in the base $\gamma \in H^*(B, \mathbb{R})$.

An equivalent description of the generalized BKK Theorem can be given as follows. Suppose the torus T has rank n and the real dimension of B is k . Similar to the toric case, a virtual polytope Δ defines a cohomology class $\rho(\Delta) \in H^2(E_\Sigma, \mathbb{R})$ on the T -toric bundle $p: E_\Sigma \rightarrow B$. In Theorem 4.3, for any given j , we define a map which associates to a virtual polytope Δ a cohomology class $\gamma_{2j}(\Delta) \in H^{2j}(B, \mathbb{R})$ such that

$$\rho(\Delta)^{n+j} \cdot p^*(\gamma) = \gamma_{2j}(\Delta) \cdot \gamma,$$

for any $\gamma \in H^{k-2j}(B, \mathbb{R})$. Here “ \cdot ” denotes the cup product on the respective cohomology ring. We call the class $\gamma_{2j}(\Delta)$ *the horizontal part* of $\rho(\Delta)^{n+j}$. We refer to Section 4 for further details.

It should be mentioned that our statements above are true not only in the algebraic category, but more generally hold for smooth manifolds.

Furthermore, unlike the toric case or the case studied in [Kav11], the cohomology ring of E_Σ is not generated in degree 2 in general. However, finite-dimensional graded commutative algebras which satisfy Poincaré duality accept a description generalizing the one used by Pukhlikov and the second author (see Theorem 7.16).

From the generalized BKK-theorem and our description of graded commutative algebras with Poincaré duality, we obtain a description of the cohomology ring of toric bundles (see Theorem 8.1). This description is well suited to compute cohomology rings of toric bundles over a fixed base manifold B . In particular, a computation of the ring of conditions of horospherical homogeneous spaces naturally follows (Corollary 8.4). Recall that the ring of conditions is an intersection ring for (not necessarily complete) homogeneous spaces. Furthermore, for a connected complex reductive group G a homogeneous space G/H is called *horospherical* if H is a closed subgroup in G containing a maximal unipotent subgroup.

We conclude our manuscript by applying our results to a certain class of toric bundles (see Theorem 9.10). In particular, we recover the results of [KK20] (and thus the results of [Kav11]) in the case of toric bundles over complete flag varieties.

Organization of the paper. Section 2 provides further details for the two descriptions of the cohomology ring of toric varieties mentioned above. Section 3 collects results about the cohomology ring of toric bundles and horospherical varieties. Section 4 contains the statement of the generalized BKK theorem. Section 5 provides further details about virtual polytopes and the related convex chains. Section 6 contains the proof of the generalized BKK theorem. Section 7 studies graded commutative finite-dimensional algebras satisfying Poincaré duality. Section 8 describes the cohomology ring of toric bundles (in general) and computes the ring of conditions of horospherical homogeneous spaces. Section 9 computes the cohomology ring of certain classes of toric bundles including toric bundles over full flag varieties G/B and projective bundles.

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2. THE COHOMOLOGY RING OF A TORIC VARIETY

We assume fundamental knowledge of toric geometry and refer to [CLS11] for further details and references. For the reader's convenience, we give further details of the two descriptions of the cohomology ring of toric varieties mentioned in Section 1. We continue to use the notation from the Introduction. In particular, M denotes a lattice, i.e., a finitely generated free abelian group, i.e., $M \simeq \mathbb{Z}^n$, $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual lattice, and $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ their dual pairing. Suppose that X_{Σ} is a smooth projective toric variety given by a smooth projective fan $\Sigma \subseteq N_{\mathbb{R}} := N \otimes \mathbb{R}$.

2.1. Intersection products in the Stanley-Reisner description. The Stanley-Reisner description of the cohomology ring yields an algorithm to compute products of cohomology classes in the top degree of $H^*(X_{\Sigma}, \mathbb{R})$: As R_{Σ} is generated in degree 1, it suffices to consider monomials $x_{i_1}^{k_1} \dots x_{i_t}^{k_t}$. Recall that the graded piece of top degree of R_{Σ} , is one-dimensional and generated by $x_{i_1} \dots x_{i_n}$ for any collection i_1, \dots, i_n of indices such that the rays $\rho_{i_1}, \dots, \rho_{i_n}$ generate a full-dimensional cone in Σ . Indeed, all such monomials $x_{i_1} \dots x_{i_n}$ yield the same element in R_{Σ} . Therefore, the evaluation of a monomial $x_{i_1}^{k_1} \dots x_{i_t}^{k_t}$ in the top degree amounts to expressing it as a linear combination of square free monomials.

For a monomial $x_{i_1}^{k_1} \dots x_{i_t}^{k_t}$ let $\sum_{i=1}^t (k_i - 1)$ be its *multiplicity*, so that being square free is equivalent to having multiplicity 0. To simplify notation, consider the monomial $x_1^{k_1} \dots x_t^{k_t}$ (always possible by reordering the variables). Suppose $x_1^{k_1} \dots x_t^{k_t}$ is a monomial with multiplicity $m > 0$ and $k_1 > 1$. The goal is to express it as a linear combination of monomials of smaller multiplicity. If ρ_1, \dots, ρ_t do not form a cone, then the monomial is equal to zero in R_{Σ} and we are done. Otherwise, the set $\{e_1, \dots, e_t\}$ can be extended to a lattice basis of N , and thus there is $\chi \in M$ such that $\chi(e_1) = 1$ and $\chi(e_j) = 0$, for $j = 2, \dots, t$. Note that χ induces a linear relation in J , so that we obtain

$$x_1^{k_1} \dots x_t^{k_t} = x_1^{k_1-1} \dots x_t^{k_t} \cdot \left(- \sum_{k=t+1}^s \chi(e_k) x_k \right).$$

This is a linear combination of monomials of multiplicity $m - 1$. Applying this procedure inductively, we end up with a linear combination of square free monomials, and thus obtain an evaluation of $x_1^{k_1} \dots x_t^{k_t}$.

2.2. The virtual polytope description. As noted in the introduction, an arbitrary line bundle on X_{Σ} can be expressed as a differences of ample line bundles which, by the toric dictionary, correspond to polytopes whose normal fan coarsens Σ . Recall that a fan Σ' in $N_{\mathbb{R}}$ *coarsens* Σ if every cone of Σ is contained in a cone of Σ' and $|\Sigma'| = |\Sigma|$. One also says that Σ *refines* Σ' . In particular, every cone of Σ' is a union of cones of Σ . One of the crucial ideas in [PK92b] is to reformulate this observation on line bundles into convex geometry. Therefore let us recall the definition of the space of virtual polytopes:

Let \mathcal{P}_{Σ}^+ be the set of polytopes in $M_{\mathbb{R}} := M \otimes \mathbb{R}$ such that Σ refines their normal fans. Recall that the *Minkowski sum* of two polytope P, Q in $M_{\mathbb{R}}$ is given by

$$P + Q = \{p + q : p \in P, q \in Q\},$$

and that the normal fan of a Minkowski sum is the coarsest common refinement of the individual fans (see [Zie95, Proposition 7.12]). Thus the Minkowski sum endows the structure of a monoid on \mathcal{P}_{Σ}^+ . It is well-known that a commutative semigroup can be embedded in a group if the operation is *cancelative*, i.e., $P_1 + Q = P_2 + Q$ implies $P_1 = P_2$. It is straightforward to show that the Minkowski sum satisfies this property. We denote the smallest group \mathcal{P}_{Σ} containing the monoid \mathcal{P}_{Σ}^+ the group of *virtual polytopes* (associated to the fan Σ). Clearly \mathcal{P}_{Σ}^+ accepts a natural multiplication map by the nonnegative reals $\mathbb{R}_{\geq 0}$ which straightforwardly extended to a scalar multiplication by \mathbb{R} on the group of virtual polytopes, thus turning \mathcal{P}_{Σ} into a real vector space. Moreover, note that this also shows that \mathcal{P}_{Σ}^+ is a cone in \mathcal{P}_{Σ} , and as it generates the ambient vector space it is full-dimensional.

For us the following alternative description of this vector space will be useful. Recall that one associates to a non-empty compact convex set $A \subseteq M_{\mathbb{R}}$ its *support function*:

$$h_A : N_{\mathbb{R}} \rightarrow \mathbb{R}; \quad x \mapsto h_A(x) := \sup\{\langle a, x \rangle : a \in A\}$$

In other words, given an exterior normal vector $x \neq 0$, the position of the corresponding supporting hyperplane is determined by its support function h_A , namely $A \subseteq \{y \in M_{\mathbb{R}} : \langle y, x \rangle \leq h_A(x)\}$ with the hyperplane $\{y \in M_{\mathbb{R}} : \langle y, x \rangle = h_A(x)\}$ intersecting A in a non-empty face. The crucial observation is that h_A uniquely describes A . Indeed, one can show that $A = \{x \in M_{\mathbb{R}} : \langle x, u \rangle \leq h_A(u) \text{ for all } u \in N_{\mathbb{R}}\}$.

Using the language of support functions \mathcal{P}_{Σ}^+ and \mathcal{P}_{Σ} can be reformulated as follows:

$$\begin{aligned} \mathcal{P}_{\Sigma}^+ &= \{h : N_{\mathbb{R}} \rightarrow \mathbb{R} : \text{convex, piecewise linear function on } \Sigma\}, \\ \mathcal{P}_{\Sigma} &= \{h : N_{\mathbb{R}} \rightarrow \mathbb{R} : \text{piecewise linear function on } \Sigma\}, \end{aligned}$$

where a function $h: N_{\mathbb{R}} \rightarrow \mathbb{R}$ is piecewise linear on Σ if $h_{\sigma} := h|_{\sigma}$ can be extended to a linear function on $\text{span}\{\sigma\}$ for any $\sigma \in \Sigma$. In addition, such a function is *convex* if $h(u) \geq h_{\sigma}(u)$ for any $\sigma \in \Sigma$ and $u \in \text{span}\{\sigma\}$.

With the above observation on line bundles we have that $\mathcal{P}_{\Sigma}/M_{\mathbb{R}} \simeq H^2(X_{\Sigma}, \mathbb{R})$ where $M_{\mathbb{R}}$ is interpreted as a linear subspace of \mathcal{P}_{Σ} , namely the subspace of linear functions. Indeed, any $m \in M_{\mathbb{R}}$ can be expressed as a difference $(P + m) - P$ for any polytope $P \in \mathcal{P}_{\Sigma}^+$. Note that the volume yields a well-defined polynomial function on $\mathcal{P}_{\Sigma}/M_{\mathbb{R}}$. The following theorem is implicit in [PK92b] and forms a key idea to get a description of the cohomology ring $H^*(X_{\Sigma}, \mathbb{R})$. The following version is taken from [Kav11, Theorem 1.1].

Theorem 2.1. *Suppose $A = \bigoplus_{i=0}^n A_i$ is a graded finite dimensional commutative algebra over a field \mathbb{K} of characteristic 0 such that A is generated (as an algebra) by the elements A_1 of degree one, $A_0 \simeq A_n \simeq \mathbb{K}$, and the bilinear map $A_i \times A_{n-i} \rightarrow A_n$ is non-degenerate for any $i = 0, \dots, n$ (Poincaré duality). Then*

$$A \simeq \mathbb{K}[t_1, \dots, t_r] / \{p(t_1, \dots, t_r) : p(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r})f(x_1, \dots, x_r) = 0\}$$

where we identify A_1 with \mathbb{K}^r via a basis v_1, \dots, v_r and define $f: A_1 \simeq \mathbb{K}^r \rightarrow \mathbb{K}$ as the polynomial given by $f(x_1, \dots, x_r) = (x_1 v_1 + \dots + x_r v_r)^n \in A_n \simeq \mathbb{K}$.

Remark 2.2. Instead of identifying A_1 with \mathbb{K}^r , the previous theorem accepts a basis-free formulation in terms of $\text{Diff}(A_1)$, the ring of differential operators with constant coefficients: $A \simeq \text{Diff}(A_1) / \{p \in \text{Diff}(A_1) : p(f) = 0\}$.

In Section 7, we study algebras with Poincaré duality which are not necessarily generated in degree 1. In particular, we prove Theorem 7.16 which is a generalization of Theorem 2.1. Then in Theorem 9.1, we show that Theorem 2.1 is indeed a corollary of Theorem 7.16.

Under the identification $\mathcal{P}_{\Sigma}/M_{\mathbb{R}} \simeq H^2(X_{\Sigma}, \mathbb{R})$ together with BKK theorem, we get that the function $H^2(X_{\Sigma}, \mathbb{R}) \rightarrow H^{2n}(X_{\Sigma}, \mathbb{R}) \simeq \mathbb{R}; x \mapsto x^n$ corresponds to the volume polynomial on \mathcal{P}_{Σ}/M , and thus we obtain (using the formulation from Remark 2.2):

$$H^*(X_{\Sigma}, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_{\Sigma}/M_{\mathbb{R}}) / \text{Ann}(\text{Vol}) \simeq \text{Diff}(\mathcal{P}_{\Sigma}) / \text{Ann}(\text{Vol})$$

where $\text{Ann}(\text{Vol})$ denotes the ideal of differential operators which annihilate the volume polynomial $\text{Vol}: \mathcal{P}_{\Sigma} \rightarrow \mathbb{R}$.

3. TORIC BUNDLES AND HOROSPHERICAL VARIETIES

In this section, we collect several facts on toric bundles which will be used below. We also introduce horospherical varieties which yield an important class of toric bundles.

3.1. Toric bundles. Let G be a topological group and let $p: E \rightarrow B$ be a G -principal bundle over a topological space B , i.e., $p: E \rightarrow B$ is a fiber bundle with structure group G equipped with a G -atlas such that the action of G on the fibers $p^{-1}(b)$ for any $b \in B$ is free and transitive. For convenience, we will identify the fibers of p with G , so that we obtain a (right) action of G on E which preserves the fibers and is transitive and free on them. To any G -principal bundle $p: E \rightarrow B$ and any topological space X equipped with a continuous action by G , one associates a fiber bundle by introducing a (right) action on the product $E \times X$:

$$(e, x) \cdot g := (e \cdot g, g^{-1} \cdot x).$$

The *associated fiber bundle* is given as the quotient $E \times_G X := (E \times X)/G$. It is a fiber bundle with fiber X . If $G = T$ is an algebraic torus, then a T -principal bundle is also called a *torus bundle*.

Crucial to the understanding of the cohomology of fiber bundles is the following theorem (see, for instance, [BT82, Theorem 5.11] or [Hat02, Theorem 4D.1]):

Theorem 3.1 (Leray–Hirsch). *Let E be a fibre bundle with fibre F over a compact manifold B . If there are global cohomology classes u_1, \dots, u_r on E whose restrictions $i^*(u_i)$ form a basis for the cohomology of each fibre F (where $i: F \rightarrow E$ is the inclusion), then we have an isomorphism of vector spaces:*

$$H^*(B, \mathbb{R}) \otimes H^*(F, \mathbb{R}) \rightarrow H^*(E, \mathbb{R}); \quad \sum_{i,j} b_i \otimes i^*(u_j) \mapsto \sum_{i,j} p^*(b_i) \cdot u_j.$$

Corollary 3.2. *If T is an algebraic torus, $p: E \rightarrow B$ a T -toric bundle as in Theorem 3.1, and X a smooth projective T -toric variety, then as a group the cohomology of $E_X = E \times_T X$ is given by*

$$H^*(E_X, \mathbb{R}) \simeq H^*(B, \mathbb{R}) \otimes H^*(X, \mathbb{R}).$$

Proof. As a group, the cohomology of smooth projective toric varieties is generated by classes Poincaré dual to orbit closures. For any orbit closure \overline{O} of X , let $E_{\overline{O}} = E \times_T \overline{O}$ be a submanifold of E_X . Note that E_X is compact, and thus Poincaré duality applies. The cohomology classes u_1, \dots, u_r Poincaré dual to these submanifolds satisfy the condition of Theorem 3.1. The statement follows. \square

Corollary 3.2 yields a description of the cohomology *group* of E_X . Crucial for this description is the map which associates to any orbit closure \overline{O} of X the Poincaré dual of the corresponding T -invariant submanifold of E_X . By restricting this map to divisors and extending using linearity, we obtain

$$\rho: \mathcal{P}_\Sigma \rightarrow H^2(E_X, \mathbb{R}) \quad (\text{where } \Sigma \text{ is the fan of } X = X_\Sigma)$$

which plays an important role in our description of the cohomology ring of E_X . We provide more details for ρ . Recall that ρ_1, \dots, ρ_r denote the rays of Σ with integer generators e_1, \dots, e_r . Let D_1, \dots, D_r be the corresponding divisors in X . We also write D_i for the submanifold $E \times_T D_i$ of E_X . Then for $\Delta \in \mathcal{P}_\Sigma$, we have:

$$\rho(\Delta) = \sum_{i=1}^r h_\Delta(e_i)[D_i] \in H^2(E_X, \mathbb{R}),$$

where $[D_i]$ is the class Poincaré dual to $D_i \subseteq E_X$ and $h_\Delta: N_\mathbb{R} \rightarrow \mathbb{R}$ is the support function of Δ .

The following observation about $\rho(\cdot)$ will be crucial for our approach. Any character $\lambda \in M$ defines a one-dimensional representation \mathbb{C}_λ of T , namely $t \cdot z = \lambda(t)z$ for $t \in T$, and $z \in \mathbb{C}_\lambda$. If \mathcal{L}_λ denotes the associated complex line bundle on B , i.e. $\mathcal{L}_\lambda \simeq E \times_T \mathbb{C}_\lambda$, then $\mathcal{L}_{\lambda+\mu} = \mathcal{L}_\lambda \otimes \mathcal{L}_\mu$, and thus we obtain a group homomorphism:

$$c: M \rightarrow H^2(B, \mathbb{Z}), \quad \lambda \mapsto c_1(\mathcal{L}_\lambda),$$

where $c_1(\mathcal{L}_\lambda)$ is the first Chern class. By linearity, we extend the homomorphism to a map of vector spaces:

$$c: M_\mathbb{R} \rightarrow H^2(B, \mathbb{R}).$$

Proposition 3.3. *Let Σ be a smooth complete fan, and $p: E_X \rightarrow B$ as before. Then for any character $\lambda \in M$:*

$$p^*c(\lambda) = \rho(\lambda),$$

where on the right hand side of the equality λ is regarded as a virtual polytope.

Proof. A character λ of T defines a T -invariant divisor $\text{div}(\lambda)$ on X which can be expressed as $\text{div}(\lambda) = \sum_{i=1}^r k_i D_i$ where the $D_i \subseteq X$ are distinct irreducible divisors and $k_i \in \mathbb{Z}$. Every divisor D_i comes with a (naturally) linearized line bundle \mathcal{L}_i and a global regular section s_i such that $\text{div}(s_i) = D_i$. The sections s_i yield generic global (smooth) sections of the associated line bundle $E \times_T \mathcal{L}_i \rightarrow E \times_T X$ with degeneracy locus $E \times_T D_i$. By [GH94, Gauss–Bonnet Formula II, p. 413], $c_1(E \times_T \mathcal{L}_i)$ is Poincaré dual to $E \times_T D_i$. Using linearity of c_1 , we get $c_1(E \times_T \mathcal{L}_\lambda) = \rho(\lambda)$. It is straightforward to verify that $E \times_T \mathcal{L}_\lambda = p^* \mathcal{L}_\lambda$, and so

$$p^*c(\lambda) = c_1(p^* \mathcal{L}_\lambda) = c_1(E \times_T \mathcal{L}_\lambda) = \rho(\lambda). \quad \square$$

We conclude this section with the following description of the cohomology ring of toric bundles by Sankaran and Uma. We note that this description is true in greater generality as stated here and we refer the reader to [SU03, Theorem 1.2] for the complete statement. The following version will be sufficient for what follows.

Theorem 3.4. *We continue to use the notation from above. If X_Σ is smooth and projective, the cohomology ring $H^*(E_\Sigma, \mathbb{R})$ is isomorphic (as an $H^*(B, \mathbb{R})$ -algebra) to the quotient of $H^*(B, \mathbb{R})[x_1, \dots, x_r]$ by*

$$\left(\langle x_{j_1} \cdots x_{j_k} : \rho_{j_1}, \dots, \rho_{j_k} \text{ do not span a cone of } \Sigma \rangle + \left\langle c(\lambda) - \sum_{i=1}^n \langle e_i, \lambda \rangle x_i : \lambda \in M \right\rangle \right).$$

Note the similarities with the Stanley-Reisner description of the cohomology ring of toric varieties. Indeed, the first ideal in Theorem 3.4 corresponds to the Stanley-Reisner ideal of the corresponding toric variety. In particular, the algorithm from Section 2.1 can be used to compute products in the top degree of $H^*(E_X, \mathbb{R})$.

3.2. Horospherical varieties. In this section, we recall horospherical varieties which provide an important source of examples of toric bundles. We conclude with a description of their associated cohomology ring.

Let G be a connected reductive complex algebraic group. A closed subgroup $H \subseteq G$ is called *horospherical* if it contains a maximal unipotent subgroup U of G and $E := G/H$ is called a *horospherical homogeneous space*. Let $B \subseteq G$ be a Borel subgroup whose unipotent radical is U . It is known that $P := N_G(H)$ is a parabolic subgroup containing B . The natural quotient $p: E \rightarrow G/P$ is a torus bundle with fibre the torus $T = P/H$. Let M be the character lattice of T . Let $\Sigma \subseteq N_\mathbb{R} := \text{Hom}_\mathbb{Z}(M, \mathbb{R})$ be a fan with corresponding toric variety $X = X_\Sigma$. The associated toric bundle $p: E_X \rightarrow G/P$ is a horospherical variety. In general, an irreducible normal G -variety Y together with an open equivariant embedding $G/H \hookrightarrow Y$ is called a *horospherical variety*. Recall that a morphism of G -varieties $\varphi: Y \rightarrow Y'$ is called *equivariant* if $\varphi(g \cdot y) = g \cdot \varphi(y)$ for any $g \in G$ and $y \in Y$. We refer to [Kno91, Tim11] for further details and references on horospherical varieties.

We conclude this section with a combinatorial description of the cohomology ring $H^*(E_X, \mathbb{R})$. Recall from the previous section that any character $\lambda \in M$ gives rise to a line bundle \mathcal{L}_λ on G/P , so that we obtain a group homomorphism $c: M \rightarrow H^2(G/P, \mathbb{Z})$. The following statement combinatorially describes the cohomology ring of E_X provided X_Σ is smooth and projective. It is a special case of a more general result.

Theorem 3.5 ([SU03, Theorem 1.2]). *Suppose Σ has rays ρ_1, \dots, ρ_r with primitive vectors $v_1, \dots, v_n \in N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ along the edges ρ_i . If X_{Σ} is smooth and projective, then the cohomology ring $H^*(E_X, \mathbb{R})$ is isomorphic as an $H^*(G/P, \mathbb{R})$ -algebra to the quotient of $H^*(G/P, \mathbb{R})[x_1, \dots, x_r]$ by the sum of ideals*

$$\langle x_{j_1} \cdots x_{j_k} : \rho_{j_1}, \dots, \rho_{j_k} \text{ do not span a cone of } \Sigma \rangle + \left\langle c(\lambda) - \sum_{i=1}^r \langle v_i, \lambda \rangle x_i : \lambda \in M \right\rangle.$$

4. BKK THEOREMS FOR TORIC BUNDLES

This section provides an overview of the main results of this paper: Theorems 4.1 and 4.3. Details and proofs will be given in Section 6, and in Section 8 these theorems will be used to describe the cohomology ring of toric bundles.

As before, let $p: E \rightarrow B$ be a principal torus bundle with respect to a torus $T \simeq (\mathbb{C}^*)^n$ over a compact smooth orientable manifold B of real dimension k . Let M be the character lattice of T and $\Sigma \subseteq N_{\mathbb{R}} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{R})$ be a smooth complete fan which gives rise to a toric variety $X = X_{\Sigma}$. Let E_X be the total space of the associated toric bundle. Note that E_X is a compact smooth orientable manifold of real dimension $k + 2n$. To keep notation simple we denote the projection map of the toric bundle by $p: E_X \rightarrow B$ as well.

Our main theorems show that a choice of a natural number $i \leq \frac{k}{2}$ and $\gamma \in H^{k-2i}(B, \mathbb{R})$ gives rise to a BKK -type theorem. First, we define two functions I_{γ} and F_{γ} on \mathcal{P}_{Σ} as follows.

Let $f_{\gamma}: M_{\mathbb{R}} \rightarrow \mathbb{R}$ be given by

$$f_{\gamma}(x) = c(x)^i \cdot \gamma,$$

where “ \cdot ” denotes the cup product of the cohomology ring $H^*(B, \mathbb{R})$. Since $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$ is a linear map, f_{γ} is a homogeneous polynomial of degree i on $M_{\mathbb{R}}$. This leads to the definition of I_{γ} :

$$I_{\gamma}: \mathcal{P}_{\Sigma} \rightarrow \mathbb{R}; \quad I_{\gamma}(\Delta) := \int_{\Delta} f_{\gamma}(x) d\mu \quad \text{for } \Delta \in \mathcal{P}_{\Sigma}^+,$$

where μ denotes the Lebesgue measure on $M_{\mathbb{R}}$ normalized with respect to the lattice M , i.e. a cube spanned by an affine lattice basis of M has volume 1. Note that we gave an explicit formula for I_{γ} on the full-dimensional cone \mathcal{P}_{Σ}^+ . By Theorem 5.5, I_{γ} extends to a homogeneous polynomial of degree $n + i$ on \mathcal{P}_{Σ} .

Recall the definition of $\rho: \mathcal{P}_{\Sigma} \rightarrow H^2(E_X, \mathbb{R})$ from Section 3.1. This leads to the definition of the function F_{γ} :

$$F_{\gamma}: \mathcal{P}_{\Sigma} \rightarrow \mathbb{R}; \quad F_{\gamma}(\Delta) := \rho(\Delta)^{n+i} \cdot p^*(\gamma).$$

Clearly, F_{γ} is a homogeneous polynomial of degree $n + i$ on \mathcal{P}_{Σ} .

The main result of this section is the following analog of the BKK theorem for toric bundles. Indeed, it expresses certain intersection numbers of cohomology classes as mixed integrals.

Theorem 4.1. *The polynomials I_{γ} and F_{γ} are proportional with coefficient of proportionality given by:*

$$(n + i)! \cdot I_{\gamma}(\Delta) = i! \cdot F_{\gamma}(\Delta) \quad \text{for any } \Delta \in \mathcal{P}_{\Sigma}.$$

In particular, the polarizations of I_{γ} and F_{γ} are proportional multilinear forms, i.e. for any $\Delta_1, \dots, \Delta_{n+i} \in \mathcal{P}_{\Sigma}$

$$(n + i)! \cdot I_{\gamma}(\Delta_1, \dots, \Delta_{n+i}) = i! \cdot F_{\gamma}(\Delta_1, \dots, \Delta_{n+i}).$$

For the reader’s convenience, we recall the concept of polarization (or equivalently mixed integrals). Let V be a (possibly infinite-dimensional) vector space. Here $V = \mathcal{P}_{\Sigma}$ is finite-dimensional, however eventually we want to consider \mathcal{P} which is an infinite-dimensional vector space.

Definition 4.2. The *Lie derivative* $L_v f(x)$ of a function $f: V \rightarrow \mathbb{R}$ with respect to a vector $v \in V$ at a point $x \in V$ is the limit

$$L_v f(x) = \lim_{t \rightarrow 0} \frac{f(x + t \cdot v) - f(x)}{t},$$

provided this limit exists.

Note that for a finite-dimensional vector space V the concept of polynomials $f: V \rightarrow \mathbb{R}$ is well-known. In the next section, we recall how to extend the concept of polynomials to the infinite-dimensional case. As the concept of polarization works both in the finite- and the infinite-dimensional case, we provide details for the general situation here. The careful reader may want to consult Definition 5.1 before reading further.

Recall that for a homogeneous polynomial $f: V \rightarrow \mathbb{R}$ of degree m , the *polarization* of f is the unique symmetric multilinear form $g: V^m \rightarrow \mathbb{R}$ such that $g(v, \dots, v) = f(v)$. It is a well-known fact that for any vector space V and any homogeneous polynomial f of degree m , the polarization exists and can be defined as follows:

$$g(v_1, \dots, v_m) = \frac{1}{m!} L_{v_1} \cdots L_{v_m} f.$$

For a homogeneous polynomial $f: M_{\mathbb{R}} \rightarrow \mathbb{R}$, the polarization of I_f is called the *mixed integral* of f .

Note that in the case of a trivial torus bundle $p: T \rightarrow \text{pt}$, Theorem 4.1 reduces to the classical BKK theorem. We conclude this section with an alternative interpretation of Theorem 4.1 which can be favourable for certain

applications. By the Leray-Hirsch Theorem (see Corollary 3.2), $H^*(E_X, \mathbb{R}) \simeq H^*(B, \mathbb{R}) \otimes H^*(X, \mathbb{R})$, and so for $\Delta \in \mathcal{P}_\Sigma$, the cycle $\rho(\Delta)^{n+i} \in H^{2n+2i}(E_X, \mathbb{R})$ can be written as

$$\rho(\Delta)^{n+i} = b_{2n+2i} \otimes x_0 + b_{2n+2i-2} \otimes x_2 + \dots + b_{2i+2} \otimes x_{2n-2} + b_{2i} \otimes x_{2n},$$

with $b_s \in H^s(B, \mathbb{R})$ and $x_r \in H^r(X, \mathbb{R})$. As X is a smooth, complete toric variety, its cohomology groups in odd degrees vanish, and therefore $x_{2k+1} = 0$ for any k . If x_{2n} is normalized such that it is dual to a point, we call b_{2i} the *horizontal part* of $\rho(\Delta)^{n+i}$. Equivalently, the horizontal part b_{2i} of $\rho(\Delta)^{n+i}$ is the unique class in $H^{2i}(B, \mathbb{R})$ such that

$$\rho(\Delta)^{n+i} \cdot p^*(\eta) = b_{2i} \cdot \eta,$$

for any $\eta \in H^{k-2i}(B, \mathbb{R})$. Then Theorem 4.1 accepts the following reformulation.

Theorem 4.3. *For any $\Delta \in \mathcal{P}_\Sigma$, the horizontal part of $\rho(\Delta)^{n+i}$ can be computed as*

$$b_{2i} = \frac{(n+i)!}{i!} \int_{\Delta} c(x)^i dx.$$

Note that $c(\cdot)^i: M_{\mathbb{R}} \rightarrow H^{2i}(B, \mathbb{R})$ is a vector valued map whose components (after choosing suitable coordinates) are given by homogeneous polynomials of degree i . Thus the integral in Theorem 4.3 exists. Furthermore, although we show that Theorem 4.1 implies Theorem 4.3, in fact they are equivalent.

Proof. Since $H^*(B, \mathbb{R})$ satisfies Poincaré duality, it suffices to check that for any $\gamma \in H^{k-2i}(B, \mathbb{R})$, we have

$$i! \cdot \gamma \cdot b_{2i} = (n+i)! \cdot \gamma \cdot \int_{\Delta} c(x)^i dx,$$

Recall from Theorem 3.1, that there are $u_1, \dots, u_r \in H^*(E_X, \mathbb{R})$ such that the restrictions $i^*(u_i)$ form a basis for the cohomology of each fibre X where $i: X \rightarrow E_X$ is the inclusion. In particular, there are $y_i \in \mathbb{R}u_1 \oplus \dots \oplus \mathbb{R}u_r$ such that $i^*(y_i) = x_i$. Since x_{2n} is Poincaré dual to a point (say to a torus fixed point $x \in X$), it follows that $y_{2n} = E \times_T \{x\}$, i.e. $y_{2n} = [S]^*$ is the class Poincaré dual to a section of p .

Let $[\text{pt}]^* \in H^k(B, \mathbb{R})$ be the class dual to a point in B . For $\gamma \in H^{k-2i}(B, \mathbb{R})$, let $\gamma \cdot b_{2i} = a \cdot [\text{pt}]^*$ for some real number $a \in \mathbb{R}$. Then,

$$\begin{aligned} p^*(\gamma) \cdot \rho(\Delta)^{n+i} &= p^*(\gamma) \cdot (p^*(b_{2n+2i}) \cdot y_0 + p^*(b_{2n+2i-2}) \cdot y_2 + \dots + p^*(b_{2i+2}) \cdot y_{2n-2} + p^*(b_{2i}) \cdot y_{2n}) \\ &= p^*(\gamma \cdot b_{2i}) \cdot y_{2n} = a \cdot p^*([\text{pt}]^*) \cdot E \times_T \{x\} = a \cdot [X]^* \cdot [S]^* = a \in H^{2n+k}(E_X, \mathbb{R}) \simeq \mathbb{R}, \end{aligned}$$

where $[X]^*$ is the class dual to a fibre of p . Hence, if we interpret $\gamma \cdot b_{2i}$ and $p^*(\gamma) \cdot \rho(\Delta)^{n+i}$ as real numbers (multiples of classes dual to a point), we get

$$\gamma \cdot b_{2i} = p^*(\gamma) \cdot \rho(\Delta)^{n+i} = F_\gamma(\Delta).$$

On the other hand,

$$\gamma \cdot \int_{\Delta} c(x)^i dx = \int_{\Delta} \gamma \cdot c(x)^i dx = I_\gamma(\Delta).$$

The statement follows by Theorem 4.1. □

Like Theorem 4.1, Theorem 4.3 admits a polarized version.

5. CONVEX CHAINS

In this subsection, we recall how to extend the definition of $I_\gamma: \mathcal{P}_\Sigma^+ \rightarrow \mathbb{R}$ from to all of \mathcal{P}_Σ . In exposition we mostly follow [PK92a]. Suppose that $\Sigma \subseteq N_{\mathbb{R}}$ is a *simplicial fan*, i.e., any cone in Σ is spanned by part of a basis of $N_{\mathbb{R}}$. Then the primitive generators e_1, \dots, e_s of the rays $\Sigma(1) = \{\rho_i = \mathbb{R}_{\geq 0}e_i: i = 1, \dots, s\}$ naturally induce coordinates on \mathcal{P}_Σ : a piecewise linear function $h \in \mathcal{P}_\Sigma$ is uniquely determined by the tuple $(h(e_1), \dots, h(e_s))$. We denote the corresponding coordinates on $\mathcal{P}_\Sigma \simeq \mathbb{R}^s$ by (h_1, \dots, h_s) .

We want to work with the space of virtual polytopes \mathcal{P} . This is an infinite-dimensional vector space obtained as inverse limit $\mathcal{P} = \varprojlim \mathcal{P}_\Sigma$ where the limit is taken over all complete fans Σ (partially ordered with respect to refinement of fans). As mentioned above, we consider polynomials, like I_γ , on this space. Let us recall the following notion of polynomials on a (possibly infinite-dimensional) real vector space V :

On the vector space of functions $g: V \rightarrow \mathbb{R}$, introduce the (*forward*) *difference operator*:

$$D_v g(x) := g(x+v) - g(x) \quad \text{where } v \in V.$$

Recall that a real topological vector space U is a real vector space equipped with a topology that makes the vector space operations $+: U \times U \rightarrow U$ and $\cdot: \mathbb{R} \times U \rightarrow U$ continuous. It is a known fact that every finite-dimensional Hausdorff topological vector space has the usual topology. From now on, we consider all finite-dimensional subspaces $U \subseteq V$ to be equipped with a topology such that U is a Hausdorff topological vector space.

Definition 5.1. A function $f: V \rightarrow \mathbb{R}$ is called a *polynomial of degree $\leq m$* if for any $v_1, \dots, v_{m+1} \in V$

$$D_{v_1} \cdots D_{v_{m+1}} f(x) = 0 \quad \text{for any } x \in V,$$

and the restriction $f|_U: U \rightarrow \mathbb{R}$ is continuous for any finite-dimensional vector space $U \subseteq V$.

Remark 5.2. It is straightforward to show that, if $f: V \rightarrow \mathbb{R}$ is a polynomial and $v_1, \dots, v_n \in V$ are linearly independent, then for any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have

$$(1) \quad f(\lambda_1 v_1 + \dots + \lambda_n v_n) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} D^\alpha f(0) \binom{\lambda_1}{\alpha_1} \cdots \binom{\lambda_n}{\alpha_n}$$

where $D^\alpha := D_{v_1}^{\alpha_1} \cdots D_{v_n}^{\alpha_n}$ and $\binom{\lambda}{\alpha} = \frac{\lambda(\lambda-1)\cdots(\lambda-\alpha+1)}{\alpha!}$ denotes the binomial coefficient. Note that Definition 5.1 deviates from [PK92a, Section 2, Definition 1] by the additional assumption of continuity. Indeed, without this assumption, equation (1) would be true for $\lambda_1, \dots, \lambda_n \in \mathbb{Q}$ (which is sufficient in [PK92a]). However, in our setting it is more natural to work over the field of real numbers, which makes this further assumption necessary.

In view of Remark 5.2, we may rephrase Definition 5.1 as follows:

Proposition 5.3. *A function $f: V \rightarrow \mathbb{R}$ is a polynomial of degree $\leq m$ if and only if its restriction to any finite dimensional subspace $U \subseteq V$ is a polynomial of degree $\leq m$ in the usual sense, i.e., an element in $\text{Sym}(U^*)$.*

Recall the objective of this section: Any smooth function $f: M_{\mathbb{R}} \rightarrow \mathbb{R}$ defines a function I_f on the cone of convex polytopes via (here μ denotes the standard Lebesgue measure on $M_{\mathbb{R}}$)

$$I_f: \mathcal{P}^+ \rightarrow \mathbb{R}; I_f(\Delta) = \int_{\Delta} f(x) d\mu.$$

Our goal is to extend I_f to the whole space of virtual polytopes \mathcal{P} . We recall how this is done following [PK92a].

Observe that $I_f: \mathcal{P}^+ \rightarrow \mathbb{R}$ is a *finitely additive measure* (also called a *valuation*) on the cone of convex polytopes, i.e. for any two convex polytopes $P, Q \in \mathcal{P}^+$ such that $P \cup Q$ is in \mathcal{P}^+ , we have

$$I_f(P \cup Q) = I_f(P) + I_f(Q) - I_f(P \cap Q).$$

A fundamental idea in [PK92a] is the introduction of the group of *convex chains* $Z(M_{\mathbb{R}})$, i.e. the additive group consisting of functions of the form $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{P_i}$ for integers n_i and convex polytopes $P_i \in \mathcal{P}^+$ where $\mathbb{1}_{P_i}$ denotes the characteristic function of the set P_i . The *degree* of a convex chain $\alpha \in Z(M_{\mathbb{R}})$ is defined to be $\sum_{i=1}^k n_i$ if $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{P_i}$ for some $P_i \in \mathcal{P}^+$ (this is a well-defined number by [PK92a, Proposition-Definition 2.1]). In [PK92a, Proposition-Definition 2.3], Pukhlikov and the second author observe that the Minkowski sum of polytopes induces a ring structure on $Z(M_{\mathbb{R}})$ via

$$\mathbb{1}_P \star \mathbb{1}_Q = \mathbb{1}_{P+Q} \quad \text{for } P, Q \in \mathcal{P}^+.$$

Clearly, the semigroup \mathcal{P}^+ can be embedded in the algebra of convex chains $Z(M_{\mathbb{R}})$. Furthermore, in [PK92a, Section 6] it is shown that \mathcal{P} can be identified with the set of invertible convex chains of degree 1.

Observe that finitely additive measures (as defined above) are in correspondence with homomorphisms of additive groups $\varphi: Z(M_{\mathbb{R}}) \rightarrow \mathbb{R}$. A finitely additive measure is *polynomial of degree $\leq m$* if for each $\alpha \in Z(M_{\mathbb{R}})$ the function $M_{\mathbb{R}} \rightarrow \mathbb{R}$ given by $\lambda \mapsto \varphi(\tau(\lambda, \alpha))$ is polynomial of degree $\leq m$ where $\tau(\lambda, \cdot): Z(M_{\mathbb{R}}) \rightarrow Z(M_{\mathbb{R}})$ denotes the linear map induced by translation by λ , i.e. $\tau(\lambda, \alpha)(x) = \alpha(x - \lambda)$. A fundamental result about polynomial finitely additive measures is the following theorem:

Theorem 5.4 ([PK92a, Corollary 7.5]). *A polynomial finitely additive measure $\varphi: Z(M_{\mathbb{R}}) \rightarrow \mathbb{R}$ of degree $\leq k$ restricted to the space of virtual polytopes $\varphi: \mathcal{P} \rightarrow \mathbb{R}$ is a polynomial of degree $\leq \dim(M_{\mathbb{R}}) + k$ in the sense of Definition 5.1 (however the continuity assumption might not be satisfied in general).*

We now apply the results of [PK92a] summarized above to our situation. Note that $I_f: \mathcal{P}^+ \rightarrow \mathbb{R}$ can be naturally extended to a homomorphism of additive groups $Z(M_{\mathbb{R}}) \rightarrow \mathbb{R}$ via

$$I_f(\alpha) = \int_{M_{\mathbb{R}}} f(x) \cdot \alpha(x) d\mu = \sum_{i=1}^k n_i \int_{P_i} f(x) d\mu \quad \text{for } \alpha = \sum_{i=1}^k n_i \mathbb{1}_{P_i} \in Z(M_{\mathbb{R}}).$$

Furthermore, $I_f: Z(M_{\mathbb{R}}) \rightarrow \mathbb{R}$ is a polynomial finitely additive measure of degree $\leq \deg(f)$ which follows by the observation that the following map is a polynomial of degree $\leq \deg(f)$ for any $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{P_i} \in Z(M_{\mathbb{R}})$:

$$I_f(\tau(\cdot, \alpha)): M_{\mathbb{R}} \rightarrow \mathbb{R}; \quad \lambda \mapsto \int_{\tau(\lambda, \alpha)} f(x) d\mu = \sum_{i=1}^k n_i \cdot \int_{P_i + \lambda} f(x) d\mu = \sum_{i=1}^k n_i \cdot \int_{P_i} f(x - \lambda) d\mu.$$

By Theorem 5.4, the restriction of I_f to the space of virtual polytopes is a polynomial of degree $\leq \dim(M_{\mathbb{R}}) + \deg(f)$. The continuity assumption of Definition 5.1 can be seen as follows.

By [PK92a, Corollary 2.2], the natural inclusion $\mathcal{P}^+ \rightarrow Z(M_{\mathbb{R}})$; $P \mapsto \mathbb{1}_P$ extends to an isomorphism of groups from \mathcal{P} (with Minkowski addition) onto the invertible elements \mathcal{P}^* of $Z(M_{\mathbb{R}})$ of degree 1 (equipped with the multiplication operation). The scalar multiplication on \mathcal{P} naturally carries over to \mathcal{P}^* :

$$(\mathbb{1}_P)^\lambda = \begin{cases} \mathbb{1}_{\lambda P} & \text{if } \lambda \geq 0, \\ \mathbb{1}_{-\lambda P}^{-1} & \text{if } \lambda < 0 \end{cases} \quad \text{for any convex polytope } P \subseteq M_{\mathbb{R}} \text{ and every real scalar } \lambda.$$

We obtain an isomorphism of vector spaces between $\mathcal{P} \simeq \mathcal{P}^*$. By our choice of the topology from above, restricting this isomorphism to corresponding finite-dimensional subspaces yields a homeomorphism. Let $U \subseteq \mathcal{P}^*$ be a finite-dimensional subspace and choose a basis $\alpha_1, \dots, \alpha_n \in U$. Then two elements $\lambda = \alpha_1^{\lambda_1} \star \dots \star \alpha_n^{\lambda_n}$ and $\mu = \alpha_1^{\mu_1} \star \dots \star \alpha_n^{\mu_n}$ in U are close to each other if and only if the scalars λ_i and μ_i are close to each other for any index i . It is straightforward to show that $I_f(\lambda)$ and $I_f(\mu)$ are close to each other, i.e. I_f is continuous.

Recall that a polynomial $f: V \rightarrow \mathbb{R}$ on a (possibly infinite-dimensional) vector space V is called *homogeneous of degree m* if for all $v \in V$ and every $t \in \mathbb{R}$, we have $f(t \cdot v) = t^m \cdot f(v)$.

To summarize the above, let us state the following theorem.

Theorem 5.5. *If $f: M_{\mathbb{R}} \rightarrow \mathbb{R}$ is a homogeneous polynomial of degree m , then the function $I_f: \mathcal{P}^+ \rightarrow \mathbb{R}$; $\Delta \mapsto I_f(\Delta) = \int_{\Delta} f(x) d\mu$ admits a unique extension to a homogeneous polynomial of degree $n + m$ on \mathcal{P} .*

6. PROOF OF THE BKK THEOREMS

In this section we prove Theorem 4.1. For the reader's convenience, we recall the used notation. Let $p: E \rightarrow B$ be a principal torus bundle with respect to an n -dimensional torus T over a smooth compact orientable manifold B of real dimension k . The character lattice of T we denote by M . Let $\Sigma \subseteq N_{\mathbb{R}} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{R})$ be a smooth complete fan with rays ρ_1, \dots, ρ_s . The primitive ray generators we denote by e_1, \dots, e_s . Let h_1, \dots, h_s be the basis of \mathcal{P}_{Σ} consisting of piecewise linear functions which vanish on every ray of Σ except one where it evaluates with 1 on the corresponding primitive ray generator. Let $X = X_{\Sigma}$ be the toric variety corresponding to Σ and denote the total space of the associated toric bundle by E_X . To keep notation simple we use the same notation $p: E_X \rightarrow B$ for the projection map of the toric bundle. Fix $i \leq \frac{k}{2}$ and a class $\gamma \in H^{k-2i}(B, \mathbb{R})$. The rays ρ_i correspond to divisors in X which give rise to divisors D_i in E_X . For a virtual polytope $\Delta \in \mathcal{P}_{\Sigma}$, we introduced

$$\rho(\Delta) = \sum_{i=1}^s h_{\Delta}(e_i)[D_i] \in H^2(E_X, \mathbb{R}),$$

where $[D_i]$ is the class dual to $D_i \subseteq E_X$ and $h_{\Delta}: N_{\mathbb{R}} \rightarrow \mathbb{R}$ is the support function of Δ . Further, we introduce

$$F_{\gamma}: \mathcal{P}_{\Sigma} \rightarrow \mathbb{R}; F_{\gamma}(\Delta) = \rho(\Delta)^{n+i} \cdot p^*(\gamma).$$

We noticed that F_{γ} is a homogeneous polynomial of degree $n + i$ on \mathcal{P}_{Σ} .

Recall that for any character $\lambda \in M$ we have an associated complex line bundle \mathcal{L}_{λ} on B . Taking Chern classes and extending by linearity, we obtain a morphism of vector spaces $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$. Let $f_{\gamma}: M_{\mathbb{R}} \rightarrow \mathbb{R}$ be the function $f_{\gamma}(x) = c(x)^i \cdot \gamma$. We defined a map $I_{\gamma}: \mathcal{P}_{\Sigma} \rightarrow \mathbb{R}$ which is explicitly given for $\Delta \in \mathcal{P}_{\Sigma}^+$ by

$$I_{\gamma}(\Delta) = \int_{\Delta} f_{\gamma}(x) d\mu.$$

In the previous section, we saw how to extend this definition to all of \mathcal{P}_{Σ} . Furthermore, by Theorem 5.5, I_{γ} is a homogeneous polynomial of degree $n + i$ on \mathcal{P}_{Σ} .

We prove Theorem 4.1 by induction on the parameter $0 \leq i \leq \frac{k}{2}$.

Let us start with the base case $i = 0$. If $i = 0$, then $\gamma \in H^k(B, \mathbb{R})$ is a multiple of the class dual to a point. For simplicity, let us assume that this multiple is 1. So $p^*(\gamma)$ is the class dual to a fibre, which is a toric variety, and thus $\rho(\Delta)^n \cdot p^*(\gamma)$ coincides with the degree of the divisor in X corresponding to Δ . By the classical BKK theorem, this can be computed by the normalized volume of Δ which equals to $n! \cdot I_{\gamma}(\Delta)$. In other words, for $i = 0$, Theorem 4.1 reduces to the classical BKK theorem.

As induction hypothesis, suppose that we know Theorem 4.1 for some $i - 1 \geq 0$. The induction step consists of proving that Theorem 4.1 is also true for i . Since both F_{γ} and I_{γ} are homogeneous polynomials of the same degree $n + i$, in order to show equality between $(n + i)! \cdot I_{\gamma}(\Delta)$ and $i! \cdot F_{\gamma}(\Delta)$, it suffices to show that all their partial derivatives of order n coincide. In other words, it suffices to consider differential monomials $\partial_{i_1}^{k_1} \dots \partial_{i_r}^{k_r}$ where $\partial_i = \partial/\partial h_i$ are the partial derivatives along the coordinate vectors of $\mathcal{P}_{\Sigma} \simeq \mathbb{R}^s$ and $\sum_{i=1}^r k_i = n$. Let us call the number $\sum_{i=1}^r (k_i - 1)$ the *multiplicity of the monomial* $\partial_{i_1}^{k_1} \dots \partial_{i_r}^{k_r}$. In particular, a monomial has multiplicity 0 if and only if it is square free.

The proof of equality of the partial derivatives of order n of $(n + i)! \cdot I_{\gamma}(\Delta)$ and $i! \cdot F_{\gamma}(\Delta)$ is by induction on the multiplicity m of the applied differential monomial. We will refer to the induction over i as the ‘‘outer induction’’ and we are going to call the induction over m the ‘‘inner induction’’.

The base case of the inner induction (i.e., the case of square free differential monomials) is treated in the next two subsections. The result of these calculations is summarized in Proposition 6.4.

6.1. Differentiation of I_γ . The main result of this subsection is Lemma 6.1 below which computes the partial derivative of I_γ with respect to square free monomial. Here, we work with a slightly more general situation than needed for the proof of Theorem 4.1: we assume that Σ is a simplicial fan, and we define the function $I_f: \mathcal{P} \rightarrow \mathbb{R}$ via $I_f(\delta) = \int_\delta f d\mu$ for any smooth function f on $M_{\mathbb{R}}$. In the proof of Theorem 4.1, we use $f = f_\gamma$.

Lemma 6.1. *Let $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset and k_1, \dots, k_r positive integers. If Δ is a polytope in the interior of \mathcal{P}_Σ^+ and $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone in Σ , then we have*

$$\partial_{i_1}^{k_1} \cdots \partial_{i_r}^{k_r} (I_f|_{\mathcal{P}_\Sigma}) (\Delta) = 0.$$

However, if $r = n$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a cone in Σ dual to the vertex $A \in \Delta$, we have

$$\partial_I (I_f|_{\mathcal{P}_\Sigma}) (\Delta) = f(A) \cdot |\det(e_{i_1}, \dots, e_{i_n})|.$$

Here $\det(e_{i_1}, \dots, e_{i_n})$ denotes the determinant of the matrix whose j -th column is given by the vector e_{i_j} . Furthermore, ∂_I denotes the partial derivative $\partial/\partial h_{i_1} \cdots \partial/\partial h_{i_n}$ along the coordinate vectors.

The proof of Lemma 6.1 relies on a folklore result on convex chains. We continue to use the notation from above. Fix real numbers $\{\lambda_i\}_{i \in I}$ and define the virtual polytope $h + \sum_{i \in I} \lambda_i h_i$ where h denotes the support function of the polytope Δ . If $\{\lambda_i\}_{i \in I}$ are sufficiently small, the virtual polytope $h + \sum_{i \in I} \lambda_i h_i$ is in the interior of \mathcal{P}_Σ^+ in which case it corresponds to the polytope obtained from Δ by moving the facets corresponding to the rays ρ_{i_j} according to the coefficients λ_{i_j} . For any subset $J \subseteq I$ we introduce the virtual polytope $\Delta(J) := h + \sum_{j \in J} \lambda_j h_j$. Furthermore, we define the convex chain

$$\sigma(\lambda_{i_1}, \dots, \lambda_{i_r}) = \sum_{J \subseteq I} (-1)^{r+|I_-|+|J|} \cdot \mathbf{1}_{\Delta(J)} \quad \text{where } I_- := \{i \in I: \lambda_i < 0\}.$$

The following folklore result will be needed in the proof of Lemma 6.1.

Proposition 6.2. *Let $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset and let $\{\lambda_i\}_{i \in I}$ be sufficiently small real numbers. If any $\lambda_i = 0$ for $i \in I$ or $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone in Σ , then $\sigma(\lambda_{i_1}, \dots, \lambda_{i_r}) = 0$. If all $\lambda_i \neq 0$ for $i \in I$, and the cardinality of I is $n = \dim(N_{\mathbb{R}})$, and $\rho_{i_1}, \dots, \rho_{i_n}$ span a cone in Σ dual to the vertex $A \in \Delta$, then*

$$\sigma(\lambda_{i_1}, \dots, \lambda_{i_n}) = \mathbf{1}_{A + \Pi(\lambda_{i_1}, \dots, \lambda_{i_n})}.$$

Here, $A + \Pi(\lambda_{i_1}, \dots, \lambda_{i_n})$ denotes the half-open parallelepiped spanned by the vectors $\lambda_{i_1} e_{i_1}, \dots, \lambda_{i_n} e_{i_n}$ shifted by the vertex A :

$$\Pi(\lambda_{i_1}, \dots, \lambda_{i_n}) = \left\{ \sum_{i \in I} a_i \lambda_i e_i : 0 < a_i \leq 1 \text{ if } \lambda_i > 0 \text{ and } 0 \leq a_i < 1 \text{ if } \lambda_i < 0 \right\}.$$

See Figures 1 and 2 for an illustration of Lemma 6.2 in dimension 2.

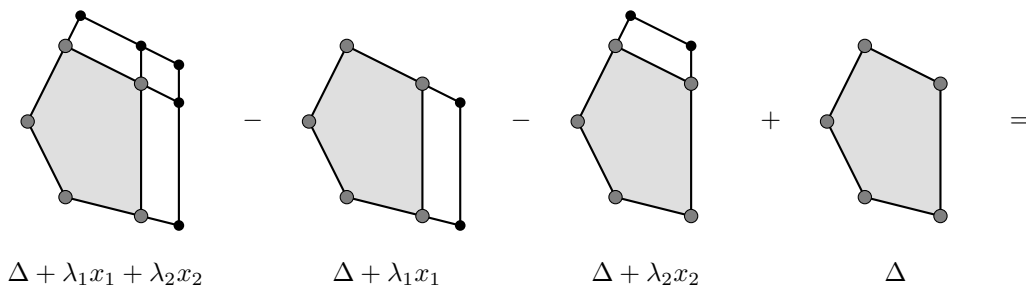


FIGURE 1. Alternating sum of characteristic functions for two adjacent edges.

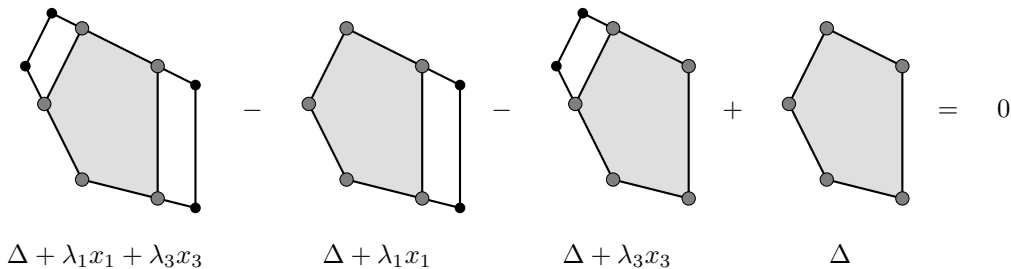


FIGURE 2. Alternating sum of characteristic functions for two disjoint edges.

Sketch of proof. For the reader's convenience we include a sketch of proof.

It is straightforward to check that $\sigma(\lambda_{i_1}, \dots, \lambda_{i_r})$ vanishes if any $\lambda_i = 0$ for $i \in I$. Hence suppose no λ_i vanishes for $i \in I$. It turns out to be more convenient to work with the case that all $\lambda_i > 0$ for $i \in I$. We can always reduce to this case as follows. Let $I_- := \{i \in I : \lambda_i < 0\}$ and set $\Delta' = h + \sum_{i \in I_-} \lambda_i h_i$. Let h' be the support function of Δ' and define $\Delta'(J) = h' + \sum_{j \in J} |\lambda_j| h_j$ for any subset $J \subseteq I$. Consider the convex chain

$$\sigma'(|\lambda_{i_1}|, \dots, |\lambda_{i_r}|) = \sum_{J \subseteq I} (-1)^{r+|J|} \mathbb{1}_{\Delta'(J)}.$$

It is straightforward to verify that $\sigma(\lambda_{i_1}, \dots, \lambda_{i_r}) = (-1)^{|I_-|} \cdot \sigma'(|\lambda_{i_1}|, \dots, |\lambda_{i_r}|)$, and thus we may assume from now on that $\lambda_i > 0$ for every $i \in I$.

Suppose that the facet F_i of Δ corresponding to the ray $\rho_i \in \Sigma$ is given by the supporting hyperplane $H_i = \{x \in M_{\mathbb{R}} : \langle x, e_i \rangle = c_i\}$ for some $c_i \in \mathbb{R}$. Note that the polytope $\Delta(I)$ is obtained from Δ by moving the supporting hyperplanes H_i for $i \in I$ outwards. We say that a point $x \in \Delta(I)$ is *beyond the hyperplane* H_i for $i \in I$ if $\langle x, e_i \rangle > c_i$. We get a map from the points of $\Delta(I)$ to the powerset 2^I of I :

$$\Delta(I) \rightarrow 2^I; x \mapsto \mathcal{I}(x) := \{i \in I : \langle x, e_i \rangle > c_i\}.$$

For $J \subseteq I$ define the region $\mathcal{C}(J)$ as the set of points $x \in \Delta(I)$ where $\mathcal{I}(x) = J$ is constant. The regions $\mathcal{C}(J)$ form a (disjoint) decomposition of $\Delta(I)$ into (possibly half-open) polytopes. Points in these regions behave well with respect to the characteristic functions $\mathbb{1}_{\Delta(J)}$ which appear in the convex chain $\sigma(\lambda_{i_1}, \dots, \lambda_{i_r})$:

$$\mathbb{1}_{\Delta(J)} = \begin{cases} 1 & \text{if } \mathcal{I}(x) \subseteq J, \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x \in \Delta(I) \text{ and } J \subseteq I.$$

For $x \in \Delta(I)$, a straightforward computation yields

$$\sigma(\lambda_{i_1}, \dots, \lambda_{i_r})(x) = \begin{cases} 0 & \text{if } |\mathcal{I}(x)| < r, \\ (-1)^r & \text{if } |\mathcal{I}(x)| = r. \end{cases}$$

Hence, $\sigma(\lambda_{i_1}, \dots, \lambda_{i_r}) = (-1)^r \cdot \mathbb{1}_{\mathcal{C}(I)}$. It is straightforward to show that $\mathcal{C}(I) = \emptyset$ if $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone in Σ . If $|I| = n = \dim(N_{\mathbb{R}})$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a cone in Σ dual to the vertex $A \in \Delta$, then a straightforward computation shows that $\mathcal{C}(I) = A + \Pi(\lambda_{i_1}, \dots, \lambda_{i_n})$. \square

Proof of Lemma 6.1. To keep notation simple assume $\rho_{i_j} = \rho_j$.

For a monomial $\partial_1^{k_1} \dots \partial_r^{k_r}$, let $\partial_I = \partial_1 \dots \partial_r$ be the corresponding square free monomial. It is enough to show that $\partial_I F_{\gamma}(\Delta) = 0$ in order to prove that $\partial_1^{k_1} \dots \partial_r^{k_r} F_{\gamma}(\Delta) = 0$.

For a smooth function g on \mathcal{P}_{Σ} , the partial derivative $\partial_I g(x)$ can be expressed as

$$(2) \quad \partial_I g(x) = \lim_{\lambda_1, \dots, \lambda_r \rightarrow 0} \left(\frac{1}{\lambda_1 \dots \lambda_r} D_{\lambda_1 h_1} \dots D_{\lambda_r h_r} g(x) \right),$$

where $D_{\lambda_i h_i}$ denotes the difference operator introduced above and $\lambda_1, \dots, \lambda_r \in \mathbb{R}$.

Applying equality (2) to the function $I_f|_{\mathcal{P}_{\Sigma}}$ one gets

$$\partial_I I_f(\Delta) = \lim_{\lambda_1, \dots, \lambda_r \rightarrow 0} \frac{1}{\lambda_1 \dots \lambda_r} \int_{M_{\mathbb{R}}} f(x) \cdot (-1)^{|I_-|} \cdot \sigma(\lambda_1, \dots, \lambda_r)(x) d\mu$$

where $\sigma(\lambda_1, \dots, \lambda_r)$ is the convex chain defined above and $I_- = \{i = 1, \dots, r : \lambda_i < 0\}$. By Proposition 6.2, the convex chain $\sigma(\lambda_1, \dots, \lambda_r)$ vanishes if ρ_1, \dots, ρ_r do not form a cone in Σ . By the same proposition, if $r = n$ and ρ_1, \dots, ρ_n generate a cone in Σ dual to the vertex $A \in \Delta$, it is equal to the characteristic function $\mathbb{1}_{A + \Pi(\lambda_1, \dots, \lambda_n)}$ of the half-open parallelepiped spanned by the vectors $\lambda_1 e_1, \dots, \lambda_n e_n$ at the point A . In this latter case, i.e. ρ_1, \dots, ρ_n span a cone in Σ , let $T: \mathbb{R}^n \rightarrow M_{\mathbb{R}}$ be the diffeomorphism given by $T(x_1, \dots, x_n) = x_1 e_1 + \dots + x_n e_n + A$. Then by a straightforward calculation, we obtain:

$$\int_{A + \Pi(\lambda_1, \dots, \lambda_n)} f(x) d\mu = (-1)^{|I_-|} \cdot \int_0^{\lambda_1} \dots \int_0^{\lambda_n} f(T(x_1, \dots, x_n)) \cdot |\det(e_1, \dots, e_n)| dx_1 \dots dx_n,$$

where $\det(e_1, \dots, e_n)$ is the determinant of the Jacobian matrix of T . So the lemma follows by the fundamental theorem of calculus:

$$\lim_{\lambda_1, \dots, \lambda_n \rightarrow 0} \frac{1}{\lambda_1 \dots \lambda_n} \int_{A + \Pi(\lambda_1 e_1, \dots, \lambda_n e_n)} (-1)^{|I_-|} \cdot f(x) d\mu = f(A) \cdot |\det(e_1, \dots, e_n)|. \quad \square$$

6.2. **Differentiation of F_γ .** Next we verify the base case of square free differential monomials for F_γ :

Lemma 6.3. *Let $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset and k_1, \dots, k_r positive integers. If Δ is a polytope in the interior of \mathcal{P}_Σ^+ such that $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone in Σ , we have*

$$\partial_{i_1}^{k_1} \dots \partial_{i_r}^{k_r} (F_\gamma|_{\mathcal{P}_\Sigma}) (\Delta) = 0.$$

If $r = n = \dim(N_{\mathbb{R}})$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a cone in Σ dual to the vertex $A \in \Delta$, we have

$$\partial_I (F_\gamma|_{\mathcal{P}_\Sigma}) (\Delta) = \frac{(n+i)!}{i!} f_\gamma(A).$$

Proof. To keep notation simple assume $i_j = j$ for $j = 1, \dots, r$.

For a monomial $\partial_1^{k_1} \dots \partial_r^{k_r}$, let $\partial_I = \partial_1 \dots \partial_r$ be the corresponding square free monomial. It is enough to show that $\partial_I F_\gamma(\Delta) = 0$ in order to prove that $\partial_1^{k_1} \dots \partial_r^{k_r} F_\gamma(\Delta) = 0$.

We compute the expansion of the polynomial F_γ at Δ . This amounts to expressing $F_\gamma(\Delta + \sum_{i=1}^s \lambda_i h_i)$ in terms of the monomials $\lambda_1^{\alpha_1} \dots \lambda_s^{\alpha_s}$ for non-negative integers $\alpha_1, \dots, \alpha_s$. A straightforward computation yields:

$$\begin{aligned} F_\gamma \left(\Delta + \sum_{i=1}^s \lambda_i h_i \right) &= \left(\rho(\Delta) + \sum_{i=1}^s \lambda_i [D_i] \right)^{n+i} \cdot p^*(\gamma) \\ &= \sum_{\alpha_0 + \alpha_1 + \dots + \alpha_s = n+i} \binom{n+i}{\alpha_0, \alpha_1, \dots, \alpha_s} \cdot \rho(\Delta)^{\alpha_0} \cdot [D_1]^{\alpha_1} \dots [D_s]^{\alpha_s} \cdot p^*(\gamma) \cdot \lambda_1^{\alpha_1} \dots \lambda_s^{\alpha_s} \\ &= \frac{(n+i)!}{(n+i-r)!} \cdot \rho(\Delta)^{n+i-r} \cdot [D_1] \dots [D_r] \cdot p^*(\gamma) \cdot \lambda_1 \dots \lambda_r + (\text{other terms}) \end{aligned}$$

where $\binom{n+i}{\alpha_0, \alpha_1, \dots, \alpha_s} = \frac{(n+i)!}{\alpha_0! \alpha_1! \dots \alpha_s!}$ denotes the usual multinomial coefficient. The derivative $\partial_I F_\gamma(\Delta)$ is equal to the coefficient in front of the monomial $\lambda_1 \dots \lambda_r$ in the expression $F_\gamma(\Delta + \sum_{i=1}^s \lambda_i h_i)$:

$$\partial_I F_\gamma(\Delta) = \frac{(n+i)!}{(n+i-r)!} \cdot \rho(\Delta)^{n+i-r} \cdot [D_1] \dots [D_r] \cdot p^*(\gamma).$$

Since Σ is a smooth fan, the divisors D_1, \dots, D_r intersect transversely in E_X . So the product $[D_1] \dots [D_r]$ is the class Poincaré dual to the set theoretic intersection of these divisors. In the case that e_1, \dots, e_r do not generate a cone in Σ the set theoretic intersection of D_1, \dots, D_r is empty, and so $\partial_I F_\gamma(\Delta) = 0$.

For the proof of the second part, let $r = n = \dim(N_{\mathbb{R}})$. We have

$$\partial_I F_\gamma(\Delta) = \frac{(n+i)!}{i!} \cdot \rho(\Delta)^i \cdot [D_1] \dots [D_n] \cdot p^*(\gamma).$$

If e_1, \dots, e_n generate a cone in Σ dual to the vertex A of Δ , then $[D_1] \dots [D_n] = E_A$, where $E_A = E \times_T A$ is the torus invariant submanifold in E_X corresponding to A . In particular, the restriction of the projection map $p: E_A \rightarrow B$ is a diffeomorphism.

Let $h_{\tilde{\Delta}}$ be the support function of the polytope $\tilde{\Delta} = \Delta - A$ which is the translation of the polytope Δ for which the vertex A is at the origin. Since the vertex of $\tilde{\Delta}$ corresponding to A is at the origin, we get

$$h_{\tilde{\Delta}}(e_1) = \dots = h_{\tilde{\Delta}}(e_n) = 0, \quad \text{and so} \quad \rho(\tilde{\Delta}) = \sum_{j>n} h_{\tilde{\Delta}}(e_j) \cdot [D_j].$$

Hence, by the first part, we get $\rho(\tilde{\Delta}) \cdot [D_1] \dots [D_n] = 0$ as there is no cone in Σ with more than n -rays. Therefore:

$$\rho(\Delta)^i \cdot [D_1] \dots [D_n] \cdot p^*(\gamma) = \rho(\tilde{\Delta} + A)^i \cdot [D_1] \dots [D_n] \cdot p^*(\gamma) = \rho(A)^i \cdot E_A \cdot p^*(\gamma).$$

By Proposition 3.3, $\rho(A) = p^*c(A)$. Since $p: E_A \rightarrow B$ is a diffeomorphism, we get:

$$\rho(A)^i \cdot p^*(\gamma) \cdot E_A = (p^*c(A))^i \cdot p^*(\gamma) \cdot E_A = c(A)^i \cdot \gamma = f_\gamma(A),$$

and therefore $\partial_I F_\gamma(\Delta) = \frac{(n+i)!}{i!} f_\gamma(A)$. □

The base case of the inner induction is an immediate corollary of Lemmas 6.1 and 6.3:

Corollary 6.4 (Base case of the inner induction). *For any $i \leq \frac{k}{2}$, any $\gamma \in H^{k-2i}(B, \mathbb{R})$ and any square free differential monomial $\partial_I = \partial_{i_1} \dots \partial_{i_n}$ of order n (where $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, s\}$), we have:*

$$\partial_I ((n+i)! \cdot I_\gamma(\Delta)) = \partial_I (i! \cdot F_\gamma(\Delta)).$$

6.3. The inner induction step. In this subsection, the index set $I \subseteq \{1, \dots, s\}$ is considered a multiset. As induction hypothesis, suppose that $\partial_I((n+i)! \cdot I_\gamma(\Delta)) = \partial_I(i! \cdot F_\gamma(\Delta))$ for any differential monomial ∂_I of multiplicity $m-1 \geq 0$. It remains to show that the equality is true for differential monomials of multiplicity m . As before, to keep notation simple assume $i_j = j$ for $j = 1, \dots, r$, so that $\partial_I = \partial_1^{k_1} \dots \partial_r^{k_r}$ for positive integers k_1, \dots, k_r . By reordering the coordinates of \mathcal{P}_Σ , we may assume $k_1 > 1$. By Lemmas 6.1 and 6.3, it suffices to consider the case where the vectors e_1, \dots, e_r form a cone in Σ (as otherwise $\partial_I I_\gamma(\Delta) = 0 = \partial_I F_\gamma(\Delta)$).

The plan is to express ∂_1 in terms of a Lie derivative L_v (for some $v \in M_{\mathbb{R}}$) and other partial derivatives. Then the (inner) induction step will follow by an explicitly computation of $L_v I_\gamma(\Delta)$ and $L_v F_\gamma(\Delta)$.

As e_1, \dots, e_r form a cone in the smooth fan Σ , they can be completed to a basis of $N_{\mathbb{R}}$. We take v to be the first vector of the dual basis of $M_{\mathbb{R}}$. Then $\langle v, e_1 \rangle = 1$, and $\langle v, e_j \rangle = 0$ for $j = 2, \dots, r$. Since $v \in M_{\mathbb{R}}$ (considered as an element of \mathcal{P}_Σ) is given by $v = \sum_{i=1}^s \langle v, e_i \rangle h_i$, we obtain that $L_v = \sum_{i=1}^s \langle v, e_i \rangle \partial_i$, and thus $\partial_1 = L_v - \sum_{j>r} \langle v, e_j \rangle \partial_j$. We get:

$$\begin{aligned} \partial_I &= \partial_1^{k_1} \dots \partial_r^{k_r} = \left(L_v - \sum_{j>r} \langle v, e_j \rangle \partial_j \right) \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \\ &= L_v \cdot \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} - \sum_{j>r} \langle v, e_j \rangle \cdot \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j. \end{aligned}$$

Since $k_1 > 1$ and $j > r$, each monomial in the sum $\sum_{j>r} \langle v, e_j \rangle \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j$ has multiplicity $m-1$, so by the induction hypothesis (of the inner induction), we get

$$(n+i)! \cdot \left(\sum_{j>r} \langle v, e_j \rangle \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j \right) \cdot I_\gamma(\Delta) = i! \cdot \left(\sum_{j>r} \langle v, e_j \rangle \partial_1^{k_1-1} \cdot \partial_2^{k_2} \dots \partial_r^{k_r} \cdot \partial_j \right) \cdot F_\gamma(\Delta).$$

It remains to consider the first summand:

Lemma 6.5. *In the situation above, we have $(n+i)! \cdot L_v I_\gamma(\Delta) = i! \cdot L_v F_\gamma(\Delta)$.*

Proof. A direct calculation shows:

$$\begin{aligned} L_v I_\gamma(\Delta) &= \partial_t \Big|_{t=0} \left(\int_{\Delta+tv} c(x)^i \cdot \gamma \, d\mu \right) = \partial_t \Big|_{t=0} \left(\int_{\Delta} c(x+tv)^i \cdot \gamma \, d\mu \right) = \\ &= \partial_t \Big|_{t=0} \left(\int_{\Delta} \left(\sum_{a=0}^i \binom{i}{a} t^a \cdot c(v)^a c(x)^{i-a} \right) \cdot \gamma \, d\mu \right) = \int_{\Delta} i \cdot c(v) c(x)^{i-1} \cdot \gamma \, d\mu \\ &= i \cdot \int_{\Delta} c(x)^{i-1} \cdot (c(v) \cdot \gamma) \, d\mu = i \cdot I_{c(v) \cdot \gamma}(\Delta). \end{aligned}$$

Similarly, by a direct calculation and Proposition 3.3, we get

$$\begin{aligned} L_v F_\gamma(\Delta) &= \partial_t \Big|_{t=0} \left(\rho(\Delta+tv)^{n+i} \cdot p^*(\gamma) \right) = \partial_t \Big|_{t=0} \left((\rho(\Delta) + t\rho(v))^{n+i} \cdot p^*(\gamma) \right) = \\ &= \partial_t \Big|_{t=0} \left(\left(\sum_{a=0}^{n+i} \binom{n+i}{a} t^a \cdot \rho(v)^a \cdot \rho(\Delta)^{n+i-a} \right) \cdot p^*(\gamma) \right) = (n+i) \cdot \rho(v) \cdot \rho(\Delta)^{n+i-1} \cdot p^*(\gamma) \\ &= (n+i) \cdot \rho(\Delta)^{n+i-1} \cdot p^*(c(v) \cdot \gamma) = (n+i) \cdot F_{c(v) \cdot \gamma}(\Delta). \end{aligned}$$

Since $c(v) \cdot \gamma \in H^{k-2(i-1)}(B, \mathbb{R})$, by the induction hypothesis of the outer induction, we have

$$(n+i-1)! \cdot I_{c(v) \cdot \gamma}(\Delta) = (i-1)! \cdot F_{c(v) \cdot \gamma}(\Delta).$$

Therefore,

$$(n+i)! \cdot L_v I_\gamma(\Delta) = (n+i)! \cdot i \cdot I_{c(v) \cdot \gamma}(\Delta) = i! \cdot (n+i) \cdot F_{c(v) \cdot \gamma}(\Delta) = i! \cdot L_v F_\gamma(\Delta). \quad \square$$

7. GRADED-COMMUTATIVE n -SELF-DUAL ALGEBRAS

7.1. Introduction. Suppose A is a cohomology ring of a compact oriented manifold which can be represented as factor-algebra of some other algebra B . In this section we discuss the question what extra information is needed to determine A out of B ? For the reader's convenience, we recall several crucial notions.

Definition 7.1. A graded \mathbb{K} -algebra $A = A^0 \oplus A^1 \oplus \dots \oplus A^k \oplus \dots$ over a field \mathbb{K} is called *graded-commutative* if for any homogeneous elements x, y , the following relation hold:

$$xy = (-1)^{\deg(x)+\deg(y)} yx$$

where $\deg(x)$ (resp. $\deg(y)$) denotes the degree of x (resp. of y).

Such algebras are fundamental in geometry. For example, the cohomology ring $A = H^*(X, \mathbb{K})$ of a CW complex X is known to be a graded-commutative \mathbb{K} -algebra. However, the cohomology ring of compact oriented manifolds satisfies further important properties:

Definition 7.2. Let n be a natural number. A graded-commutative \mathbb{K} -algebra is *Poincaré n -self-dual* (or just, is *n -self-dual*) if the following conditions hold:

- (1) The algebra A has a multiplicative unit element e which is homogeneous of degree zero, i.e. $e \in A^0 \subseteq A$.
- (2) The homogeneous components A^k for $k > n$ vanish, i.e. $A^k = 0$ for $k > n$, and $\dim_{\mathbb{K}}(A^n) = 1$.
- (3) The pairing $A^k \times A^{n-k} \rightarrow A^n$ induced by multiplication in the algebra A is non-degenerate for $0 \leq k \leq n$.

Remark 7.3. By the non-degeneracy of the pairing $A^k \times A^{n-k} \rightarrow A^n$, we have $\dim_{\mathbb{K}}(A^k) = \dim_{\mathbb{K}}(A^{n-k})$ for every $0 \leq k \leq n$ such that the component A^k is a finite dimensional space. In particular, $\dim_{\mathbb{K}}(A^0) = 1$.

The following example plays a key role in our discussion.

Example 7.4. Let M be a connected compact oriented n -dimensional manifold. By Poincaré duality, the cohomology ring $A = H^*(M, \mathbb{K})$ is an n -self-dual graded-commutative \mathbb{K} -algebra. Moreover, A is equipped with two extra structures:

- (1) A \mathbb{K} -linear function $\ell_*: A \rightarrow \mathbb{K}$ given as follows. By linearity, it suffices to define ℓ_* on homogeneous elements. For homogeneous α of degree n , we let $\ell_*(\alpha)$ be equal to the value of the cohomology class α on the fundamental class of the oriented manifold M . If the degree of α is not n , we set $\ell_*(\alpha) = 0$.
- (2) A \mathbb{K} -bilinear intersection form F_{ℓ_*} on A defined by the identity $F_{\ell_*}(\alpha, \beta) = \ell_*(\alpha \cdot \beta)$ (recall that we denote the cup product of the cohomology ring A by “ \cdot ”). Note that F_{ℓ_*} is non-degenerate (the non-degeneracy of F_{ℓ_*} is equivalent to the Poincaré duality on A).

Consider another graded-commutative \mathbb{K} -algebra $B = B^0 \oplus B^1 \oplus \dots \oplus B^k \oplus \dots$ with multiplicative unit element $e \in B^0$ and $\dim_{\mathbb{K}}(B^0) = 1$. Suppose that the cohomology ring $A = H^*(M, \mathbb{K})$ from Example 7.4 is a factor-algebra of B .

Question 7.5. We are going to investigate the following questions:

- (1) What extra information on B is needed to determine the algebra A ?
- (2) What extra information on B is needed to determine the intersection form F_{ℓ_*} on A ?

We provide answers to both questions in Subsections 7.2 and 7.3 respectively.

7.2. Graded-commutative n -self-dual factor algebras. In this section, let $B = B^0 \oplus B^1 \oplus \dots \oplus B^k \oplus \dots$ be a graded-commutative \mathbb{K} -algebra with multiplicative unit element $e \in B^0$ and $\dim_{\mathbb{K}}(B^0) = 1$. We want to describe all n -self dual factor-algebras of B . To that extent we introduce the following notion.

Definition 7.6. An ideal $I \subseteq B$ is called *n -self-dual* (or *n -sd ideal* for short) if I is a two-sided homogeneous ideal and the factor-algebra $A = B/I$ is n -self-dual.

Clearly, factor-algebras of B which are n -self-dual are in correspondence with n -sd ideals. Therefore part (1) of Question 7.5 boils down to describing all n -sd ideals in B . Our solution consists of the following steps:

- (1) If I is an n -sd ideal then $I \cap B^n$ is a hyperplane in B^n (see Lemma 7.7).
- (2) Conversely, for any hyperplane $L \subseteq B^n$ there is a unique n -sd ideal $I = I(L)$ such that $I \cap B^n = L$ (see Lemmas 7.12 and 7.13).
- (3) We present an explicit construction of the n -sd ideal $I(L)$ (see Definition 7.11).

Lemma 7.7. *If $I \subseteq B$ is an n -sd ideal then $L = I \cap B^n$ is a hyperplane in B^n .*

Proof. Since $B^n/(I \cap B^n) = A^n$ and $\dim(A^n) = 1$, L is a hyperplane in B^n . □

As hyperplanes are in correspondence with linear functions, we introduce the following:

Definition 7.8. A linear function $\ell: B \rightarrow \mathbb{K}$ on a graded-commutative \mathbb{K} -algebra B is *n -homogeneous* if it is not identically equal to zero on B , but for any $m \neq n$ its restriction to the homogeneous component B^m vanishes. Denote by $L_{\ell} \subseteq B^n$ the hyperplane in B^n defined by identity $L_{\ell} = \{\ell = 0\} \cap B^n$.

Clearly, any hyperplane $L \subseteq B^n$ is equal to the hyperplane L_{ℓ} for a unique (up to a non-zero scalar multiple) n -homogeneous linear function ℓ . Next, we explain how to obtain an n -self-dual factor-algebra from an n -homogeneous linear function $\ell: B \rightarrow \mathbb{K}$.

Definition 7.9. For an n -homogeneous linear function $\ell: B \rightarrow \mathbb{K}$ let $I_1(L_{\ell})$ and $I_2(L_{\ell})$ be the subsets of B defined by the following conditions:

- (1) an element $a \in B$ belongs to $I_1(L_{\ell})$ if and only if $\ell(ab) = 0$ for all $b \in B$;
- (2) an element $b \in B$ belongs to $I_2(L_{\ell})$ if and only if $\ell(ab) = 0$ for all $a \in B$.

Clearly, the sets $I_1(L_{\ell})$ and $I_2(L_{\ell})$ depend only on the hyperplane $L = L_{\ell} \subseteq B^n$.

Lemma 7.10. For an n -homogeneous linear function ℓ , $I_1(L_\ell) = I_2(L_\ell)$ is a two-sided homogeneous ideal in B .

Proof. We first prove that the set $I_1(L_\ell)$ is a *right* ideal in B . If $a \in I_1(L_\ell)$ and $b \in B$, then for any $c \in B$ the identity $\ell((ab)c) = \ell(a(bc)) = 0$ holds, and thus $ab \in I_1(L_\ell)$. If $a, b \in I_1(L_\ell)$ then $\ell(ac) = \ell(bc) = 0$ for any $c \in B$. Hence, $\ell((a+b)c) = 0$ which implies $a+b \in I_1(L_\ell)$.

Next we prove that $I_1(L_\ell)$ is homogeneous, i.e. if $a = \sum_{i=0}^d a_i$ for $a_i \in B^i$ is in $I_1(L_\ell)$, then $a_i \in I_1(L_\ell)$ for every i . For $c = \sum_{m=0}^d c_m$ with $c_m \in B^m$, we have $\ell(a_i c) = \sum_{m=0}^d \ell(a_i c_m)$, and thus it suffices to show $\ell(a_i c_m) = 0$ for any homogeneous $c_m \in B_m$. Since ℓ is n -homogeneous, $\ell(a_i c_m) = 0$ for every i with $i+m \neq n$. However, from $0 = \ell(ac_m) = \sum_{i=0}^d \ell(a_i c_m)$, it follows $\ell(a_{n-m} c_m) = 0$. Thus $a_i \in I_1(L_\ell)$.

Analogously, it follows that $I_2(L_\ell)$ is a homogeneous *left* ideal in B .

Let us show that $I_1(L_\ell) \subseteq I_2(L_\ell)$. Let $a = \sum_{j=0}^d a_j$ be in $I_1(L_\ell)$ and $b = \sum_{i=0}^d b_i$ be in B where $a_i, b_i \in B^i$. Then $\ell(ba) = \sum_{i=0}^d \sum_{j=0}^d \ell(b_i a_j) = \sum_{i=0}^d \sum_{j=0}^d (-1)^{i+j} \ell(a_j b_i)$. Since $I_1(L_\ell)$ is homogeneous, we have $a_j \in I_1(L_\ell)$, and thus $\ell(a_j b_i) = 0$ for all couples i, j . It follows $\ell(ba) = 0$ which means that $I_1(L_\ell) \subseteq I_2(L_\ell)$.

Similarly, we can prove that $I_2(L_\ell) \subseteq I_1(L_\ell)$ and the statement is proven. \square

Definition 7.11. For an n -homogeneous linear function ℓ , let $I(L_\ell)$ be the two-sided ideal $I_1(L_\ell) = I_2(L_\ell)$. The ideal $I(L_\ell)$ depends only on the hyperplane $L = L_\ell$, and so the notation $I(L) = I(L_\ell)$ makes sense.

Lemma 7.12. For any hyperplane $L \subseteq B^n$ the set $I(L) \subseteq B$ is an n -sd ideal.

Proof. By Lemma 7.10, $I(L)$ is a two-sided homogeneous ideal. Set $A = B/I(L)$.

If $k > n$, then for any $b_k \in B^k$ and any $c = \sum_{i=0}^d c_i$ in B with $c_i \in B^i$, we have $\ell(b_k c) = \sum_{i=0}^d \ell(b_k c_i) = 0$ (since ℓ is n -homogeneous). Hence, $B^k = I(L) \cap B^k$ for $k > n$, or equivalently A^k vanishes for $k > n$.

It remains to show that the pairing $A^k \times A^{n-k} \rightarrow A^n$ induced by multiplication is non-degenerate. Suppose $a \in A^k$ is in the kernel of this pairing, i.e. $ab = 0$ for any $b \in A^{n-k}$. Let $a^* \in B^k$ be an element whose image in A is equal to a . Then for any $b^* \in B^{n-k}$, we have $a^* b^* \in I(L)$, and in particular, $\ell(a^* b^*) = 0$. Since ℓ is n -homogeneous, for any $c \in B$, we have $\ell(a^* c) = \ell(a^* c_{n-k})$ where $c_{n-k} \in B^{n-k}$ is the homogeneous component of c of degree $n-k$. Thus $a^* \in I_1(L_\ell) = I(L)$, or equivalently $a = 0$ in A , i.e. the pairing is non-degenerate in the first component. A similar argument shows non-degeneracy in the second component. \square

Lemma 7.13. For any hyperplane $L \subseteq B^n$ there is a unique n -sd ideal $I \subseteq B$ such that $I \cap B^n = L$. This ideal I coincides with the ideal $I(L)$.

Proof. Let ℓ be an n -homogeneous linear function such that $L = L_\ell$.

We start by showing that $I(L) \cap B^n = L$. The inclusion “ \subseteq ” is straightforward. For the reverse inclusion suppose $a \in L$. Then for any $b = \sum_{i=0}^d b_i$ in B with $b_i \in B^i$, we have $\ell(ab) = \sum_{i=0}^d \ell(ab_i) = \ell(ab_0) = b_0 \ell(a) = 0$, and thus $a \in I(L) \cap B^n$. This shows the existence-part of the statement.

To show uniqueness, let $I \subseteq B$ be an n -sd ideal with $I \cap B^n = L$. As both I and $I(L)$ are homogeneous, it suffices to show that $I \cap B^k = I(L) \cap B^k$ for every k . If $a \in I(L) \cap B^k = I_1(L_\ell) \cap B^k$, then its image in the factor-algebra $A = B/I$ belongs to the kernel of the pairing $A^k \times A^{n-k} \rightarrow A^n = B/L$. Thus $a \in I \cap B^k$ and $I_1(L_\ell) \cap B^k \subseteq I \cap B^k$. Conversely, if $a \notin I_1(L_\ell) \cap B^k$, then there is $b \in B^{n-k}$ such that $ab \notin L$. Thus the image of the element $ab \in B^n$ is not equal to zero in $A = B/I$ and $a \notin I \cap B^k$. \square

7.3. Frobenius forms and n -self-dual factor-algebras. In this section, we investigate the second part of Question 7.5. For the reader’s convenience, we provide a summary at the end of the section.

Consider a bilinear form $F: L_1 \times L_2 \rightarrow \mathbb{K}$ on a product of \mathbb{K} -linear spaces L_1 and L_2 . Recall the *left radical* $R_1 \subseteq L_1$ respectively the *right radical* $R_2 \subseteq L_2$ of F :

- (1) $a \in R_1$ if and only if $F(a, b) = 0$ for all $b \in L_2$;
- (2) $b \in R_2$ if and only if $F(a, b) = 0$ for all $a \in L_1$.

Definition 7.14. Let $\ell: B \rightarrow \mathbb{K}$ be an n -homogeneous linear function on a graded-commutative algebra B . The *Frobenius bilinear form* $F_\ell: B \times B \rightarrow \mathbb{K}$ associated with ℓ is the form defined by the identity $F_\ell(a, b) = \ell(a \cdot b)$.

Theorem 7.15. Let $\ell: B \rightarrow \mathbb{K}$ be an n -homogeneous linear function on a graded-commutative algebra B .

- (1) The n -sd ideal $I(L_\ell) = I_1(L_\ell) = I_2(L_\ell)$ coincides with the left and also the right radical of F_ℓ .
- (2) There is a unique n -homogeneous linear function ℓ_* on the factor algebra $A = B/I(L)$ such that $\ell = \rho^* \ell_*$ where $\rho: B \rightarrow A$ is the natural homomorphism. Furthermore, for any $a, b \in B$ the relation $F_\ell(a, b) = F_{\ell_*}(\rho(a), \rho(b))$ holds.
- (3) The Frobenius form F_{ℓ_*} on A is non-degenerate.

Proof. By definition the left radical (resp. the right radical) of the form F_ℓ coincides with $I_1(L_\ell)$ (resp. with $I_2(L_\ell)$). By Lemma 7.10, $I(L_\ell) = I_2(L_\ell) = I_1(L_\ell)$, and thus the first statement follows (see also Definition 7.11).

For the second statement, observe that the kernel $L \subseteq B^n$ of the restriction of the function $\ell|_{B^n}$ coincides with the kernel of the surjective map $\rho: B^n \rightarrow A^n$ (see Lemma 7.13). Thus there is a unique function $\ell_*: A^n \rightarrow \mathbb{K}$ such that $\ell = \rho^* \ell_*$. The relation $F_\ell(a, b) = F_{\ell_*}(\rho(a), \rho(b))$ follows by definition.

For the last statement, observe that the form F_{ℓ_*} is obtained from the form F_ℓ by taking the quotient of B by the left and right radical of the form F_ℓ . Thus F_{ℓ_*} is a non-degenerate form. \square

Let us summarize the results from this section. In order to reconstruct an n -sd factor-algebra $A = B/I$ equipped with an intersection form on A it suffices to fix a Frobenius form F_ℓ on B corresponding to an n -homogeneous linear function ℓ on B . The algebra A is equal to $B/I(L_\ell)$ (note that the algebra A depends only on the hyperplane $L = L_\ell$, i.e. proportional n -homogeneous functions define the same algebra A). The form F_ℓ is induced by the unique non-degenerate intersection form F_{ℓ_*} on A . The set of n -sd factor-algebras $A = B/I$ equipped with non-degenerate intersection forms is in one-to-one correspondence with the set of n -homogeneous linear functions on B .

Theorem 7.16. *Let B be a graded-commutative algebra over a field \mathbb{K} , and let A be an n -sd factor-algebra of B with a chosen isomorphism $\phi: A_n \rightarrow \mathbb{K}$. Let $\rho: B \rightarrow A$ be the natural homomorphism. Then*

$$A \simeq B/I(L_\ell),$$

where ℓ is the n -homogeneous linear function defined by

$$\ell: B_n \rightarrow \mathbb{K}, \quad \ell(b) = \phi(\rho(b)),$$

and extended by 0 to B_i with $i \neq n$.

Moreover, for any n -homogeneous linear function ℓ , the algebra $A = B/I(L_\ell)$ is n -self dual and the corresponding Frobenius form F_{ℓ_*} on A is non-degenerate.

8. APPLICATIONS

8.1. The Cohomology ring of toric bundles. In this section, we apply the results from Sections 6 and 7 to give a description of the cohomology ring of a toric bundle. Let, as before, $p: E \rightarrow B$ be a principal torus bundle, Σ a smooth projective fan with $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$, $X = X_\Sigma$ the corresponding toric variety, and E_X the corresponding toric bundle. By the Leray-Hirsch theorem (see Theorem 3.1) the cohomology ring $H^*(E_X, \mathbb{R})$ is a quotient of the polynomial algebra $R[x_1, \dots, x_r]$, where $R = H^*(B, \mathbb{R})$ is the cohomology ring of the base.

Theorem 8.1. *In the notation from above the cohomology ring of E_X is given by*

$$H^*(E_X, \mathbb{R}) \simeq R[x_1, \dots, x_r]/I(L_\ell)$$

where $\ell: R[x_1, \dots, x_r] \rightarrow \mathbb{R}$ is a $(k+2n)$ -homogeneous linear function defined by:

$$\ell(\gamma \cdot x_{i_1} \cdots x_{i_s}) = I_\gamma(\rho_{i_1}, \dots, \rho_{i_s})$$

for any monomial $\gamma \cdot x_{i_1} \cdots x_{i_s}$ with $\deg(\gamma) + 2s = k + 2n$.

Proof. By Theorem 7.16, as $H^*(E_X, \mathbb{R})$ is a graded commutative $(k+2n)$ -self dual factor algebra of $R[x_1, \dots, x_r]$, it is given by $R[x_1, \dots, x_r]/I(L_\ell)$ for a $(k+2n)$ -homogeneous linear function ℓ which is obtained by pairing cohomology classes with the fundamental class of E_X . Hence the statement follows by Theorem 4.1. \square

8.2. Ring of conditions. The ring of conditions constitutes an intersection ring for horospherical homogeneous spaces. We refer to [DCP85] for further details and references. The exposition given here follows [Hof19].

Let G be a reductive complex algebraic group and $H \subseteq G$ a closed subgroup. A cycle of codimension k of G/H is a formal linear combination $k_1 X_1 + \dots + k_r X_r$ of irreducible closed subvarieties X_i of G/H with $k_i \in \mathbb{Z}$. We write $\mathcal{Z}^k(G/H)$ for the \mathbb{Z} -module of cycles of codimension k . Two cycles X, Y are said to *intersect properly* if either their intersection is empty or the dimension of each irreducible component of $X \cap Y$ equals $\dim(X) + \dim(Y) - \dim(G/H)$. We need the following result by Kleiman [Kle74, Corollary 4]:

Theorem 8.2 (Kleiman's transversality theorem). *For any two irreducible subvarieties $X, Y \subseteq G/H$ there exists a dense open $U \subseteq G$ such that $g \cdot X$ and Y intersect properly for each $g \in U$. If X, Y have complementary dimensions, $g \cdot X \cap Y$ is a finite union of points and $|g \cdot X \cap Y|$ is constant for generic $g \in G$.*

Theorem 8.2 gives rise to an intersection pairing between algebraic cycles of complementary codimensions:

$$\mathcal{Z}^k(G/H) \times \mathcal{Z}^{\dim(G/H)-k}(G/H) \rightarrow \mathbb{Z}; \quad (X, Y) \mapsto (X \cdot Y) := |g \cdot X \cap Y| \quad (\text{for generic } g \in G),$$

where $X, Y \subseteq G/H$ are assumed to be irreducible subvarieties.

Two algebraic cycles $X, Y \in \mathcal{Z}^k(G/H)$ are said to be equivalent if for any algebraic cycle of complementary codimension $Z \in \mathcal{Z}^{\dim(G/H)-k}(G/H)$ the intersection products coincide, i.e. $(X \cdot Z) = (Y \cdot Z)$. We denote the group of equivalence classes by $C^k(G/H)$ ("the group of conditions of codimension k "). The intersection pairing factors through the equivalence relation, so that we obtain a pairing: $C^k(G/H) \times C^{\dim(G/H)-k}(G/H) \rightarrow \mathbb{Z}$.

De Concini and Procesi showed that $C^*(G/H) := \bigoplus_{k=0}^{\dim(G/H)} C^k(G/H)$ can be equipped with a ring structure:

Theorem 8.3 ([DCP85, Section 6.3]). *Suppose G/H is a horospherical homogeneous space. Define an intersection product on $C^*(G/H)$: $[X] \cdot [Y] := [gX \cap Y]$ for generic $g \in G$ where $X, Y \subseteq G/H$ are irreducible subvarieties. This intersection product is well-defined and there is a canonical isomorphism of graded rings*

$$C^*(G/H) = \varinjlim_E H^*(E, \mathbb{Z})$$

where the limit is taken over complete (or equivalently smooth) horospherical embeddings $G/H \hookrightarrow E$.

Theorem 8.3 is a special case of [DCP85, Section 6.3] which suffices for our purposes. Furthermore, the intersection product in Theorem 8.3 might not be well-defined for arbitrary homogeneous spaces G/H . We refer to [Hof19, Exercise 4.1] for a counterexample. In this paper we will exclusively work with the ring of conditions with real coefficients. Therefore, we introduce the notation $C_{\mathbb{R}}^*(G/H)$ for the tensor product $C^*(G/H) \otimes_{\mathbb{Z}} \mathbb{R}$.

For the rest of this section let $H \subseteq G$ be a horospherical subgroup. In Section 3.2, we saw that $P = N_G(H)$ is a parabolic subgroup and that $E = G/H$ is a principal torus bundle over G/P with respect to the torus $T = P/H$. For a T -toric variety X , we saw that the associated toric bundle E_X is a horospherical variety. Since any toric variety is dominated by a smooth projective one, the same is true for these horospherical varieties E_X . Furthermore, using the Luna-Vust theory of spherical embeddings (see, for instance, [Kno91, Theorem 3.3]), it follows that an (arbitrary) horospherical variety $G/H \hookrightarrow Y$ is dominated by some horospherical variety E_X as described above. Thus, the smooth projective E_X form a cofinal set for the direct limit in Theorem 8.3.

We now give a description of the ring of conditions $C_{\mathbb{R}}^*(G/H)$ using Theorem 8.1. Let $R = H^*(G/P, \mathbb{R})$ be the cohomology ring of the flag variety G/P and $\text{Sym}^*(\mathcal{P})$ be the symmetric algebra of virtual polytopes. Furthermore set $k = \dim_{\mathbb{C}}(G/P)$ and $n = \dim_{\mathbb{C}}(P/H)$.

Corollary 8.4. *With the notation above, $C^*(G/H)$ is an $2(n+k)$ -self dual factor algebra of $R \otimes \text{Sym}(\mathcal{P})$:*

$$C_{\mathbb{R}}^*(G/H) = R \otimes \text{Sym}(\mathcal{P})/I_{\ell}, \text{ with } \ell(\gamma \otimes \Delta_{i_1} \cdots \Delta_{i_s}) = I_{\gamma}(\Delta_{i_1}, \dots, \Delta_{i_s}).$$

In particular, $\ell(\gamma \otimes \Delta_{i_1} \cdots \Delta_{i_s}) = 0$ unless $\deg(\gamma) + 2s = 2(n+k)$.

We conclude this section with an alternative description of $C_{\mathbb{R}}^*(G/H)$ using the description of the cohomology ring from Section 3.2. The challenge is to find a good description of the ring in Theorem 3.4 as we want to take the direct limit over all smooth projective E_X . The following approach is inspired by [Bri96] (see also [Hof19]).

As in Section 3.2, let M be the character lattice of the torus $T = P/H$ and set $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and $N_{\mathbb{R}} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{R})$. Let Σ be a smooth projective fan in $N_{\mathbb{R}}$. A map $f: N_{\mathbb{R}} \rightarrow \mathbb{R}$ is *piecewise polynomial* if for any $\sigma \in \Sigma$, the map $f|_{\sigma}: \sigma \rightarrow \mathbb{R}$ extends to a polynomial function on the linear space $\text{span}_{\mathbb{R}}\{\sigma\}$, i.e. a piecewise polynomial function f on Σ is a collection of compatible polynomial functions $f_{\sigma}: \sigma \rightarrow \mathbb{R}$. In particular, such a function is continuous. We denote by PP_{Σ} the set of all piecewise polynomial functions on Σ which is a ring under pointwise addition and multiplication. Let $\text{Sym}(M_{\mathbb{R}})$ be the symmetric algebra of polynomial functions on $N_{\mathbb{R}}$. Note that PP_{Σ} is a positively graded \mathbb{R} -algebra with graded subalgebra $\text{Sym}(M_{\mathbb{R}})$. Indeed, any piecewise polynomial function uniquely decomposes into a sum of homogeneous piecewise polynomial functions.

We can now reformulate Theorem 3.4:

Proposition 8.5 ([Hof19, Proposition 4.10]). *If E_X is a horospherical variety obtained as associated toric bundle of a smooth projective toric variety X , the cohomology ring $H^*(E_X, \mathbb{R})$ is isomorphic as an $H^*(G/P, \mathbb{R})$ -algebra to the quotient of $H^*(G/P, \mathbb{R}) \otimes \text{PP}_{\Sigma}$ by the ideal $\langle c(m) \otimes 1 - 1 \otimes \langle \cdot, m \rangle : m \in M \rangle$.*

Let PP be the set of all piecewise polynomial functions on smooth projective fans in $N_{\mathbb{R}}$, i.e., $\text{PP} = \bigcup_{\Sigma} \text{PP}_{\Sigma}$ where the union is taken over all smooth projective fans $\Sigma \subseteq N_{\mathbb{R}}$.

Theorem 8.6 ([Hof19, Theorem 4.11]). *We have that*

$$C_{\mathbb{R}}^*(G/H) \simeq (H^*(G/P, \mathbb{R}) \otimes \text{PP}) / \langle c(m) \otimes 1 - 1 \otimes \langle \cdot, m \rangle : m \in M \rangle,$$

where $\langle \cdot, m \rangle \in \text{Sym}(M_{\mathbb{R}})$ is a globally linear function on $N_{\mathbb{R}}$.

9. EXAMPLES

In this section, we apply the results obtained above to special cases where the description becomes particularly transparent. We determine how the description of the cohomology ring of toric bundle simplifies if the cohomology ring of the base variety B is generated in degree 2 and the map $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$ is surjective. Full flag varieties provide an ample class of varieties satisfying these assumptions. By using the theory of string polytopes, we obtain an explicit description of the cohomology ring of toric bundles over a full flag variety. The section is concluded by a reproduction of the famous description of the cohomology ring of projective bundles.

9.1. Assumptions on the base manifold. The results above become particularly nice if the cohomology ring $H^*(B, \mathbb{R})$ is generated by the image of the map $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$. In this case, the map $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$ is surjective, and $H^*(B, \mathbb{R})$ is generated in degree 2. As all odd degrees of $H^*(B, \mathbb{R})$ vanish, we get that the (real) dimension of B is even, say $k = 2r$. Furthermore, for any smooth projective fan Σ , the cohomology ring $H^*(E_X, \mathbb{R})$ of the toric bundle E_X associated to the toric variety $X = X_{\Sigma}$, is also generated in degree 2. If dimension of the torus is n , then the real dimension of E_X is $2(n+r)$ and in particular even.

Since all odd degrees of $H^*(E_X, \mathbb{R})$ vanish, we consider the grading $H^{*/2}(E_X, \mathbb{R})$ on this ring where all degrees are divided by 2. Note that in this specific situation the property of graded-commutativity coincides with the ring being commutative. These notational changes will make the results below look more ergonomic.

Our goal is to extend the description of the cohomology ring of smooth projective toric varieties in terms of differential forms and the volume polynomial to this situation (see 2.1). For the reader's convenience, we recall the necessary objects.

Let V be a finite-dimensional vector space over a field \mathbb{K} and let $\text{Diff}(V)$ be the ring of differential operators on V with constant coefficients. A choice of a basis for V , induces an isomorphism of $\text{Diff}(V)$ with the ring of polynomials in symbols ∂_i corresponding to the partial derivatives along the i -th coordinate (induced by the chosen basis). The ring $\text{Diff}(V)$ acts naturally on the ring $\text{Sym}(V^*)$ of polynomials on V . In order to describe the cohomology ring of the toric bundle E_X in the case that c is surjective, we use the following theorem which is an analogue of Theorem 7.16 for commutative algebras which are generated in degree 1. It is implicitly contained in the work of Pukhlikov and the second author (see [PK92b], see also [Kav11, Theorem 1.1.]).

Theorem 9.1. *Let A be a commutative finite dimensional graded algebra over \mathbb{K} . Write $A = \bigoplus_{i=0}^n A_i$ where A_i is the i -th graded piece of A . Suppose the following conditions hold:*

- (1) *A is generated (as an algebra) by A_1 , and $A_0 \simeq A_n \simeq \mathbb{K}$.*
- (2) *The bilinear map $A_i \times A_{n-i} \rightarrow A_n \simeq \mathbb{K}$ given by $(u, v) \mapsto u \cdot v$ is non-degenerate for all $i = 0, \dots, n$.*

If $\rho: V \rightarrow A_1$ is a surjective linear map, then A (as a graded algebra) is isomorphic to the algebra $\text{Diff}(V)/\text{Ann}(f)$. Here, $f(v) := \rho(v)^n \in A_n \simeq \mathbb{K}$ is a homogeneous polynomial of degree n on V and $\text{Ann}(f) = \{d \in \text{Diff}(V): df = 0\}$ is its annihilator.

We provide a proof to show how Theorem 9.1 follows from Theorem 7.16.

Proof. Since the algebra A is generated in degree 1, the surjective linear map $\rho: V \rightarrow A_1$ gives rise to a surjection $p: \text{Diff}(V) \simeq \text{Sym}(V) \rightarrow A$. Hence, by Theorem 7.16, there is an isomorphism $A \simeq \text{Diff}(V)/I(L_{\ell})$ where $\ell = \phi \circ p: \text{Diff}(V) \rightarrow \mathbb{K}$ ($\phi: A_n \rightarrow \mathbb{K}$ a fixed isomorphism). So it remains to show that $I(L_{\ell}) = \text{Ann}(f)$.

The map $\ell: \text{Diff}(V) \rightarrow \mathbb{K}$ is defined by its values on the n -th graded piece $\text{Diff}^n(V)$ of $\text{Diff}(V)$. For $v_1, \dots, v_n \in V$, the value $\ell(L_{v_1} \cdots L_{v_n})$ at the corresponding differential monomial is given by polarization of $f: V \rightarrow \mathbb{K}$ (recall that L_{v_i} denotes the Lie derivative with respect to the vector v_i ; see Definition 4.2):

$$\ell(L_{v_1} \cdots L_{v_n}) = \frac{1}{n!} L_{v_1} \cdots L_{v_n} f.$$

Extending by linearity from the product of linear forms to all of $\text{Diff}^n(V)$, we obtain

$$\ell(d) = \frac{1}{n!} df \quad \text{for any } d \in \text{Diff}^n(V).$$

Clearly, for any $j > n$, we have $I(L_{\ell}) \cap \text{Diff}^j(V) = \text{Ann}(f) \cap \text{Diff}^j(V) = \text{Diff}^j(V)$. Since both ideals are homogeneous, it suffices to check $I(L_{\ell}) \cap \text{Diff}^j(V) = \text{Ann}(f) \cap \text{Diff}^j(V)$ for $j \leq n$.

Let $d \in \text{Ann}(f)$ be a differential operator of degree $j \leq n$. Then $d' df = 0$ for any $d' \in \text{Diff}^j(V)$ for any i , and thus $d \in I(L_{\ell})$. Otherwise, if $d \notin \text{Ann}(f)$, then df is a non zero polynomial of degree $n-j$ and hence there exists $d' \in \text{Diff}^{n-j}(V)$ with $d' df \neq 0$. It follows that $\ell(d' d) \neq 0$ which implies $d \notin I(L_{\ell})$. So $I(L_{\ell}) = \text{Ann}(f)$ and the theorem is proved. \square

With Theorem 9.1, we can prove the following statement.

Theorem 9.2. *Let $p: E \rightarrow B$ be a torus bundle for which the map $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$ is surjective. Let Σ be a smooth projective fan with associated toric variety X . Then the cohomology ring $H^*(E_X, \mathbb{R})$ is given by*

$$H^{*/2}(E_X, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_{\Sigma})/\text{Ann}(I),$$

where I is the homogeneous polynomial on \mathcal{P}_{Σ} defined by $I(\Delta) = \int_{\Delta} c(x)^r dx$.

In the proof of Theorem 9.2, we need the following observation.

Lemma 9.3. *In the situation above, the map $\rho: \mathcal{P}_{\Sigma} \rightarrow H^2(E_X, \mathbb{R})$ is surjective.*

Proof. By the Leray-Hirsch Theorem (see Theorem 3.1), we have

$$H^2(E_X, \mathbb{R}) \simeq H^2(B, \mathbb{R}) \oplus H^2(X, \mathbb{R}) \simeq (H^2(B, \mathbb{R}) \oplus \mathcal{P}_{\Sigma}) / \{c(\lambda) - \rho(\lambda) = 0: \lambda \in M\},$$

where the second isomorphism follows by Theorem 3.5. The statement follows from the surjectivity of the map $c: M_{\mathbb{R}} \rightarrow H^2(B, \mathbb{R})$. \square

Proof of Theorem 9.2. By Theorem 9.1 and Lemma 9.3, the cohomology ring $H^*(E_X, \mathbb{R})$ of E_X is given by $H^{*/2}(E_X, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_\Sigma)/\text{Ann}(f)$, where $f(\Delta) = \rho(\Delta)^{n+r}$. By Theorem 4.1 (with $i = r$), for any $\Delta \in \mathcal{P}_\Sigma$,

$$r! \cdot \rho(\Delta)^{n+r} = (n+r)! \cdot I(\Delta).$$

Hence, $\text{Ann}(f) = \text{Ann}(I)$, and so the theorem is proven. \square

The reader might wonder how strong the assumptions in Theorem 9.2 are: torioidal horospherical varieties over a full flag variety provide an ample class of examples which satisfy the conditions of Theorem 9.2. A prominent example in this class is the following. Let G be a semisimple algebraic group, B a Borel subgroup in G , $U \subset B$ the unipotent radical of B , and $T \subset B$ a maximal torus. Consider the toric bundle

$$p: E = G/U \rightarrow G/B.$$

In [Bor53], Borel showed that the cohomology ring of the full flag variety G/B is isomorphic to the ring of coinvariants of the Weyl group $W = N_G(T)/T$ of G , i.e. the following factor ring where the action of W on the ring of polynomials on the Lie algebra of T is given by the diagonal action of the coadjoint action on $M_{\mathbb{R}}$:

$$H^*(G/B, \mathbb{R}) \simeq \text{Sym}(M_{\mathbb{R}})/\text{Sym}(M_{\mathbb{R}})^W.$$

More precisely, the above isomorphism is induced by the map $c: M_{\mathbb{R}} \rightarrow H^2(G/B, \mathbb{R})$, and so c is surjective.

9.2. Relation to string polytopes. We continue to examine the example at the end of the previous section, i.e. the torus bundle $p: E = G/U \rightarrow G/B$. If $G = \text{SL}_n$, then Theorem 9.2 has a beautiful reformulation.

For λ a dominant weight, let $\text{GZ}_\lambda \subseteq \mathbb{R}^{N=n(n-1)/2}$ be the corresponding *Gelfand-Zetlin polytope* (see [GC50]). From the defining inequalities, it can be straightforwardly deduced that Gelfand-Zetlin polytopes are additive:

$$\text{GZ}_{\lambda+\mu} = \text{GZ}_\lambda + \text{GZ}_\mu \quad \text{for any dominant weights } \lambda, \mu.$$

By linearity, we extend the definition of Gelfand-Zetlin polytopes to all of the positive Weyl chamber Λ^+ , i.e. we define GZ_λ for any $\lambda \in \Lambda^+$: if λ is in the positive Weyl chamber, we can write $\lambda = \sum_{i=1}^r a_i \cdot \mu_i$ for positive real numbers a_i and dominant (integral) weights μ_i ; set $\text{GZ}_\lambda = \sum_{i=1}^r a_i \cdot \text{GZ}_{\mu_i}$ (Minkowski addition). By the additivity of Gelfand-Zetlin polytopes, it follows that this definition is well-defined.

The Gelfand-Zetlin polytopes form a linear family of polytopes, a concept which we recall from [KV18, Definition 1.2]. Let V, W be two real vector spaces (not necessarily finite-dimensional) and let $C \subseteq V$ be a full-dimensional convex cone. Note that the cone C isn't necessarily closed. We only assume that $c_1x_1 + c_2x_2 \in C$ for any $x_1, x_2 \in C$ and all non-negative real numbers $c_1, c_2 \geq 0$. An \mathbb{R} -linear map $\Delta: C \rightarrow \mathcal{P}_W^+$ is called a *linear family of polytopes* where “linearity” is explained in the following folklore result:

Proposition 9.4. *A map $\Delta: C \rightarrow \mathcal{P}_W^+$ is called linear if either of the following equivalent conditions is satisfied:*

- (1) $\Delta(c_1x_1 + c_2x_2) = c_1\Delta(x_1) + c_2\Delta(x_2)$ for any $x_1, x_2 \in C$ and all non-negative real numbers $c_1, c_2 \geq 0$.
- (2) *There is a (unique) linear map $V \rightarrow \mathcal{P}_W$ such that its restriction to C coincides with $\Delta(\cdot)$.*

For the reader's convenience, we include a proof for Proposition 9.4.

Proof. The implication “(2) \Rightarrow (1)” is straightforward. It remains to show the reverse implication.

Choose a (Hamel) basis of V , say $\{v_i\}_{i \in I}$. Since C is full-dimensional, we may (and will) assume that $\{v_i\}_{i \in I} \subseteq C$. We define a linear map $F: V \rightarrow \mathcal{P}_W$ by setting $F(v_i) = \Delta(v_i)$ for $i \in I$ and show that $F|_C = \Delta$. Let $x \in C$ expressed as $x = \sum_{j \in J} \lambda_j v_j$ for a finite subset $J \subseteq I$ and $\lambda_j \in \mathbb{R}$. Consider the partition of J :

$$J_- := \{j \in J: \lambda_j < 0\} \quad \text{and} \quad J_+ := \{j \in J: \lambda_j > 0\}.$$

Then $x + \sum_{j \in J_-} (-\lambda_j)v_j = \sum_{j \in J_+} \lambda_j v_j$, and thus

$$\Delta\left(x + \sum_{j \in J_-} (-\lambda_j)v_j\right) = \Delta\left(\sum_{j \in J_+} \lambda_j v_j\right) \Rightarrow \Delta(x) + \sum_{j \in J_-} (-\lambda_j)\Delta(v_j) = \sum_{j \in J_+} \lambda_j \Delta(v_j) \Rightarrow \Delta(x) = \sum_{j \in J} \lambda_j \Delta(v_j),$$

where the second implication follows by our assumption that (1) holds. The statement follows by the observation that $F(x) = \sum_{j \in J} \lambda_j \Delta(v_j)$. \square

Example 9.5. It is straightforward to show that Gelfand-Zetlin polytopes $\text{GZ}: (\Lambda^+)^{\circ} \rightarrow \mathcal{P}_{\mathbb{R}^N}^+$ form a linear family of polytopes. Here, $(\Lambda^+)^{\circ}$ denotes the interior of the positive Weyl chamber.

For a linear family of polytopes $\Delta: C \rightarrow \mathcal{P}_W^+$, we construct its *lift* $\tilde{\Delta}: \mathcal{C} \rightarrow \mathcal{P}_{V \oplus W}^+$ where \mathcal{C} denotes the following convex cone (C° denotes the interior of C)

$$\mathcal{C} := \{Q \in \mathcal{P}_V^+ : Q \subseteq C^{\circ}\}.$$

Proposition 9.6. *\mathcal{C} is a full-dimensional convex cone.*

Proof. A straightforward calculation shows that \mathcal{C} is a convex cone. To show that $\mathcal{C} \subseteq \mathcal{P}_V$ is full-dimensional, let $Q = Q_1 - Q_2 \in \mathcal{P}_V$ where $Q_1, Q_2 \in \mathcal{P}_V^+$. Since $C \subseteq V$ is full-dimensional, there is a translation vector $v \in V$ such that $Q_1 + v, Q_2 + v \subseteq C$. The statement follows from the observation that $Q = (Q_1 + v) - (Q_2 + v)$. \square

We define the lift $\tilde{\Delta}: \mathcal{C} \rightarrow \mathcal{P}_{V \oplus W}^+$ of the linear family of polytopes $\Delta: C \rightarrow \mathcal{P}_V^+$ by setting for any $Q \in \mathcal{C}$:

$$\tilde{\Delta}(Q) := \{(x, y) \in V \oplus W : x \in Q, y \in \Delta(x)\}.$$

Proposition 9.7. *Suppose V is finite-dimensional. Then for any $Q \in \mathcal{C}$, $\tilde{\Delta}(Q)$ is a polytope, i.e. $\tilde{\Delta}(Q) \in \mathcal{P}_{V \oplus W}^+$. Furthermore, $\tilde{\Delta}: \mathcal{C} \rightarrow \mathcal{P}_{V \oplus W}^+$ is linear in the sense of Proposition 9.4.*

Proof. Let $Q \in \mathcal{C}$. Consider the H -representation of Q , i.e.

$$Q = \bigcap_{i=1}^s \{x \in V : \langle x, n_i \rangle \leq b_i\} \quad \text{where } n_i \in V^* \text{ are the facet normals and } b_i \in \mathbb{R}.$$

To determine the H -representation of $\tilde{\Delta}(Q)$, we also need the H -representation of $\Delta(x)$ for $x \in C^\circ$. By [KV18, Proposition 1.3], the polytopes $\Delta(x)$ for $x \in C^\circ$ share the same normal fan (indeed, Kaveh and Villeda assume C to be polyhedral, but their argument can be easily generalized for arbitrary convex cones). In particular,

$$\Delta(x) = \bigcap_{j=1}^{\sigma} \{y \in W : \langle y, \nu_j \rangle \leq \beta_j(x)\} \quad \text{where } \nu_j \in W^* \text{ are the facet normals and } \beta_j(x) \in \mathbb{R}.$$

Since for $x, x' \in C^\circ$, $\Delta(x)$, $\Delta(x')$ and $\Delta(x) + \Delta(x')$ share the same normal fan (see for instance [Zie95, Proposition 7.12]), it straightforwardly follows from the linearity of Δ that the $\beta_j: C^\circ \rightarrow \mathbb{R}$ are linear in the sense of Proposition 9.4. In particular, β_j can be uniquely extended to a linear function on V , i.e. $\beta_j \in V^*$.

From the above it follows that $\tilde{\Delta}(Q) \subseteq V \oplus W$ is a polytope as $\tilde{\Delta}(Q)$ is bounded with H -representation:

$$\tilde{\Delta}(Q) = \{(x, y) \in V \oplus W : \langle x, n_i \rangle \leq b_i \text{ for } i = 1, \dots, s \text{ and } \langle y, \nu_j \rangle \leq \langle x, \beta_j \rangle \text{ for } j = 1, \dots, \sigma\}.$$

Hence, $\tilde{\Delta}(Q) \in \mathcal{P}_{V \oplus W}^+$. By using the linearity of Δ (in the sense of Proposition 9.4) a straightforward calculation shows that $\tilde{\Delta}: \mathcal{C} \rightarrow \mathcal{P}_{V \oplus W}$ is linear too. \square

We denote the composition of the (linear) lift $\tilde{\Delta}: \mathcal{C} \rightarrow \mathcal{P}_{V \oplus W}^+$ with the volume polynomial on $\mathcal{P}_{V \oplus W}$ by $\widetilde{\text{Vol}}$. The polynomial $\widetilde{\text{Vol}}$ has degree $\dim(V) + \dim(W)$.

Let us apply this construction of linear families of virtual polytopes to the linear family of Gelfand-Zetlin polytopes $GZ: \Lambda^+ \rightarrow \mathcal{P}_{\mathbb{R}^N}^+$. The polynomial $\widetilde{\text{Vol}}$ has degree $\dim(M_{\mathbb{R}}) + N = \dim_{\mathbb{C}}(E)$, and thus its degree coincides with the degree of the polynomial $I: \mathcal{P}_{M_{\mathbb{R}}} \rightarrow \mathbb{R}$. Indeed, these two polynomials coincide up to a constant scalar multiple. We get the following result.

Theorem 9.8. *Consider the torus bundle $E = \text{SL}_n/U$. Let Σ be a smooth projective fan with corresponding toric variety X . Then the cohomology ring $H^*(E_X, \mathbb{R})$ is given by*

$$H^{*/2}(E_X, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_{\Sigma}) / \text{Ann}(\widetilde{\text{Vol}}).$$

Proof. The statement follows from Theorem 9.2 and the fact that $c(\lambda)^N = N! \cdot \text{Vol}(GZ_{\lambda})$ (see for example [AB05, Remark 3.9 (iii)] or [Kav11, Theorem 2.5]). \square

As a straightforward corollary, we obtain the following result.

Corollary 9.9. *The ring of conditions $C_{\mathbb{R}}^*(\text{SL}_n/U)$ is given by*

$$C_{\mathbb{R}}^*(\text{SL}_n/U) \simeq \text{Diff}(\mathcal{P}_{M_{\mathbb{R}}}) / \text{Ann}(\widetilde{\text{Vol}}).$$

9.3. Toric bundles over a full flag variety G/B . In general, suppose G is an (arbitrary) semisimple Lie group and let $U \subseteq B \subseteq G$ be as above. We use the same letter N to denote the (complex) dimension of the full flag variety G/B . In this case, for any character λ in the positive Weyl chamber, one can define a *string polytope* $S_{\lambda} \subseteq \mathbb{R}^N$ such that $c(\lambda)^N = N! \cdot \text{Vol}(S_{\lambda})$ (see [AB05, Remark 3.9 (iii)] or [Kav11, Theorem 2.5]). The construction of string polytopes depends on the choice of a reduced expression \underline{w}_0 of the longest element w_0 in the Weyl group. Fix some reduced expression \underline{w}_0 .

String polytopes S_{λ} are not necessarily additive, i.e. $S_{\lambda+\mu} = S_{\lambda} + S_{\mu}$ might not be true. However, there exists a convex cone $\mathcal{C}_{\underline{w}_0} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}^N$ called *weighted string cone* such that

$$S_{\lambda} = \pi^{-1}(\lambda) \cap \mathcal{C}_{\underline{w}_0},$$

where $\pi: M_{\mathbb{R}} \oplus \mathbb{R}^N \rightarrow M_{\mathbb{R}}$ is the natural projection (see [Lit98, BZ01]). By [KV18, Proposition 1.4], it follows that the map which sends weights to the corresponding string polytopes is piecewise linear with respect to a fan \mathcal{F} which subdivides the positive Weyl chamber. That is, if Λ^+ denotes the positive Weyl chamber, then the map $S: \Lambda^+ \rightarrow \mathcal{P}_{\mathbb{R}^N}; \lambda \mapsto S_{\lambda}$ is piecewise linear with respect to \mathcal{F} . In particular, if σ is a full dimensional

cone of \mathcal{F} , then $S: \sigma \rightarrow \mathcal{P}_{\mathbb{R}^N}^+$ is a linear family of polytopes. By the construction from above, we obtain its corresponding lift $\mathcal{P}_{M_{\mathbb{R}}} \rightarrow \mathcal{P}_{M_{\mathbb{R}} \oplus \mathbb{R}^N}$ which gives rise to a polynomial $\widetilde{\text{Vol}}$ of degree $\dim(M_{\mathbb{R}}) + N = \dim(G/U)$. The same arguments as above yield generalizations of Theorem 9.8 and Corollary 9.9:

Theorem 9.10. *Consider the torus bundle $E = G/U \rightarrow G/B$. Let Σ be a smooth projective fan with associated toric variety X . Then the cohomology ring $H^*(E_X, \mathbb{R})$ is given by*

$$H^{*/2}(E_X, \mathbb{R}) \simeq \text{Diff}(\mathcal{P}_{\Sigma}) / \text{Ann}(\widetilde{\text{Vol}}).$$

Furthermore, the ring of conditions $C^*(G/U)$ is given by

$$C_{\mathbb{R}}^*(G/U) \simeq \text{Diff}(\mathcal{P}_{M_{\mathbb{R}}}) / \text{Ann}(\widetilde{\text{Vol}}).$$

In this case the ring of conditions $C_{\mathbb{R}}^*(G/U)$ coincides with the ring of complete intersections defined in [KK20] and our description of $C_{\mathbb{R}}^*(G/U)$ is analogous to the one given there.

Remark 9.11. Note that the lift of the linear family of string polytopes depends on the choice of a maximal cone σ of \mathcal{F} . However, since for λ in the positive Weyl chamber the volume of S_{λ} coincides (up to a constant multiple) with the degree of the corresponding line bundle \mathcal{L}_{λ} , the lifted volume polynomial $\widetilde{\text{Vol}}$ is independent of the choice of σ of \mathcal{F} . Indeed, for polytopes Q in the interior of the positive Weyl chamber, $\widetilde{\text{Vol}}(Q) = \int_Q \frac{1}{N!} c(\lambda)^N d\lambda$.

9.4. Projective bundles. Let $p: E \rightarrow B$ be a torus bundle for a torus $T \cong (\mathbb{C}^*)^n$ as before and Σ be the toric fan of the projective space \mathbb{P}^n . After an appropriate choice of coordinates in $N \simeq \mathbb{Z}^n$, the fan Σ coincides with the normal fan of the standard simplex. More precisely, Σ has rays generated by e_1, \dots, e_n, e_{n+1} , where

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1), e_{n+1} = (-1, \dots, -1),$$

and the cones of Σ are spanned by any subset of these rays. By Theorem 3.4, the cohomology ring of E_X can be computed as

$$H^*(E_X, \mathbb{R}) \simeq \frac{H^*(B, \mathbb{R})[H_1, \dots, H_{n+1}]}{\langle H_1 \cdots H_{n+1} \rangle + \langle c(\lambda_i) - H_i + H_{n+1} : i = 1, \dots, n \rangle} \simeq \frac{H^*(B, \mathbb{R})[H_{n+1}]}{(H_{n+1} + c(\lambda_1)) \cdots (H_{n+1} + c(\lambda_n))},$$

where $\lambda_1, \dots, \lambda_n$ denotes the bases of M dual to e_1, \dots, e_n .

We conclude this section by relating this description to the standard description of the cohomology ring of projective bundles (see for instance [Ful98, Theorem 3.3]). Let D_{n+1} be the divisor on E_X corresponding to the ray ρ_{n+1} . Then $E_X \setminus D_{n+1}$ has the structure of a vector bundle V over B . Furthermore, E_X coincides with the projectivization of the vector bundle $V \oplus \mathcal{O}$ where \mathcal{O} denotes the trivial line bundle on B . The following theorem reproduces the classical description of the cohomology ring of E_X .

Theorem 9.12. *The cohomology ring of E_X can be computed as:*

$$H^*(E_X, \mathbb{R}) \simeq \frac{H^*(B, \mathbb{R})[t]}{t^{n+1} + c_1(V \oplus \mathcal{O}) \cdot t^n + \dots + c_n(V \oplus \mathcal{O}) \cdot t + c_{n+1}(V \oplus \mathcal{O})}.$$

Here $c_1(V \oplus \mathcal{O}), \dots, c_{n+1}(V \oplus \mathcal{O})$ denote the Chern classes of the vector bundle $V \oplus \mathcal{O}$ of rank $n+1$ over B .

Proof. The vector bundle V is the direct sum of the line bundles \mathcal{L}_{λ_i} corresponding to the basis of $M \simeq \mathbb{Z}^n$ dual to e_1, \dots, e_n , i.e $V = \mathcal{L}_{\lambda_1} \oplus \dots \oplus \mathcal{L}_{\lambda_n}$. Therefore, the Chern classes of $V \oplus \mathcal{O}$ can be computed as

$$c_i(V \oplus \mathcal{O}) = s_i(c(\lambda_1), \dots, c(\lambda_n)),$$

where s_i is the i -th elementary symmetric polynomial in n variables. The theorem follows from the fact that

$$(H_{n+1} + c(\lambda_1)) \cdots (H_{n+1} + c(\lambda_n)) = H_{n+1}^{n+1} + s_1(c(\lambda_1), \dots, c(\lambda_n)) \cdot H_{n+1}^n + \dots + s_n(c(\lambda_1), \dots, c(\lambda_n)) \cdot H_{n+1}.$$

□

Remark 9.13. The choice of the coordinates in which the fan Σ has standard form is not unique, but the resulting rings are isomorphic.

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