GENERALIZED VIRTUAL POLYTOPES AND QUASITORIC MANIFOLDS

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Abstract. In this paper we develop a theory of volume polynomials of generalized virtual polytopes based on the study of topology of affine subspace arrangements in a real Euclidean space. We apply this theory to obtain a topological version of the BKK Theorem, the Stanley-Reisner and Pukhlikov-Khovanskii type descriptions for cohomology rings of generalized quasitoric manifolds.

1. Introduction

In [PK92a] Pukhlikov and the first author generalized the classical theory of finitely-additive measures of convex polytopes and proposed a geometric construction for a virtual polytope as a Minkowski difference of two convex polytopes. Using this notion, in [PK92b] the same authors proved a Riemann-Roch type theorem linking integrals and integer sums of quasipolynomials over convex chains from a certain family. As a byproduct, they obtained a description for a cohomology ring of a complex nonsingular projective toric variety via a volume polynomial of a virtual polytope. A theory of mixed volumes of virtual convex bodies was developed in [Tim99] in order to produce an 'elementary' proof of the classical g-theorem, motivated by the ideas of [PK92b] and the approach of [McM93].

A topogical generalization of a complex nonsingular projective toric variety is known in toric topology as a (quasi)toric manifold. It was introduced and studied alongside with its real counterpart, a small cover, in [DJ91]: in particular, it was shown that the Stanley-Reisner description for cohomology rings holds for quasitoric manifolds. Since that time quasitoric manifolds and their generalization, torus manifolds [Mas99, HM03], have been studied intensively in toric topology and found numerous valuable applications in homotopy theory [CMS08, HKS16, HK17], unitary [BPR07, LW16] and special unitary bordism [LP16, LLP18], hyperbolic geometry [BP16, BEM+17, BGLV20], and other areas of research.

A remarkable property of torus manifolds is that they acquire a combinatorial description in a similar way to that in the case of toric varieties. Namely, instead of a (rational polyhedral) fan, it is based on the notions of a multi-fan and a multi-polytope, introduced and studied in [HM03]. A multi-fan is a collection of cones, which can overlap each other, unlike it was in the classical case of an ordinary fan. A multi-polytope is an arbitrary finite collection of rays emanating from the origin in the real Euclidean space alongside with positive numbers, the distances to the normal affine hyperplanes from the origin, one number for each ray. In [AM16] the theory of multi-poytopes was applied to prove a version of the BKK Theorem and the Pukhlikov-Khovanskii description for cohomology rings of quasitoric manifolds. On the other hand, a Stanley-Reisner type description for the cohomology of certain torus manifolds was obtained in [MP06] using methods and tools of the theory of manifolds with corners and equivariant topology.

Smooth structures on quasitoric manifolds were constructed in [BPR07] by means of a topological analogue of the Cox construction, in which a coordinate subspace arrangement is replaced by a moment-angle manifold. By the result of [PU12], a moment-angle-complex over a starshaped sphere has a smooth structure. This allowed us in [KLM21] to introduce generalized quasitoric manifolds as quotient spaces of moment-angle-complexes over starshaped spheres by freely acting compact tori of maximal possible rank.

This paper is devoted to developing the theory of generalized virtual polytopes and applying it in order to obtain a topological version of the BKK Theorem, the Stanley-Reisner and Pukhlikov-Khovanskii type descriptions for intersection rings of generalized quasitoric manifolds.

Generalized virtual polytopes and affine subspace arrangements. The first part of the paper is devoted to the theory of generalized virtual polytopes and integration over them, based on studying the homotopy types of unions of affine subspace arrangements in a real Euclidean space. The construction and the theory of generalized virtual polytopes were motivated by the properties of integral functionals on the space of smooth convex bodies. We discuss smooth convex bodies in Section 2.

Let Q be a polynomial of degree $\leq k$ (homogeneous polynomial of degree k) on $\mathbb{R}^n$, $\omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$, and let C_s be the cone of strictly convex bodies $\Delta \subset \mathbb{R}^n$ with smooth boundary. Then the function

$$F(\Delta) = \int_\Delta Q\omega$$

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on the cone C_s is a polynomial of degree $\le k+n$ (homogeneous polynomial of degree $k+n$).

Now, to extend the integration functional to the vector space generated by the cone C_s, we introduce the notion of a *virtual convex body* as a formal difference of convex bodies (with the usual identification $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4 \Leftrightarrow \Delta_1 + \Delta_4 = \Delta_2 + \Delta_3$). Then the following statement summarizes the results of Section 2:

let M be the space of virtual convex bodies representable as a difference of convex bodies from the cone C_s. Then the functional F on C_s can be extended as an integral of the form $Q\omega$ over the chain of virtual convex bodies. Moreover, such an extension is a polynomial on M.

In Section 3 we study the homological properties of unions X of (finite) arrangements of affine subspaces $\{L_i\}$ in a real Euclidean space $L = \mathbb{R}^n$ by means of the nerves K_X of their (closed) coverings by L_i's.

Given two affine subspace arrangements indexed by the same finite set I, we say that the nerve K_X of the collection $\{L_i\}$ *dominates* the nerve K_Y of the collection $\{M_i\}$ if

$$\bigcap_{j\in J} L_j \neq \varnothing \text{ implies that } \bigcap_{j\in J} M_j \neq \varnothing \text{ for any } J \subset I,$$

and we write $K_X \ge K_Y$ in this case. Furthermore, we say that a continuous map $f\colon X \to Y$ is *compatible* with K_X and K_Y if

$$x \in L_{i_1} \cap \ldots \cap L_{i_k} \quad \text{then} \quad f(x) \in M_{i_1} \cap \ldots \cap M_{i_k}.$$

Our main tool in the study of the homological properties of unions of affine subspaces is the statement that the following conditions hold:

(i) If a map $f\colon X \to Y$ compatible with K_X and K_Y exists, the condition $K_X \ge K_Y$ holds;
(ii) if a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then it is unique up to a homotopy;
(iii) if K_X is isomorphic to K_Y and a map $f\colon X \to Y$ compatible with K_X and K_Y exists, then f provides a homotopy equivalence between X and Y.

We then prove that any union X of affine subspaces has the so called *good triangulation* (see Definition 3.6) and use this fact to show that if $K_X \ge K_Y$, then there is a map $f\colon X \to Y$ compatible with K_X and K_Y.

Now, suppose we have an arrangement of hyperplanes $\{H_i\}$ in $L = R^n$. We call it *non-degenerate* if there is no proper linear subspace $V \subset \mathbb{R}^n$ which is parallel to all H_i's. Then the union X of such an arrangement has the homotopy type of a wedge of $(n-1)$-dimensional spheres, in which the number of spheres is equal to the number of bounded regions in $L \setminus X$, see also Theorem 4.9. Therefore, each cycle $\Gamma \in H_{n-1}(X, \mathbb{Z})$ is equal to a linear combination $\Gamma = \sum \lambda_j \partial\Delta_j$, where each coefficient λ_j equals the winding number of the cycle Γ around a point $a_j \in \Delta_j \setminus \partial\Delta_j$. Here, Δ_j denotes a closure of the bounded open polyhedron, which is a bounded component of $L \setminus X$.

In Section 4 we study the homotopy properties of unions X of (finite) arrangements of affine subspaces $\{L_i\}$ in a real Euclidean space $L = \mathbb{R}^n$ by means of the methods developed in Section 3 and the theory of smooth convex bodies in L.

We say that the two hyperplane arrangements, $\mathcal{H}_1$ and $\mathcal{H}_2$ are *combinatorially equivalent* if the corresponding nerves $K_{\mathcal{H}_1}, K_{\mathcal{H}_2}$ are isomorphic. Let $\mathcal{H} = \{H_1, \ldots, H_s\}, \mathcal{H}' = \{H'_1, \ldots, H'_s\}$ be two combinatorially equivalent hyperplane arrangements, and let $X = \bigcup H_i$ and $Y = \bigcup H'_i$ be the corresponding unions of hyperplanes. Then there exists a canonical homotopy equivalence $f\colon X \to Y$. Moreover, we show that for any (finite) simplicial complex K there exists a (finite) affine subspace arrangement $\{L_i\}$ such that the nerve of the (closed) covering of X by the L_i's is homotopy equivalent to X and has the homotopy type of K.

In order to study the homotopy type of a union of affine subspaces in $\mathbb{R}^n$, we consider a finite union $U \subset \mathbb{R}^n$ of open convex bodies: $U = \bigcup U_i$. Our goal is then to study the homotopy type of the set $\mathbb{R}^n \setminus U$. We will do it making use of the following notion from convex geometry. By a *tail cone* $\mathrm{tail}(U)$ of a convex body U we mean the set of points $v \in \mathbb{R}^n$ such that for any $a \in U$ and $t \ge 0$, the inclusion $a + tv \in U$ holds.

It is easy to see that, for any convex set $U \subset \mathbb{R}^n$, its tail cone $\mathrm{tail}(U)$ has the following properties:

- The set $\mathrm{tail}(U)$ is a convex closed cone in $\mathbb{R}^n$. A convex set U is bounded if and only if $\mathrm{tail}(U)$ is the origin $O \in \mathbb{R}^n$;
- If $\mathrm{tail}(U)$ is a vector space V, then for any transversal space V' (i.e. for any V' such that $\mathbb{R}^n = V \oplus V'$), the set U can be represented in the form $U = U' \oplus V$, where $U' = U \cap V'$ is a bounded convex set. That is, if $\mathrm{tail}(U)$ is a vector space, then one has: $U = U' \oplus \mathrm{tail}(U)$ for a certain bounded convex set U'.

Our main result here is the following theorem:

the set $\mathbb{R}^n \setminus U$ is homotopy equivalent to the set $\mathbb{R}^n \setminus \bigcup\{a_i + \mathrm{tail}(U_i)\}$, where the summation is taken over all i such that $\mathrm{tail}(U_i)$ is a vector space.

Now, assume that all the linear spaces $V_i = \mathrm{tail}(U_i)$ above are equal to the same linear space V and denote by T a transversal subspace to V, i.e. such a linear subspace of $\mathbb{R}^n$ that $\mathbb{R}^n = T \oplus V$. Then the set $\mathbb{R}^n \setminus U$ is homotopy equivalent to $T \setminus \{b_i\}$, where $b_i := T \cap \{a_i + V_i\}$. This statement totally describes the homotopy type of the set $\mathbb{R}^n \setminus \bigcup H_i$, where $\{H_i\}$ is any collection of affine hyperplanes in $\mathbb{R}^n$. Indeed, the complement $\mathbb{R}^n \setminus \bigcup H_i$ is a union of open convex sets. Moreover, the maximal linear subspaces contained in $\mathrm{tail}(U_i)$ are the same for each U_i: each of them is equal to the intersection of the linear spaces $\tilde{H}_i$ parallel to the affine hyperplanes H_i.

Volumes of generalized virtual polytopes and intersection rings of generalized quasitoric manifolds. In the second part of the paper we apply the theory of volume polynomials of generalized virtual polytopes to study the cohomology rings of generalized quasitoric manifolds.

First, we work out the monomial and linear relations between characteristic submanifolds of codimension 2 in the intersection rings of generalized quasitoric manifolds. Then we prove a topological version of the BKK Theorem, based on the properties of the volume polynomial for a generalized virtual polytope, which yields a convex theoretic formula for the self-intersection polynomial on the second cohomology of a generalized quasitoric manifold. Finally, we make use of the BKK Theorem as well as the polynomial ring description of a Poincaré duality algebra worked out in [PK92b, KM21] to obtain the Pukhlikov-Khovanskii type description of the cohomology ring of a generalized quasitoric manifold.

In Section 5 we introduce the notion of a generalized virtual polytope and study the properties of integral functionals on the space of generalized virtual polytopes. Let Δ be a triangulation of an $(n-1)$-dimensional sphere on the vertex set $V(\Delta)=\{v_1,\dots,v_m\}$. In what follows, we will identify a simplex of Δ with the set of its vertices view as a subset in $\{1,2,\dots,m\}$.

A map $\lambda\colon V(\Delta)\to(\mathbb{R}^n)^*$ is called a *characteristic map* if for any vertices $v_{i_1},\dots,v_{i_r}$ belonging to the same simplex of Δ the images $\lambda(v_{i_1}),\dots,\lambda(v_{i_r})$ are linearly independent (over $\mathbb{R}$). Similarly, one can define the notion of an integer *characteristic map* $\lambda\colon V(\Delta)\to(\mathbb{Z}^n)^*$.

Such a map defines an m-dimensional family of hyperplane arrangements $\mathcal{AP}$ in the following way. For any $h=(h_1,\dots,h_m)\in\mathbb{R}^m$, the arrangement $\mathcal{AP}(h)$ is given by

$$\mathcal{AP}(h)=\{H_1,\dots,H_m\} \text{ with } H_i=\{\ell_i(x)=h_i\},$$

where we denote by ℓ_i the linear function $\lambda(v_i)$ for each $i\in[m]:=\{1,2,\dots,m\}$. Given a subset $I\subset[m]$, we also denote by $H_I=\bigcap_{j\in I}H_j$ and by Γ_I the face dual to the simplex $I\in\Delta$ in the polyhedral complex $\Delta^\perp$ dual to the simplicial complex Δ (facets of $\Delta^\perp$ are closed stars in Δ' of the vertices of Δ viewed as vertices of its barycentric subdivision Δ').

By a *generalized virtual polytope* we mean a map $f\colon\Delta^\perp\to\bigcup_{\mathcal{AP}(h)}H_i$ *subordinate* to the characteristic map λ; that is, for any $I\subset[m]$, we have:

$$f(\Gamma_I)\subset H_I.$$

Let U be a bounded region of $\mathbb{R}^n\setminus\bigcup_{\mathcal{AP}(h)}H_i$ and $W(U,f)$ be a winding number of a map f. For a polynomial Q on $\mathbb{R}^n$ let us consider the following integral functional on the space of generalized virtual polytopes:

$$I_Q(f):=\sum W(U,f)\int_U Q\omega.$$

The key result of Section 5 is the computation of all partial derivatives of $I_Q(f)$, which goes as follows. Let $I=\{i_1,\dots,i_r\}\subseteq[m]$ be such that $I\notin\Delta$ and $k_1,\dots,k_r$ be positive integers. Then we have

$$\partial_{i_1}^{k_1}\cdots\partial_{i_r}^{k_r}(I_Q)(f)=0.$$

However, if $r=n=\dim\Delta+1$ and I is a simplex in Δ dual to the vertex $A\in\Delta^\perp$, then we have

$$\partial_I(I_Q)(f)=\operatorname{sign}(I)Q(A)\cdot|\det(e_{i_1},\dots,e_{i_n})|.$$

We observe that the volume of the oriented image $f_h(\Delta^\perp)\subset\mathbb{R}^n$ is a function on the real vector space $\mathcal{L}=\{f_h\colon\Delta^\perp\to\mathbb{R}^n\}$ and its value $Vol(f_h)$ on a generalized virtual polytope f_h is a homogeneous polynomial in $h_1,\dots,h_m$ of degree n. It follows easily that all the partial derivatives for two homogeneous polynomials, the volume polynomial $\mathrm{Vol}(f)$ and the integral functional $I_Q(f)$, for a generalized virtual polytope f coincide (up to a constant multiple), see Corollary 5.10. This means that, up to the constant multiple, those two polynomials coincide.

We start Section 6 by recalling the notion of a generalized quasitoric manifold introduced in [KLM21]. In what follows we assume that $K=K_\Sigma$ is a *starshaped sphere*, i.e. an intersection of a complete simplicial fan Σ in $\mathbb{R}^n\simeq N\otimes_{\mathbb{Z}}\mathbb{R}$ with the unit sphere $S^{n-1}\subset\mathbb{R}^n$). In this case, the moment-angle-complex $\mathcal{Z}_K$ acquires a smooth structure, see [PU12]. Let further, $\Lambda\colon\Sigma(1)\to N$ be a characteristic map. Then the $(m-n)$-dimensional subtorus $H_\Lambda:=\ker\exp\Lambda\subset(S^1)^m$ acts freely on $\mathcal{Z}_K$ and the smooth manifold $X_{\Sigma,\Lambda}:=\mathcal{Z}_K/H_\Lambda$ is called a *generalized quasitoric manifold*.

Our description for the cohomology of $X_{\Sigma,\Lambda}$ goes in three steps:

(i) We give a cell decomposition of $X_{\Sigma,\Lambda}$ and show that $H^*(X_{\Sigma,\Lambda})$ is generated by the classes of characteristic submanifolds of codimension 2;
(ii) We show that the monomial and linear relations between classes of characteristic submanifolds of codimension 2 in $X_{\Sigma,\Lambda}$ are satisfied;
(iii) We prove a version of the BKK Theorem for $X_{\Sigma,\Lambda}$ and use it to get the Pukhlikov-Khovanskii type description of $H^*(X_{\Sigma,\Lambda})$.

It is worth mentioning that the steps (ii) and (iii) above could be used in the much more general setting of torus manifolds. However, in this more general case the algebra obtained by Pukhlikov-Khovanskii description might be different from the cohomology ring. Indeed, the algebra computed by the intersection polynomial is the Poincaré duality quotient of the subalgebra of cohomology ring generated by classes of characteristic submanifolds of codimension 2.

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2. Smooth convex bodies and the space of maps $f : S^{n-1} \to \mathbb{R}^n$

In this section we consider a motivational construction of smooth virtual convex bodies.

Consider a set of smooth maps

$$f : S^{n-1} \to \mathbb{R}^n.$$

Such a set forms a vector space under scaling and pointwise addition of functions:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (\lambda f)(x) = \lambda f(x).$$

For a strictly convex smooth body $\Delta \subset \mathbb{R}^n$, its boundary $\partial\Delta$ can be identified with the image of the unite sphere under a Gauss map

$$f_\Delta : S^{n-1} \to \partial\Delta.$$

In terms of the support function H_Δ of Δ, the map f_Δ is equal to the restriction of the gradient $\mathrm{grad} H_\Delta$ on the sphere S^{n-1}. Thus we got an inclusion of the space of strictly convex smooth bodies (and their formal differences) into the space of smooth mappings from S^{n-1} to $\mathbb{R}^n$. This inclusion respects the Minkowski addition of convex bodies.

We will be interested in integral functionals on the space of convex bodies. First notice, that one can express the integral $\int_\Delta \omega$ in terms of the corresponding map f_Δ:

$$\int_\Delta \omega = \int_{S^{n-1}} f^*\alpha,$$

where α is any form such that $d\alpha = \omega$.

Let α be a $(n-1)$-form on $\mathbb{R}^n$ given by

$$\alpha = P_1 \widehat{dx_1} \wedge \cdots \wedge dx_n + \cdots + P_n dx_1 \wedge \cdots \wedge \widehat{dx_n}.$$

Here the symbol $\widehat{dx_i}$ means that the term dx_i is missing. The following Theorem is obvious.

Theorem 2.1. *If all coefficients P_i of the form α are polynomials of degree $\leq k$ on $\mathbb{R}^n$ then the function $\int_{S^{n-1}} f^*\alpha$ on the space of smooth mappings $f : S^{n-1} \to \mathbb{R}^n$ is a polynomial of degree $\leq k + n - 1$.*

If all coefficients P_i of the form α are homogeneous polynomials of degree k, then the function $\int_{S^{n-1}} f^*\alpha$ is a homogeneous polynomial of degree $k + n - 1$ on the space of smooth mappings.

2.1. Integral functional on the space of maps and winding numbers.

For an $(n-1)$-form α on $\mathbb{R}^n$ and a smooth map $f : S^{n-1} \to \mathbb{R}^n$, one can give a different way to compute the integral $\int_{S^{n-1}} f^*\alpha$. Let $U \subset \mathbb{R}^n$ be a connected component of $\mathbb{R}^n \setminus \{f(S^{n-1})\}$.

Definition 2.2. The *winding number* $W(U, f)$ of U with respect to f is the mapping degree of the map

$$\frac{f - a}{||f - a||} : S^{n-1} \to S^{n-1} \tag{1}$$

where a is any point in U.

The mapping degree is well defined, i.e. is independent of the choice of $a \in U$, since maps (1) for different $a \in U$ are homotopy equivalent.

Proposition 2.3. *For any smooth $(n-1)$-form α on $\mathbb{R}^n$ and for any smooth mapping $f : S^{n-1} \to \mathbb{R}^n$ the following identity holds:*

$$\int_{S^{n-1}} f^*\alpha = \sum W(U, f) \int_U d\alpha$$

where the sum is taken over all connected components U of the complement $\mathbb{R}^n \setminus \{f(S^{n-1})\}$.

Proof. Follows from the Stokes's formula. □

Theorem 2.4. *Let Q be a polynomial of degree $\leq k$ (homogeneous polynomial of degree k) on $\mathbb{R}^n$ and let $\omega = dx_1 \wedge \cdots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$. Then the function*

$$\sum W(U, f) \int_U Q\omega$$

on the space of smooth mappings is a polynomial of degree $\leq k+n$ (homogeneous polynomial of degree $k+n$).

Proof. Consider a $(n-1)$ form $\alpha = P dx_2 \wedge \cdots \wedge dx_n$, where P is a degree $k+1$ polynomial such that $\partial P/\partial x_1 = Q$. Clearly, $d\alpha = Q\omega$. Thus the statement follows from Theorem 2.1 and Proposition 2.3. □

Let us denote by C_s the cone of strictly convex bodies $\Delta \subset \mathbb{R}^n$ with smooth boundary. As a corollary, we obtain the following statement.

Corollary 2.5. *Let Q and ω be the same as before. Then the function*

$$F(\Delta) = \int_\Delta Q\omega \tag{2}$$

on the cone C_s is a polynomial of degree $\leq k+n$ (homogeneous polynomial of degree $k+n$).

Proof. Indeed, for the map $f = \mathrm{grad} H_\Delta : S^{n-1} \to \mathbb{R}^n$ there are exactly two connected components of $\mathbb{R}^n \setminus \{f(S^{n-1})\}$: the component $U_1 = \mathbb{R}^n \setminus \Delta$ and the component $U_2 = int(\Delta)$. Moreover, the corresponding winding numbers are

$$W(U_1, f) = 0; \quad W(U_2, f) = 1.$$

Thus the statement follows from Theorem 2.4. □

We would like to extend the integration functional to the vector space generated by the cone C_s.

Definition 2.6. 1) A *virtual convex body* is a formal difference of convex bodies (with the usual identification $\Delta_1 - \Delta_2 = \Delta_3 - \Delta_4 \Leftrightarrow \Delta_1 + \Delta_4 = \Delta_2 + \Delta_3$);

2) The *support function of virtual convex body* $\Delta = \Delta_1 - \Delta_2$ is the difference of support functions of Δ_1 and Δ_2;

3) The *chain of virtual convex body with smooth support function* H is the set of connected components U of the complements $\mathbb{R}^n \setminus \{\mathrm{grad} H(S^{n-1})\}$ taken with the coefficients $W(U, \mathrm{grad} H)$.

The following theorem summarizes the results of this section.

Theorem 2.7. *Let L be the space of virtual convex bodies representable as a difference of convex bodies from the cone C_s. Then function (2) on C_s can be extended as an integral of the form $Q\omega$ against the chain of virtual convex bodies. Moreover, such an extension is a polynomial on L.*

3. Union of affine subspaces

In this section we study homological properties of the union of affine subspaces in a vector space $L \simeq \mathbb{R}^n$. Let I be a finite set of indexes. Consider a set $\{L_i\}$ of affine subspaces of a vector space L indexed by elements $i \in I$ and let $X = \cup_{i \in I} L_i$ be their union.

First we define the main combinatorial invariant of the union of affine subspaces. The topological space X has a natural covering by the affine spaces L_i.

Definition 3.1. The nerve K_X of the natural covering of X is the simplicial complex with vertex set indexed by I, i.e. one vertex for each index $i \in I$. A set of vertexes $v_{i_1}, \dots, v_{i_k}$ defines a simplex in K_X if and only if the intersection $L_{i_1} \cap \cdots \cap L_{i_k}$ is not empty.

Consider another set of affine spaces $\{M_i\}$ of a linear space M with the same set of indexes I and with the complex K_Y corresponding to the natural covering of Y.

Definition 3.2. We will say that the nerve K_X of the collection $\{L_i\}$ *dominates* the nerve K_Y of the collection $\{M_i\}$ if

$$\bigcap_{j \in J} L_j \neq \varnothing \text{ implies that } \bigcap_{j \in J} M_j \neq \varnothing \text{ for any } J \subset I.$$

We will write $K_X \geq K_Y$ in this case.

The nerves K_X and K_Y are *equivalent* if $K_X \geq K_Y$ and $K_Y \geq K_X$.

Note that if $K_X \geq K_Y$ then there is a natural inclusion $K_X \to K_Y$. Moreover, if K_X and K_Y are equivalent then this inclusion provides an isomorphism between these complexes.

3.1. **Maps compatible with coverings.** In this subsection we introduce our main tool in the study of union of affine subspaces. Let as before $X = \cup_{i\in I} L_i$ and $Y = \cup_{i\in I} M_i$ be two collections of affine subspaces indexed by a finite set I. First we will need the following definitions.

Definition 3.3. For a point $x \in X = \cup_{i\in I} L_i$ let $I(x)$ be the subset of indices I such that

$$x \in L_i \text{ if and only if } i \in I(x).$$

For points $x \in X$, $y \in Y$, we will say that $x \geq y$ if $I(x) \supset I(y)$.

In particular, Definition 3.3 yields the following notion.

Definition 3.4. A continuous map $f : X \to Y$ is *compatible* with K_X and K_Y if for any $x \in X$, we have $x \leq f(x)$. In other words, if

$$x \in L_{i_1} \cap \ldots \cap L_{i_k} \quad \text{then} \quad f(x) \in M_{i_1} \cap \ldots \cap M_{i_k}.$$

The following theorem is our main tool in the study of homological properties of the union of affine subspaces.

Theorem 3.5. *The following conditions hold*

(i) *If a map $f : X \to Y$ compatible with K_X and K_Y exists, the condition $K_X \geq K_Y$ holds;*
(ii) *if a map $f : X \to Y$ compatible with K_X and K_Y exists, then it is unique up to a homotopy;*
(iii) *if K_X is isomorphic to K_Y, then the map $f : X \to Y$ compatible with K_X and K_Y provides a homotopy equivalence between X and Y.*

Proof. (i) Assume a map $f : X \to Y$ compatible with K_X and K_Y exists, then $K_X \geq K_Y$. Indeed, if $L_{i_1}\cap\cdots\cap L_{i_k}$ is not empty and contains a point x the set $M_{i_1} \cap \cdots \cap M_{i_k}$ contains $f(x)$ and in particular is non-empty.

(ii) if f, g are two maps from X to Y compatible with K_X and K_Y, then for any $0 \leq t \leq 1$ the map $tf+(1-t)g$ is also a map compatible with K_X and K_Y. Indeed, for any $x \in X$ the set of points $y \in Y$ such that $I(x) \subset I(y)$ is convex.

(iii) Assume that K_X and K_Y are isomorphic and there are maps $f : X \to Y$ and $g : Y \to X$ compatible with K_X and K_Y.

Then the map $g \circ f : X \to X$ is homotopy equivalence. Indeed the identity map Id_X and the map $g \circ f$ are compatible with K_X and hence are homotopy equivalent by (ii). Similarly, the map $f \circ g : Y \to Y$ is homotopic to the identity map Id_Y. □

To prove the existence of the compatible maps we would need the following definition.

Definition 3.6. A *good triangulation* of the set $X = \cup_{i\in I} L_i$ is a triangulation such that the following condition holds.

The set of vertices of a simplex S in a good triangulation is totally ordered in the sense of Definition 3.3. In other words, there is an order of the set of vertices $\{v_{i_1}, \ldots, v_{i_s}\}$ of S such that

$$I(v_{i_1}) \subset \ldots \subset I(v_{i_s}).$$

Definition 3.7. Consider the following *natural stratification* of $X = \bigcup_{i\in I} L_i$ by open strata of different dimensions: we say that two points $x, y \in X$ belong to one stratum if

$$x \geq y \quad \textit{and} \quad y \geq x,$$

or equivalently,

$$I(x) = I(y).$$

The stratum containing a point x is the intersection $L(x)$ of the subspaces L_i for $i \in I(x)$ with removed union of spaces L_i for all $i \notin I(x)$.

Definition 3.8. A stratum U_1 of the natural stratification of X is *bigger* than stratum U_2, $U_1 \geq U_2$, of the same stratification if the closure of U_1 contains U_2.

Easy to see that $U_1 \geq U_2$ if and only if for any $x \in U_1$, $y \in U_2$ the relation $x \geq y$ holds.

Definition 3.9. A stratum U has a rank k if the longest chain of strictly decreasing strata

$$U = U_1 > \cdots > U_k$$

has the length k.

Theorem 3.10. *For any finite union $X = \bigcup L_i$ of affine subspaces L_i of a linear space L one can construct a good triangulation.*

Proof. We construct a good triangulation of X in two steps. First, we construct a triangulation compatible with natural stratification of X, i.e. a triangulation such that any open simplex is contained in some open stratum.

A triangulation compatible with natural stratification of X can be constructed inductively by first triangulating of all strata of rank one (i.e. all closed strata) and then extending it to all strata of one higher at each step.

One can construct a good triangulation then by taking a barycentric subdivision of any triangulation of X compatible with standard stratification. Indeed, the set of vertices of each simplex in this subdivision corresponds to an increasing chain of faces of a simplex in the original triangulation which are contained in an increasing chain of strata. □

Theorem 3.11. *If $K_X \geq K_Y$, then there is a map $f : X \to Y$ compatible with K_X and K_Y.*

Proof. First, let us consider a good triangulation τ of X. Then, for any vertex v of τ, let us define $f(v)$ to be any point in Y such that $I(f(x)) \supset I(x)$. Such point always exists since $K_X \geq K_Y$. Then one extends map f linearly on each simplex of τ.

The constructed map f is compatible with K_X and K_Y. Indeed, for any point $x \in X$ there is the smallest simplex S of the good triangulation such that $x \in S$. Among the vertices $V(S)$ of this simplex there is a biggest vertex v. One can see that $I(x) = I(v)$. Since $f(x)$ belongs to the linear combination of the points $f(v_i)$ for $v_i \in V(S)$ the inclusion $I(f(x)) \supset I(x)$ holds. □

Lemma 3.12. *Let $L^0 \subset L$ be such a linear space that*

$$L = L^0 + \hat{L},$$

i.e. L is a direct sum of $\hat{L}$ and L^0. Let $L_i^0 = L_i \cap L^0$ and $X^0 = X \cap L^0 = \cup L_i^0$.

Then X^0 is a non-degenerate union of the affine hyperplanes $L_i^0 \subset L^0$. Moreover, $X = X^0 \times \hat{L}$, thus X is homotopy equivalent to a wedge of $(n-1-l)$-dimensional spheres, where $l = \dim \hat{L}$.

3.2. **Barycentric subdivision and a covering of a simplicial complex.** We will need some general facts related to barycentric subdivision of a simplicial complex.

Let C' be the simplicial complex obtained by the barycentric subdivision of a given simplicial complex C. Each vertex of C' is the barycenter of some simplex of C. A set of vertices of C' belongs to one simplex of C' if and only if the simplices of C corresponding to these vertices are totally ordered with respect to inclusion.

To each vertex v of C let us associate the closed subset X_v of C' which is equal to the union of all simplices of C' containing the vertex v.

Lemma 3.13. *1) The nerve of the covering of C' by the collection of closed subsets X_v corresponding to all vertices v of C coincides with the original complex C.*

2) All sets X_v and their nonempty intersections are homotopy equivalent to a point.

Proof. 1) By definition the set of vertices v of C can by identified with the set of subsets X_v, which provides a covering of C'. If vertices $v_1, \dots, v_k$ belong to one simplex of C, then the sets $X_{v_1}, \dots, X_{v_k}$ contain the barycenter of that simplex, and therefore, these sets have a nonempty intersection. Conversely a set X_v intersects a simplex Δ of the complex C only if v is a vertex of Δ. Thus, if the intersection $X_{v_1} \cap \dots \cap X_{v_k}$ is not empty, then $v_1, \dots, v_k$ belong to some simplex Δ of C.

2) Any nonempty intersection $X_{v_1} \cap \dots \cap X_{v_k}$ can be represented as a union of some simplices of C' containing a common vertex, which is the barycenter of the simplex with the vertices $v_1, \dots, v_k$. Such union is a cone, hence it is homotopy equivalent to a point. □

3.3. **Map $f : K'_X \to Y$ For $K_X \geq K_Y$.** A continuous map $f : K'_X \to Y$ is compatible with the natural coverings

$$BK_X = \cup_{i \in I} X_{v_i} \quad \textit{and} \quad Y = \cup_{i \in I} M_i$$

if for any $i \in I$ the inclusion

$$f(X_{v_i}) \subset M_i$$

holds.

Assume that $K_X \geq K_Y$. Let K_X be the nerve of the natural covering of

$$X = \cup_{i \in I} L_i.$$

The barycentric subdivision K'_X of K_X has its own natural covering by the sets $\hat{L}_i$ equal to the union of the simplices in K'_X, which contain the vertex v_i corresponding to the spaces L_i. By Lemma 3.13 the nerve of the covering of K'_X is isomorphic to K_X. Let us generalize definition of maps between topological spaces compatible with their coverings. Let I be a finite set of indexes. Consider a set $\{X_i\}$ of closed subsets of X indexed by elements $i \in I$.

Definition 3.14. The nerve K_X of the covering $X = \bigcup X_i$ is the simplicial complex whose set of vertices V_X contains one vertex v_i for each subset X_i i.e. one vertex for each index $i \in I$. A set of vertexes $v_{i_1}, \dots, v_{i_k}$ defines a simplex in K_X if and only if the intersection

$$X_{i_1} \cap \dots \cap X_{i_k}$$

is not empty.

The following Theorem can be proved exactly as Theorem 3.5.

Theorem 3.15. *1) A map $f : K'_X \to Y$ compatible with K_X and K_Y exists if and only if the condition $K_X \geq K_Y$ holds;*

2) if a compatible with K_X and K_Y map exists then it is unique up to a homotopy.

4. Homotopy type of a union of affine subspaces

In this section we study the homotopy type of a union of affine subspaces. In particular, we show that a union of affine subspaces can have a homotopy type of any simplicial complex (Theorem 4.4), whereas a union of affine hyperplanes is always homotopic to a wedge of spheres (Theorem 4.9)

Consider a finite set $\{A_i\}$ of affine independent points in a real vector space L. Consider the simplex $T \subset L$ whose set of vertices is the set $\{A_i\}$. With each face T_J of T let L_{T_J} be its affine hull of T_J. We obtain a collection of affine subspaces in L corresponding to the faces T_J.

Recall that the subspace A of a topological space X is called a strong deformation retract if there is a homotopy $\pi(x, t) \colon X \times I \to X$ such that

(i) $\pi(x, 0) = x$ for any $x \in X$;
(ii) $\pi(x, 1) \in A$ for any $x \in X$;
(iii) $\pi(a, t) = a$ for any $a \in A$ and $t \in I$.

Lemma 4.1. *The simplex T is a strong deformation retract of the union of hyperplanes L. Moreover, the deformation retraction $\pi L \times I \to L$ can be chosen to preserve the covering of L by affine spaces L_{T_i}, i.e.*

$$\pi(x, t) \in L_{T_i} \text{ for every } x \in L_{T_i}, t \in I.$$

Proof. Each point $x \in L$ is representable in a unique way as

$$x = \sum \lambda_i A_i, \quad \text{where} \quad \sum \lambda_i = 1$$

(the numbers λ_i are the barycentric coordinates of x with respect to the simplex T).

Consider the projection $\pi \colon L \to T$ which maps a point x with the barycentric coordinates $\{\lambda_i\}$ to the point $\pi(x)$ whose i-th barycentric coordinate is equal to $\max(\lambda_i, 0)$.

It is easy to see that the map $\pi(x, t)$ defined by

$$\pi(x, t) = (1 - t)x + t\pi(x)$$

satisfies the conditions of the Lemma. □

Let $\{T_i\}$ be an ordered collection of faces of the simplex T of size N. Consider the following two sets equipped with the covering by N closed convex sets:

- the union $\bigcup_{i=1}^{N} T_i$, equipped with the covering by the faces T_i from the set $\{T_i\}$;
- the union $\bigcup_{i=1}^{N} L_{T_i}$ of affine spans L_{T_i} of faces T_i, equipped with the covering by the spaces L_{T_i}.

Theorem 4.2. *The natural embedding $\bigcup T_i \to \bigcup L_{T_i}$ makes $\bigcup T_i$ a strong deformation retract of $\bigcup L_{T_i}$. Moreover, the deformation retraction can be chosen to preserve the covering of $\bigcup L_{T_i}$ by affine spaces L_{T_i}*

Proof. Indeed as the required projection and its homotopy one can take the restriction of the homotopy from Lemma 4.1 to $\bigcup L_{T_i}$. □

4.1. Barycentric subdivision and corresponding affine subspaces.

Let Δ be a simplicial complex and let Δ' be its barycentric subdivision. In particular, any simplex Δ_i in Δ corresponds to a vertex A_{Δ_i} of Δ'.

Consider a collection of affine independent points in a vector space L identified with the vertices of Δ'. Then Δ' is naturally embedded to the simplex T generated by this collection.

For a vertex A_i of Δ let it's star $St(A_i)$ be the collection of simplices of Δ having A_i as a vertex. Each star $St(A_i)$ defines a face T_i of T be taking the convex hull of vertices of T which correspond to simplices in $St(A_i)$.

Let X_Δ be the union of all the faces $T_i \subset T$ corresponding to the vertices of Δ. The set $X = X_\Delta$ has a natural covering by the faces T_i. On the other hand, let Y be the union of affine spans L_{T_i} for all the faces T_i corresponding to the vertices of Δ. The set Y has a natural covering by the subspaces L_{T_i}.

The following statement is an immediate corollary of Theorem 4.2.

Corollary 4.3. *The set $X \subset Y$ is a deformation retract of Y. Moreover, the deformation retraction respects the coverings of X by T_i and of Y by L_{T_i}.*

Theorem 4.4. *The nerve of the covering of X by T_i can be naturally identified with the nerve of the covering of Y by L_{T_i}.*

Both of these nerves can be naturally identified with the original simplicial complex Δ.

We can consider the barycentric subdivision Δ' of Δ as a suC'omplex of the complex of all faces of the simplex T. Let Z be the union of all simplices in Δ'. The set Z is equipped with the following covering: with every vertex A_i of Δ one can associate the union Z_i of all (closed) simplices, containing the vertex A_i. In other words, Z_i is the union of all the faces of T containing the vertex A_i and belonging to the simplicial complex Δ'.

Under the embedding $Z \to X$, the sets Z_i are equal to $T_i \cap Z$.

Theorem 4.5. *There is a map $\pi\colon X \to Z$ such that the following conditions hold:*
1) the map π maps each simplex T_i to the set Z_i;
2) the map π maps each simplex from Z to itself.

Proof. The set X is stratified by its covering $X = \cup T_i$ in the following way.

Each stratum of this stratification is a nonempty intersection of some collection of the sets T_i without all nonempty intersections of the bigger collections of sets T_i. In particular this stratification also stratifies the set $Z \subset X$.

The set of all the strata of the above stratification can be naturally identified with the set of all simplices of Δ. Indeed the intersection $\cap T_{i_j}$ is nonempty if and only if there is a simplex in Δ with the vertices A_{i_j}.

In other words, the set of all the strata is in one-to-one correspondence with the set of vertices of Δ', i.e. with the set of vertices of T.

The triangulation of X by the faces of T belonging to X is compatible with the above stratification, i.e. each open simplex of this triangulation is contained in some stratum.

Now one can take the barycentric subdivision of the above triangulation. It provides a good triangulation for our stratification, i.e. for each simplex from the triangulation is compatible in the following sense: if two strata contain two vertices of a simplex of the triangulation, then one of the strata belongs to the closure of another.

Now we are ready to defined a map π. A map π is a map from X to Z which is linear on each simplex of the barycentric subdivision of the natural triangulation of X, which maps each vertex A of the triangulation to the vertex of Δ' corresponding to the stratum, containing the vertex A.

One can easily check that the map we just constructed satisfies all conditions of the Theorem, which finishes the proof. □

Theorem 4.6. *The map $\pi\colon X \to Z$ is homotopic to the identity map.*

Let $\tilde{\pi}$ be the restriction of π to Z. Then $\tilde{\pi}$ maps Z to itself and this map is homotopic to the identity map.

Proof. Observe that if $x \in T_i \subseteq X$, then $\pi(x)$ also belongs to T_i, as well as the entire segment joining these two points due to the definition of the map π. Therefore one can define a linear homotopy $F(x,t) = (1-t)x + t\pi(x)$ between the identity map and the map π.

Furthermore, the map $\tilde{\pi}$ maps any simplex of Δ' to itself. Hence one can define a linear homotopy $G(x,t) = (1-t)x + t\tilde{\pi}(x)$ between the identity map and the map π. □

4.2. **Homotopy type of union of hyperplanes.** Let $\mathcal{H} = \{H_1, \dots, H_s\}$ be a collection of affine hyperplanes in $\mathbb{R}^n$ indexed by the set $[s] = \{1, \dots, s\}$.

Definition 4.7. The *nerve* $K_{\mathcal{H}}$ of $\mathcal{H}$ is the simplicial complex on s vertices $v_1 \dots, v_s$ such that a set of vertices $v_{i_1}, \dots, v_{i_k}$ defines a simplex in $K_{\mathcal{H}}$ if and only if the intersection $H_{i_1} \cap \dots \cap H_{i_k}$ is not empty.

We will say that two hyperplane arrangements $\mathcal{H}_1, \mathcal{H}_2$ are *combinatorially equivalent* if the corresponding nerves $K_{\mathcal{H}_1}, K_{\mathcal{H}_2}$ are isomorphic.

Theorem 4.8. *Let $\mathcal{H} = \{H_1, \dots, H_s\}, \mathcal{H}' = \{H'_1, \dots, H'_s\}$ be two combinatorially equivalent hyperplane arrangements, and let $X = \bigcup H_i$ and $Y = \bigcup H'_i$ be the corresponding unions of hyperplanes. Then there exists a canonical homotopy equivalence $f\colon X \to Y$.*

Proof. As the canonical homotopy equivalence $f\colon X \to Y$ one can take any map such that

$$f(x) \in H'_j \text{ for every } x \in H_j.$$

□

In particular, there is a canonical isomorphism between homology groups of combinatorially equivalent hyperplane arrangements:

$$f_*\colon H_*(X) \to H_*(Y).$$

We will say that the collection of hyperplanes $\{H_1, \dots, H_s\}$ is *non-degenerate* if it is a non-degenerate collection of affine subspaces. That is there is no proper linear subspace $L \subset \mathbb{R}^n$ which is parallel to all H_i's.

Theorem 4.9. *Let $\mathcal{H}$ be a non-degenerate arrangement of affine hyperplanes in $\mathbb{R}^n$. Then their union X is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres.*

The number of spheres is equal to the number of bounded regions in $\mathbb{R}^n \setminus X$.

Proof. We will prove a more general result see Theorem 4.12 and Corollary 4.13. ◻

Corollary 4.10. *Let $L \supset X = \bigcup L_i$ be a non-degenerate union of affine hyperplanes L_i. Then, if $n > 1$, the group*

$$H_{n-1}(X, \mathbb{Z})$$

is a free Abelian group generated by the cycles $\partial\Delta_j$, where Δ_j is a closure of the bounded open polyhedron, which is a bounded component of $L \setminus X$. All other groups $H_i(X, \mathbb{Z})$ for $i > 0$ are equal to zero and $H_0(X, \mathbb{Z}) \cong \mathbb{Z}$.

According to the above Corollary 4.10, each cycle $\Gamma \in H_{n-1}(X, \mathbb{Z})$ is equal to a linear combination

$$\Gamma = \sum \lambda_j \partial\Delta_j.$$

Moreover, each coefficient λ_j is equal to the winding number of the cycle Γ around a point $a_j \in \Delta_j \setminus \partial\Delta_j$.

Now we are ready to proof Theorem 4.9. Let $U \subset \mathbb{R}^n$ be a finite union of open convex bodies: $U = \bigcup U_i$. We are going to study the homotopy type of the set $\mathbb{R}^n \setminus U$. First we need the following definition.

Definition 4.11. The *tail cone* of a convex body U is a set of points $v \in \mathbb{R}^n$ such that for any $a \in U$ and $t \geq 0$, the inclusion $a + tv \in U$ holds.

One can check that for any convex set $U \subset \mathbb{R}^n$ the set $\mathrm{tail}(U)$ satisfies the following conditions.

- The set $\mathrm{tail}(U)$ is a convex closed cone in $\mathbb{R}^n$. A convex set U is bounded if and only if $\mathrm{tail}(U)$ is the origin $O \in \mathbb{R}^n$.
- If $\mathrm{tail}(U)$ is a vector space L, then for any transversal space L_1 (i.e. for any L_1 such that $\mathbb{R}^n = L \oplus L_1$) the set U is representable in the form $U = U_1 \oplus L$, where $U_1 = U \cap L_1$ is a bounded convex set. That is, if $\mathrm{tail}(U)$ is a vector space, then one has: $U = U_1 \oplus \mathrm{tail}(U)$ for a certain bounded convex set U_1.

If the set $\mathrm{tail}(U_i)$ is a linear space L_i, then alongside with U_i we can also consider a shifted space $a_i + L_i \subset U_i$, where a_i is any point in U_i.

We will prove the following theorem.

Theorem 4.12. *The set $\mathbb{R}^n \setminus U$ is homotopy equivalent to the set $\mathbb{R}^n \setminus \bigcup\{a_i + L_i\}$, where the summation is taken over all i such that $\mathrm{tail}(U_i)$ is a vector space.*

Assume that in Theorem all the linear spaces L_i are equal to the same linear space L. Denote by T a transversal subspace to L, i.e. such a linear subspace of $\mathbb{R}^n$ that $\mathbb{R}^n = T \oplus L$.

Corollary 4.13. *Under the above assumptions, the set $\mathbb{R}^n \setminus U$ is homotopy equivalent to $T \setminus \{b_i\}$, where $b_i = T \cap \{a_i + L_i\}$.*

Corollary 4.13 totally describes the homotopy type of the set $\mathbb{R}^n \setminus \bigcup H_i$, where $\{H_i\}$ is any collection of affine hyperplanes in $\mathbb{R}^n$. Indeed, the complement $\mathbb{R}^n \setminus \bigcup H_i$ is a union of open convex sets. Moreover, the maximal linear subspaces contained in $\mathrm{tail}(U_i)$ are the same for each U_i: each of them is equal to the intersection of the linear spaces $\tilde{H}_i$ parallel to the affine hyperplanes H_i.

We will need some general facts about convex bodies.

Lemma 4.14. *Let $U \subset \mathbb{R}^n$ be a bounded open convex set, X be the closure of U, ∂X be the boundary of X (i.e. $X = U \cup \partial X$), and let $a \in U$ be any point in U. Then ∂X is a retract of $X \setminus \{a\}$.*

Proof. Let $\pi \colon X \setminus \{a\} \to \partial X$ be the projection of $X \setminus \{a\}$ to ∂X from the point a. The following map provides a homotopy retraction:

$$F(x, t) = (1 - t)x + t\pi(x),$$

where $x \in X \setminus \{a\}$ and $0 \leq t \leq 1$. ◻

Corollary 4.15. *Let $U \subset \mathbb{R}^n$ be an open convex set such that $\mathrm{tail}(U)$ is a vector space L. Then, by definition, for any $a \in U$ the shifted space $a + L$ belongs to U and the set X is homotopy equivalent to the set $X \setminus L$ where X is the closure of U.*

We will need the following auxiliary lemma. Let us represent $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \oplus \mathbb{R}^1$ and let us use accordingly the notation (x, y) for points in $\mathbb{R}^n$, where $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^1$.

Let $y = f(x)$ be a continuous function on $\mathbb{R}^{n-1}$. Denote by $X \subset \mathbb{R}^n$ the set of points (x, y), where $y \geq f(x)$. Then ∂X is the graph of the function f (i.e. $(x, y) \in \partial X$ if and only if $y = f(x)$).

Lemma 4.16. *The natural projection $\pi \colon X \to \partial X$ mapping a point (x, y) to $(x, f(x))$ is homotopic to the identity map.*

Proof. One can consider the following homotopy: $G(x, y, t) = (1 - t)(x, y) + t\pi(x, y)$. ◻

Now, assume that the set $\mathrm{tail}(U) \subset \mathbb{R}^n$ is not a vector space, i.e. assume that there is a vector $v \in \mathrm{tail}(U)$ such that the vector $-v$ does not belong to $\mathrm{tail}(U)$.

Let $a \in U$ be an arbitrary point. Since $-v$ is not in $\mathrm{tail}(U)$, there is a positive number τ such that $a-\tau v \in \partial X$. Let $\tilde{L}$ be the supporting hyperplane of x at the point $a - \tau v$.

Let us make an affine change of variables in $\mathbb{R}^n$ in such a way that the hyperplane $\tilde{L}$ becomes the hyperplane $y = 1$, the point $a - \tau v$ becomes the point $(0, 1)$ and the vector v becomes the standard basis vector in $\mathbb{R}^1$.

After this change of coordinates, U becomes an open convex set in $\mathbb{R}^{n-1} \oplus \mathbb{R}^1$ such that U belongs to the half space $y \geq 1$, and alongside with every point $a \in U$ our convex set U contains the entire ray $a + \tau v$, where $\tau \geq 0$ and v is the vector $(0, 1)$.

Consider the diffeomorphism g of the open half space $y > 0$ to itself defined by the following formula:

$$g(x, y) = (xy, y).$$

Lemma 4.17. *Under g the closure X of U is mapped to the domain Y defined by the following condition $(x, y) \in Y$ if and only if $y \geq f(x)$, where f is a certain continuous function on $\mathbb{R}^n$.*

Proof. First, let us consider the map $\tilde{g}\colon \partial X \to \mathbb{R}^{n-1} \oplus \{0\}$ given by the formula:

$$\tilde{g}(x, y) = (xy, 0).$$

Let us show that $\tilde{g}$ is a homeomorphism of ∂X and $\mathbb{R}^{n-1}$. For every vector $x \in \mathbb{R}^{n-1} \oplus \{0\}$, consider the set of points $\partial X_x \subset \partial X$ defined by the following condition: a point $(x_0, y_0) \in \partial X_x$ if and only if x_0 is proportional to x. It is easy to see that the set ∂X_x is homeomorphic to a line.

We can parametrize it by an oriented distance from the point $(0, 1)$ (which belongs to ∂X_x for every x) along this curve with arbitrary chosen orientation.

Now the map $\tilde{g}$ maps the curve ∂X_x to the line of vectors proportional to x. Moreover this map is monotonic and proper. Hence it provides a homeomorphism between ∂X_x and the line $\tau x, \tau \in \mathbb{R}$. This argument implies that the map $\tilde{g}\colon \partial X \to \mathbb{R}^{n-1}$ is a homeomorphism.

The image of ∂X under the diffeomorphism $g\colon (x, y) \mapsto (xy, y)$ is a graph of the function f such that $f(x)$ equals the coordinate y of the point $(x, y) := \tilde{g}^{-1}(x)$. Then the set X is mapped by this diffeomorphism to the domain in $\mathbb{R}^n$ consisting of the points (x, y), where $y \geq f(x)$. ☐

Corollary 4.18. *Let $U \subset \mathbb{R}^n$ be an open convex set such that the cone* $\mathrm{tail}(U)$ *is not a vector space. Then the boundary ∂X of the closure X of U is homotopy equivalent to X.*

5. GENERALIZED VIRTUAL POLYTOPES: DEFINITIONS AND RESULTS

Let Δ be a simplicial complex homeomorphic to the $(n-1)$-dimensional sphere. Denote by $V(\Delta) = \{v_1, \ldots, v_m\}$ the set of vertices of Δ. In what follows, we will identify simplex S of Δ with the set of vertices $I \subset V(\Delta)$ which belongs to S.

Definition 5.1. The map $\lambda\colon V(\Delta) \to (\mathbb{R}^n)^*$ is called a characteristic map if for any vertices $v_{i_1}, \ldots, v_{i_r}$ belonging to the same simplex of Δ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_r})$ are independent. In particular, for any maximal simplex $\{v_{i_1}, \ldots, v_{i_n}\}$ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_n})$ form a basis of $(\mathbb{R}^n)^*$.

The map $\lambda\colon V(\Delta) \to (\mathbb{Z}^n)^*$ is called an integer *characteristic map* if for any maximal simplex $\{v_{i_1}, \ldots, v_{i_n}\}$ of Δ the images $\lambda(v_{i_1}), \ldots, \lambda(v_{i_n})$ form a basis of the lattice $(\mathbb{Z}^n)^*$.

Let us denote by ℓ_i the linear function $\lambda(v_i)$ for any $i \in [r]$. The characteristic map λ defines an m-dimensional family of hyperplane arrangements $\mathcal{AP}$ in the following way. For any $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$, the arrangement $\mathcal{AP}(h)$ is given as

$$\mathcal{AP}(h) = \{H_1, \ldots, H_m\} \text{ with } H_i = \{\ell_i(x) = h_i\}.$$

We denote by X_h the union of hyperplanes from $\mathcal{AP}(h)$.

Given a subset $I \in [s]$, we will denote by H_I the intersection

$$H_I = \bigcap_{j \in I} H_j.$$

It follows from the definition of the characteristic map that H_I is non-empty whenever the vertices v_j with $j \in I$ belong to the same simplex.

Let $\Delta^\perp$ be a dual complex to Δ, we define a correspondence between faces of $\Delta^\perp$ and the strata H_I in the following way. A face Γ_I of $\Delta^\perp$ dual to a simplex I of Δ is associated to the stratum H_I.

Definition 5.2. We say that a map $f : \Delta^\perp \to X_h$ is subordinate to a characteristic map λ, if for any face Γ_I of $\Delta^\perp$ we have $f(\Gamma_I) \subset H_I$.

Theorem 5.3. *The space of maps $f : \Delta^\perp \to X_h$ subordinate to a characteristic map λ is a non-empty convex set. In particular, every two such maps are homotopic.*

Proof. First let us show the second part of the lemma assuming that a map $f\colon \Delta^\perp \to X_h$ subordinate to coverings exists. First notice that H_I is convex for any $I \subset [m]$. Therefore, for any two maps $f, f'\colon \Delta^\perp \to X_h$ subordinate to the coverings, any member of the linear homotopy between them is also subordinate to the covering:

$$f_t := (1-t)f + tf', \quad t \in [0,1].$$

Thus the space of maps $f\colon \Delta^\perp \to X_h$ subordinate to coverings of K' and X_h is contractible (assuming it is non-empty).

To show the existence of such maps we use the following construction. First let us choose any inner product on $\mathbb{R}^n$. This defines a set of distinguished points $x_I \in H_I$ via taking orthogonal projection of the origin in $\mathbb{R}^n$ to a subspace H_I. On the other hand, vertices of the barycentric subdivision $\Delta^\perp$ of Δ are in bijection with simplices of Δ, hence are labeled by subsets $I \subset [m]$.

We construct the map $f_h\colon \Delta^\perp \to X_h$ subordinate to the coverings in the following way. First we define the image of vertices v_I of $\Delta^\perp$ by

$$f_h(v_I) = x_I,$$

and then we extend the map by linearity. A map f_h is well defined since (Δ, Λ) is a characteristic pair (indeed, H_I is nonempty whenever I indexes a simplex in Δ) and is subordinate to the covering by $st(v_i)$, by construction. ☐

The family of maps $f_h : \Delta^\perp \to X_h$ satisfies another nice property.

Corollary 5.4. *In the situation as before, one has $f_{h+h'} = f_h + f_{h'}$.*

Proof. The statement follows from the fact that the frame points x_I used in the construction depend linearly on $h \in \mathbb{R}^n$:

$$x_{I,h+h'} = x_{I,h} + x_{I,h'}.$$

☐

With every hyperplane arrangement $\mathcal{AP}(h)$ let us associate a chain $\Delta(h) = \sum_i W(U_i, f)$, where U_i are the connected components of the compliment $\mathbb{R}^n \setminus \mathcal{AP}(h)$, and $f : \Delta^\perp \to \mathcal{AP}(h)$ is any map subordinate to the coverings. Since any two such maps are homotopic, the chain $\Delta(h)$ is well-defined.

Definition 5.5. We will call the chain $\Delta(h)$ a *generalized virtual polytope* associated to a simplicial complex Δ, characteristic map Λ and a vector $h \in \mathbb{R}^m$. We denote by $\mathcal{P}_{K,\Lambda} \simeq \mathbb{R}^m$ the space of all generalized virtual polytope associated to a simplicial complex Δ and characteristic map Λ.

Remark 5.6. Classical virtual polytopes are defined as convex chains and hence, they carry more information then a chain $\Delta(h)$: the chain $\Delta(h)$ is only a full-dimensional part of virtual polytope. However, in this paper we only interested in volumes of generalized virtual polytopes and integrals over them, so it is enough for us to work with the chain $\Delta(h)$. We will study other valuations on the space of generalized virtual polytopes in future work.

5.1. **Integration over generalized virtual polytopes.** Let α be a $(n-1)$-form on $\mathbb{R}^n$ given by

$$\alpha = P_1 \widehat{dx_1} \wedge \cdots \wedge dx_n + \cdots + P_n dx_1 \wedge \cdots \wedge \widehat{dx_n}.$$

Here the symbol $\widehat{dx_i}$ means that the term dx_i is missing. The following Theorem is obvious.

Theorem 5.7. *If all coefficients P_i of the form α are polynomials of degree k (a polynomial of degree $\leq k$) on $\mathbb{R}^n$ then the function $\int_{\Delta^\perp} f^*\alpha$ is a homogeneous polynomial of degree $k+n-1$ (a polynomial of degree $\leq k+n-1$) on the space of mappings $f : \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ subordinate to the corresponding covering.*

Proof. Analogous to the proof of Theorem 2.1 since by Corollary 5.4 the family of maps f_h can by chosen so that $f_{h_1} + f_{h_2} = f_{h_1+h_2}$. ☐

Let U be a bounded region of $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ and $W(U, f)$ be a winding number of a map f as before. The following proposition follows from the Stokes theorem.

Proposition 5.8. *Let α be as before and $d\alpha = Q\omega$ where Q is a polynomial of degree $\leq k$ (homogeneous polynomial of degree k) on $\mathbb{R}^n$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$ is the standard volume form on. Then the following identity holds*

$$\sum W(U,f) \int_U Q\omega = \int_{\Delta^\perp} f^*\alpha$$

for $f : \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ subordinate to the corresponding covering.

In particular $\sum W(U,f) \int_U Q\omega$ is a polynomial of degree $\leq k+n-1$ (homogeneous polynomial of degree $k+n-1$) on the space of mappings $f : \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ subordinate to the corresponding covering.

For a polynomial Q on $\mathbb{R}^n$ and a generalized virtual polytope $f : \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$, let us denote by $I_Q(f)$ the integral

$$\sum W(U,f)\int_U Q\omega.$$

The following lemma computes the derivatives of I_Q.

Lemma 5.9. *Let $f : \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ be a generalized virtual polytope given by the simplicial complex Δ on s vertices. Let $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset such that $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a simplex in Δ and $k_1, \dots, k_r$ positive integers, then we have*

$$\partial_{i_1}^{k_1}\cdots\partial_{i_r}^{k_r}(I_Q)(f) = 0.$$

However, if $r = n$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a simplex in Δ dual to the vertex $A \in M_\mathbb{R}$, we have

$$\partial_I(I_Q)(f) = \operatorname{sign}(I)Q(A)\cdot|\det(e_{i_1}, \dots, e_{i_n})|.$$

Proof. By linearity of derivation it is enough to compute the partial derivatives for each summand $W(U,f)\int_U Q\omega$ separately.

In the first case, when $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a simplex in Δ, the intersection of corresponding hyperplanes $H_{i_1}, \dots, H_{i_r}$ do not correspond to a vertex of U for any bounded region of $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ with $W(U,f) \neq 0$. Hence $\partial_{i_1}^{k_1}\cdots\partial_{i_r}^{k_r}(I_Q)(f) = 0$ by [HKM20, Lemma 6.1].

On the other hand, if $\rho_{i_1}, \dots, \rho_{i_n}$ span a simplex in Δ, then there is exactly one region U_i of $\mathbb{R}^n \setminus \bigcup_{\mathcal{AP}(h)} H_i$ having the intersection

$$A = H_{i_1} \cap \ldots \cap H_{i_n}$$

as a vertex. Then by [HKM20, Lemma 6.1] we get

$$\partial_I(I_Q)(f) = \partial_I\int_{U_i} Q\omega = \operatorname{sign}(I)Q(A)\cdot|\det(e_{i_1}, \dots, e_{i_n})|.$$

□

As an immediate consequence of Lemma 5.9 we obtain the following corollary.

Corollary 5.10. *Let $f : \Delta^\perp \to \bigcup_{\mathcal{AP}(h)} H_i$ be a generalized virtual polytope given by the simplicial complex Δ on s vertices. Let $I = \{i_1, \dots, i_r\} \subseteq \{1, \dots, s\}$ be a subset such that $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a simplex in Δ and $k_1, \dots, k_r$ positive integers, then we have*

$$\partial_{i_1}^{k_1}\cdots\partial_{i_r}^{k_r}\operatorname{Vol}(f) = 0.$$

However, if $r = n$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a simplex in Δ dual to the vertex $A \in M_\mathbb{R}$, we have

$$\partial_I\operatorname{Vol}(f)(f) = \operatorname{sign}(I)\cdot|\det(e_{i_1}, \dots, e_{i_n})|.$$

6. Cohomology of generalized quasitoric manifolds

In this section we will describe the cohomology rings of a class of torus manifolds called generalized quasitoric manifolds. Let $T \simeq (S^1)^n$ be a compact torus with character lattice M and $N = M^\vee$. Let K be an abstract simplicial complex of dimension $n-1$ on the vertex set $[m] = \{1, 2, \dots, m\}$. Recall that its moment-angle-complex $\mathcal{Z}_K$ is defined to be the $(m+n)$-dimensional cellular subspace in the unitary polydisc $(D^2)^m \subset \mathbb{C}^m$ given by the formula $\bigcup_{I\in K}\prod_{i=1}^m Y_i$, where $Y_i = D^2$, if $i \in I$ and $Y_i = S^1$, otherwise.

There is a natural (coordinatewise) action of the compact torus $(S^1)^m$ on $\mathcal{Z}_K$ and the orbit space $\mathcal{Z}_K/(S^1)^m$ is homeomorphic to the cone over the barycentric subdivision of K.

In what follows we assume that $K = K_\Sigma$ is a *starshaped sphere*, i.e. an intersection of a complete simplicial fan Σ in $\mathbb{R}^n \simeq N \otimes_\mathbb{Z} \mathbb{R}$ with the unit sphere $S^{n-1} \subset \mathbb{R}^n$). In this case, the moment-angle-complex $\mathcal{Z}_K$ is homeomorphic to the moment-angle manifold and therefore acquires a smooth structure.

Let further, $\Lambda : \Sigma(1) \to N$ be a *characteristic map*, i.e. such a map that the collection of vectors

$$\Lambda(\rho_1), \dots, \Lambda(\rho_k)$$

can be completed to a basis of the cocharacter lattice N, whenever $\rho_1, \dots, \rho_k$ generate a cone in Σ. Then the $(m-n)$-dimensional subtorus $H_\Lambda := \ker\exp\Lambda \subset (S^1)^m$ acts freely on $\mathcal{Z}_K$ and the smooth manifold $X_{\Sigma,\Lambda} := \mathcal{Z}_K/H_\Lambda$ will be called a *generalized quasitoric manifold.*

Our description of cohomology rings of $X_{\Sigma,\Lambda}$ will be given in three steps:

(i) We give a cell decomposition of $X_{\Sigma,\Lambda}$ and show that $H^*(X_{\Sigma,\Lambda})$ is generated by the classes of characteristic submanifolds of codimension 2;
(ii) We show that two sets of relations are satisfied between classes of characteristic submanifolds of codimension 2;

(iii) We prove a version of BKK theorem for $X_{\Sigma,\Lambda}$ and use it to get Pukhlikov-Khovanskii type description of $H^*(X_{\Sigma,\Lambda})$.

Remark 6.1. Steps (ii) and (iii) above could be used in the much more general situation of torus manifolds. However, in this more general case the algebra obtained by Pukhlikov-Khovanskii description might be different from the cohomology ring. Indeed, the algebra computed by the intersection polynomial is the Poincaré duality quotient of the subalgebra of cohomology ring generated by classes of characteristic submanifolds of codimension 2 (see [AM16] for details).

In what follows we will always assume that our generalized quasitoric manifolds are *omnioriented*; as in the case of a quasitoric manifold, we say that $X_{\Sigma,\Lambda}$ is omnioriented if an orientation is specified for $X_{\Sigma,\Lambda}$ and for each of the m codimension-2 characteristic submanifolds D_i. The choice of this extra data is convenient for two reasons. First, it allows us to view the circle fixing D_i as an element in the lattice $N = \mathrm{Hom}(S^1, T^n) \simeq \mathbb{Z}^n$. But even more importantly, the choice of omniorientation defines the fundamental class $[X_{\Sigma,\Lambda}]$ of $X_{\Sigma,\Lambda}$ as well as cohomology classes $[D_i]$ dual to the characteristic submanifolds.

We further assume that $\Sigma \subset \mathbb{R}^n$ and $N_{\mathbb{R}}$ are endowed with orientation. This defines a sign for each collection of rays $\rho_{i_1}, \dots, \rho_{i_n}$ forming a maximal cone of Σ in the following way. Let $I = \{i_1, \dots, i_n\}$ be a set of indices ordered such that the collection of rays $\rho_{i_1}, \dots, \rho_{i_n}$ is positively oriented in $\mathbb{R}^n$. Then

$$\mathrm{sign}(I) = \det(\Lambda(\rho_{i_1}), \dots, \Lambda(\rho_{i_n})) = \pm 1.$$

Finally, as before, with a characteristic pair (Σ, Λ) we associate a space of generalized virtual polytopes $\mathcal{P}_{\Sigma,\Lambda} \simeq \mathbb{R}^m$. For every generalized virtual polytope $\Delta(h) \in \mathcal{P}_{\Sigma,\Lambda}$ we associate an element of $H^2(X_{\Sigma,\Lambda})$ in the following way

$$\Delta(h) \mapsto h_1[D_1] + \ldots + h_m[D_m] \in H^2(X_{\Sigma,\Lambda}),$$

there $D_1, \dots, D_m$ are the codimension 2 characteristic submanifolds with oriented in according to the omniorientation of $X_{\Sigma,\Lambda}$.

6.1. **Cell decomposition of generalized quasitoric manifolds.** To provide the cell decomposition of the generalized quasitoric manifold $X_{\Sigma,\Lambda}$ let us first give a slightly different description of the moment-angle complex $\mathcal{Z}_K$ for a starshaped sphere $K = K_\Sigma$. The moment-angle complex is given as a disjoint union of strata $\mathcal{Z}_K = \bigsqcup_{\sigma \in \Sigma} H_\sigma$, where

$$H_\sigma = \mathcal{Z}_K \cap \left(\bigcap_{\rho_i \in \sigma} \{z_i = 0\} \right) \cap \left(\bigcap_{\rho_j \notin \sigma} \{z_j \neq 0\} \right) \subset \mathbb{C}^m.$$

Our construction of a cell decomposition of $X_{\Sigma,\Lambda}$ is a slight generalization of the Morse theoretic argument introduced in [Kho86] and applied to quasitoric manifolds in [DJ91]. Since we do not assume that Σ is a dual fan to some polytope, we cannot use the generic linear function as in [Kho86]. Instead, let us choose a vector $v \in \mathbb{R}^n$ in a general position with Σ, i.e. a vector v which belongs to the interior of a full dimensional cone of Σ.

Let $\tau_1, \dots, \tau_s$ be cones of dimension n in Σ. For a maximal cone τ, we will say that a face σ of τ is incoming with respect to vector v if the intersection $\tau \cap (\sigma + v)$ is unbounded. Let us further define the index $ind(\tau)$ of a maximal cone τ to be the number of incoming rays of τ.

For each maximal cone τ, we us associate a disjoint union of open cells of $\mathcal{Z}_K$ via

$$\widetilde{U}_\tau = \bigsqcup_{\sigma} H_\sigma,$$

where the union is taken all incoming faces σ of τ. Since each cone σ is incoming for exactly one cone of maximal dimension τ, we get $\mathcal{Z}_K = \bigsqcup_{i=1}^s \widetilde{U}_{\tau_i}$. That the cells U_τ are invariant under the action of $H \simeq (S^1)^{m-n}$ and that

$$\widetilde{U}_\tau \simeq (D^2)^{\mathrm{ind}(\tau)} \times (S^1)^{m-n}.$$

Moreover, the action of H is free and transitive on the second component of $(D^2)^{\mathrm{ind}}\sigma \times (S^1)^{m-n}$, hence we get:

$$X_{\Sigma,\Lambda} \bigsqcup_{i=1}^{s} \widetilde{U}_{\tau_i}/H,$$

where $\widetilde{U}_{\tau_i}/H \simeq (D^2)^{\mathrm{ind}(\tau)}$

Theorem 6.2. *Let $X_{\Sigma,\Lambda}$ be a generalized quasitoric manifold. Then $X_{\Sigma,\Lambda}$ has a cellular decomposition with only even dimensional cells. The cells in the decomposition are in bijection with maximal cones τ in Σ. The dimension of the cell corresponding to a cone τ is $2\mathrm{ind}(\tau)$.*

Corollary 6.3. *The Euler characteristics of $X_{\Sigma,\Lambda}$ is equal to the number of maximal cones in Σ.*

6.2. **Relations between characteristic submanifolds.** In this subsection we will show two sets of relations between classes of codimension 2 characteristic submanifolds in the cohomology ring of $X_{\Sigma,\Lambda}$. In the following proposition we show the Stanley-Raisner relations in $H^*(X_{\Sigma,\Lambda})$.

Proposition 6.4. *For codimension 2 characteristic submanifolds $D_{i_1},\dots,D_{i_n}$ in a generalized quasitoric manifold $X_{\Sigma,\Lambda}$, one has:*

$$[D_{i_1}]\cdots[D_{i_n}] = \begin{cases} \operatorname{sign}(I) & \text{if } \rho_{i_1}\dots,\rho_{i_n} \text{ form a cone in } \Sigma \\ 0 & \text{otherwise} \end{cases}$$

Proof. Indeed, in the intersection ring of the generalized quasitoric manifold $X_{\Sigma,\Lambda}$ we have: $[D_{i_1}]\cdots[D_{i_n}] = (-1)^v$, the sign of the fixed point $v = D_{i_1}\cap\dots\cap D_{i_n} \in X_{\Sigma,\Lambda}$, which compares two orientations on $\mathcal{T}_v X_{\Sigma,\Lambda}$: the one induced by coorientations of characteristic submanifolds D_i's and the one induced by the representation of $T^n := T^m/H$ in the tangent space $\mathcal{T}_v X_{\Sigma,\Lambda} \cong \mathbb{C}^n$.

On the other hand, the weights of the tangential representation of the compact torus T^n at the fixed point v form a lattice basis dual to $(\Lambda(\rho_{i_1}),\dots,\Lambda(\rho_{i_n}))$. Therefore, the sign $(-1)^v = \det(\Lambda(\rho_{i_1}),\dots,\Lambda(\rho_{i_n})) = \operatorname{sign}(I)$, which finishes the proof. □

To obtain linear relations we need to analyze further the construction of generalized quasitoric manifolds. There are coordinate line bundles. Let $L_1,\dots,L_m$ be natural $(S^1)^m$-equivariant line bundles on $\mathcal{Z}_K$, for an integer vector $\mathbf{k} = (k_1,\dots,k_m) \in \mathbb{Z}^m$ the tensor product

$$L_{\mathbf{k}} = L_1^{k_1}\otimes\dots\otimes L_m^{k_m}$$

descends to a complex line bundle $\widetilde{L}_{\mathbf{k}}$ on $X_{\Sigma,\Lambda}$. Moreover, if $\mathbf{k}\in\mathbb{Z}^M$ is such that the corresponding character acts trivially on $H_\Lambda \subset (S^1)^m$, the descendant bundle $\widetilde{L}_{\mathbf{k}}$ is topologically trivial. It is easy to see that there is a smooth section of $\widetilde{L}_{\mathbf{k}}$ with the degenerate locus given by $\sum_{i=1}^m k_i[D_i]$. By exactness of the sequence

$$0 \to M \xrightarrow{\Lambda^*} \mathbb{Z}^m \to M_{H_\Lambda} \to 0,$$

the characters $\mathbf{k}$ acting trivially on H_Λ are identified with the character lattice M of T with $k_i = \chi(v_i)$ for $\chi\in M$ and $v_i = \Lambda(\rho_i)$. Thus we obtain the following proposition.

Proposition 6.5. *For any character $\chi\in M$, the following linear relation in $H^2(X_{\Sigma,\Lambda})$ holds:*

$$\sum_{i=1}^m \chi(v_i)[D_i] = 0,$$

where $v_i := \Lambda(\rho_i)$ for $1\le i\le m$.

Proof. Indeed, the descendant line bundle $\widetilde{L}_{\chi(v_1),\dots,\chi(v_m)}$ is trivial and hence

$$c_1(\widetilde{L}_{\chi(v_1),\dots,\chi(v_m)}) = \sum_{i=1}^m \chi(v_i)[D_i] = 0.$$ □

6.3. **BKK theorem and Pukhlikov-Khovanskii description.** In particular, Theorem 6.10 means that to obtain a description of the cohomology ring of quasitoric manifold it is enough to compute the self-intersection polynomial

$$h_1[D_1]+\dots+h_m[D_m] \mapsto \langle (h_1[D_1]+\dots+h_m[D_m])^m, [X_{\Sigma,\Lambda}]\rangle$$

on the space of combination of classes of codimemsion 2 characteristic submanifolds. This is the subject of the following theorem.

Theorem 6.6. *Let $X_{\Sigma,\Lambda}$ be a generalized quasitoric manifold with codimension 2 characteristic submanifolds $D_1,\dots,D_m$. Then the following identity holds*

$$\langle (h_1[D_1]+\dots+h_m[D_m])^m, [X_{\Sigma,\Lambda}]\rangle = n!Vol(f_h),$$

where $f_h\in\mathcal{P}_{\Sigma,\Lambda}$ is a generalized virtual polytope given by parameters $h=(h_1,\dots,h_m)$.

Proof. Let us identify the space of linear combinations $h_1[D_1]+\dots+h_m[D_m]$ with the space of generalized virtual polytopes $\mathcal{P}_{\Sigma,\Lambda}$. Under this identifications both self-intersection and volume functions are homogeneous polynomials of degree n on $\mathcal{P}_{\Sigma,\Lambda}$. Let us denote them by $S:\mathcal{P}_{\Sigma,\Lambda}\to\mathbb{R}$ and $Vol:\mathcal{P}_{\Sigma,\Lambda}\to\mathbb{R}$ respectively.

To show equality $S(h) = n!Vol(h)$ it is enough to prove equality of any partial derivatives of S and Vol of degree n:

$$\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}S(h) = n!\cdot\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}Vol(h),$$

where $\partial_{i_j} = \partial/\partial h_{i_j}$ and $\sum_{j=1}^s k_{i_j} = n$.

Let us call the number $\sum_{i=1}^s(k_i-1)$ the *multiplicity of the monomial* $\partial_{i_1}^{k_1}\dots\partial_{i_s}^{k_s}$. In particular, a monomial has multiplicity 0 if and only if it is square free. We will prove the equality by induction in multiplicity of differential monomial. For square free monomials, the equality follows from the first part of Corollary 5.10 and Proposition 6.4. Indeed, for by Corollary 5.10 we have if $r = n$ and $\rho_{i_1}, \dots, \rho_{i_n}$ span a simplex in Δ dual to the vertex $A \in M_{\mathbb{R}}$, we have

$$\partial_{i_1} \dots \partial_{i_n} \mathrm{Vol}(h) = \begin{cases} \mathrm{sign}(i_1, \dots, i_n), & \text{if } \rho_{i_1}, \dots, \rho_{i_n} \text{ span a cone in } \Sigma; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand $\partial_{i_1} \dots \partial_{i_n} S(h)$ is equal to the coefficient in front of $t_{i_1} \dots, t_{i_n}$ in the polynomial $S(h + (t_1 \dots, t_m)$. We get

$$S(h + (t_1 \dots, t_n) = \langle ((h_1 + t_1)[D_1] + \dots + (h_m + t_m)[D_m])^m, [X_{\Sigma,\Lambda}] \rangle = t_{i_1} \dots t_{i_n} \cdot n! \cdot \langle D_{i_1} \dots D_{i_n}, [X_{\Sigma,\Lambda}] \rangle + \dots$$

Hence by Proposition 6.4 we get

$$\partial_{i_1} \dots \partial_{i_n} S(h) = \begin{cases} n! \cdot \mathrm{sign}(i_1, \dots, i_n), & \text{if } \rho_{i_1}, \dots, \rho_{i_n} \text{ span a cone in } \Sigma; \\ 0, & \text{otherwise.} \end{cases}$$

Now, let us assume that the equality of partial derivatives is true for all differential monomials of multiplicity $r - 1$. Let $\partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s}$ be differential monomial of multiplicity r with $k_1 \geq 1$. We can assume that $\rho_{i_1}, \dots, \rho_{i_k}$ span a cone in Σ since otherwise

$$\partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s} S(h) = n! \cdot \partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s} Vol(h) = 0.$$

In that case there is a character $\chi \in M$ such that

$$\langle \chi, \Lambda(\rho_{i_1}) \rangle = 1,\ \langle \chi, \Lambda(\rho_{i_2}) \rangle = 0,\ \dots,\ \langle \chi, \Lambda(\rho_{i_k}) \rangle = 0.$$

Therefore, since the volume is invariant under the translation of generalized virtual polytope, we get

$$\partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s} Vol(h) = -\sum_{l \neq i_j} \langle \chi, \Lambda(\rho_l) \rangle \partial_l \partial_{i_1}^{k_1 - 1} \dots \partial_{i_s}^{k_s} Vol(h)$$

and similarly by Proposition 6.5:

$$\partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s} S(h) = -\sum_{l \neq i_j} \langle \chi, \Lambda(\rho_l) \rangle \partial_l \partial_{i_1}^{k_1 - 1} \dots \partial_{i_s}^{k_s} S(h).$$

Moreover, the differential monomials on the right hand side of the expressions above have multiplicity less then r, so the equality

$$\partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s} S(h) = n! \cdot \partial_{i_1}^{k_1} \dots \partial_{i_s}^{k_s} Vol(h)$$

follows from the induction hypothesis. □

We will finish this subsection with the a different interpretation of Theorem 6.6. Let us first recall the classical interpretation of the BKK theorem for toric variety. The Newton polyhedron $\Delta(f) \subset \mathbb{R}^n$ of a Laurent polynomial $f = \sum a_i x^{k_i}$ is the convex hull of vectors k_i with $a_i \neq 0$. For a fixed polytope Δ let E_Δ be a finite dimensional vector space of Laurent polynomials f such that $\Delta(f) \subset \Delta$. BKK theorem computes the number of solutions of a system $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ of generic Laurent polynomials with fixed Newton polytopes $\Delta_1, \dots, \Delta_n$.

Theorem 6.7 (BKK Theorem)**.** *Let $f_1, \dots, f_n$ be generic Laurent polynomials with $\Delta(f_i) \subset \Delta_i$. Then all solutions of the system $f_1 = \dots = f_n = 0$ in $(\mathbb{C}^*)^n$ are non-degenerate and the number solutions is*

$$n! Vol(\Delta_1, \dots, \Delta_n),$$

where Vol is the mixed volume.

One can reformulate Theorem 6.6 in a similar way. Let $\Delta_1, \dots, \Delta_n$ be generalized virtual polytopes in $\mathcal{P}_{\Sigma,\Lambda}$ associated to a generalized quasitoric manifold $X_{\Sigma,\Lambda}$. Let L_{Δ_i} be a line bundle associated to the generalized virtual polytope Δ_i and let $E_\Delta = \Gamma(X_{\Sigma,\Lambda}, L_{\Delta_i})$ be the space of smooth sections of L_{Δ_i}. Then Theorem 6.6 can be reformulated in the following way.

Theorem 6.8. *Let $s_1, \dots, s_n$ be generic Laurent polynomials with $s_i \in E_{\Delta_i}$. Then all solutions of the system $s_1 = \dots = s_n = 0$ in $X_{\Sigma,\Lambda}$ are non-degenerate and the number solutions counted with signs is*

$$n! Vol(\Delta_1, \dots, \Delta_n),$$

where Vol is the mixed volume of generalized virtual polytopes.

Remark 6.9. Note that in algebraic case, the multiplicity of each non-degenerate root is 1, however in the case of smooth sections $s_i \in \Gamma(X_{\Sigma,\Lambda}, L_{\Delta_i})$ the multiplicity of a non-degenerate root might be -1. Nevertheless, the number solutions counted with signs still can be computed as mixed volume.

6.4. **Cohomology ring of a generalized quasitoric manifold.** In this subsection we use the approach introduced by Pukhlikov and the first author for the computation of cohomology rings. The central ingredient for such a description is an exact computation of Macalaey inverse systems for graded algebras with Poincaré duality generated in degree 1.

We will call a graded, commutative algebra $A = \bigoplus_{i=0}^{n} A_i$ over a field $\mathbb{K}$ of characteristic 0 a *Poincaré duality algebra* if

- $A_0 \simeq A_n \simeq \mathbb{K}$;
- the bilinear map $A_i \times A_{n-i} \to A_n$ is non-degenerate for any $i = 0, \dots, n$ (Poincaré duality).

The main example of Poincaré duality algebras comes from the following example. Let X be a smooth manifold of dimension $2n$. Then the algebra of even degree cohomology classes $A = \bigoplus_{i=0}^{n} H^{2i}(X)$ is a Poincaré duality algebra. In particular, since $H^{2i+1}(X_{\Sigma,\Lambda}) = 0$ for a generalized quasitoric manifold, $H^*(X_{\Sigma,\Lambda})$ is also a Poincaré duality algebra. The following theorem gives a description Poincaré duality algebras.

Theorem 6.10. *Let A be a Poincaré duality algebra which is generated (as an algebra) by the elements $A_1 = \mathbb{K}\langle v_1, \dots, v_r\rangle$ of degree one. Then*

$$A \simeq \mathbb{K}[t_1, \dots, t_r]/\{p(t_1, \dots, t_r) \in \mathbb{K}[t_1, \dots, t_r] \colon p(\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_r}) f(x_1, \dots, x_r) = 0\}$$

where we identify A_1 with $\mathbb{K}^r$ via a basis $v_1, \dots, v_r$ and define $f \colon A_1 \simeq \mathbb{K}^r \to \mathbb{K}$ as the polynomial given by $f(x_1, \dots, x_r) = (x_1 v_1 + \ldots + x_r v_r)^n \in A_n \simeq k$.

Theorem 6.10 was used in [PK92a] to give a description of cohomology ring of a smooth projective toric variety. Later it was used in [Kav11] to provide a description of cohomology ring of full flag varieties G/B. A more general version of Theorem 6.10 was recently obtained in [KM21] and used in [HKM20, KLM21] to give a description of cohomology ring of toric and quasitoric bundles.

Theorem 6.10 accepts a coordinate free reformulation. Indeed, the ring $\mathbb{K}[t_1, \dots, t_r]$ in Theorem 6.10 can be identified with the ring of differential operators with constant coefficients $\mathrm{Diff}(A_1)$ on A_1. Hence the description of algebra A becomes

$$A \simeq \mathrm{Diff}(A_1)/\mathrm{Ann}(f),$$

where $\mathrm{Ann}(f) = \{D \in \mathrm{Diff}(A_1) \,|\, D \cdot f = 0$ is the annihilator ideal of f.

Theorem 6.11. *Let $X_{\Sigma,\Lambda}$ be a generalized quasitoric manifold and let $\mathcal{P}_{\Sigma,\Lambda}$ be the space of generalized virtual polytopes associated to it. Then the cohomology ring $H^*(X_{\Sigma,\Lambda})$ can be computed as*

$$H^*(X_{\Sigma,\Lambda}) = \mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda})/\mathrm{Ann}(\mathrm{Vol}),$$

where $\mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda})$ is the ring of differential operators with constant coefficients on $\mathcal{P}_{\Sigma,\Lambda}$ and $\mathrm{Ann}(\mathrm{Vol})$ is the annihilator ideal of the volume polynomial.

Proof. By Theorem 6.2 the cohomology ring $H^*(X_{\Sigma,\Lambda})$ is generated by the classes of codimension 2 characteristic submanifolds of $X_{\Sigma,\Lambda}$. Hence there is a surjection $\mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda}) \to H^*(X_{\Sigma,\Lambda})$ with a kernel given by Theorem 6.10 as the annihilator ideal of self-intersection polynomial $S(h)$ of classes of codimension 2 characteristic submanifolds. However by Theorem 6.6 $S(h) = n!\mathrm{Vol}(h)$ and hence:

$$H^*(X_{\Sigma,\Lambda}) = \mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda})/\mathrm{Ann}(S) = \mathrm{Diff}(\mathcal{P}_{\Sigma,\Lambda})/\mathrm{Ann}(\mathrm{Vol}).$$

□

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