Overdetermined Systems of Equations on Toric, Spherical, and Other Algebraic Varieties

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Abstract

Let $E_1, \ldots, E_k$ be a collection of linear series on an algebraic variety X over $\mathbb{C}$. That is, $E_i \subset H^0(X, \mathcal{L}_i)$ is a finite dimensional subspace of the space of regular sections of line bundles $\mathcal{L}_i$. Such a collection is called overdetermined if the generic system

$$s_1 = \ldots = s_k = 0,$$

with $s_i \in E_i$ does not have any roots on X. In this paper we study solvable systems which are given by an overdetermined collection of linear series. Generalizing the notion of a resultant hypersurface we define a consistency variety $R \subset \prod_{i=1}^k E_i$ as the closure of the set of all systems which have at least one common root and study general properties of zero sets $Z_{\mathbf{s}}$ of a generic consistent system $\mathbf{s} \in R$. Then, in the case of equivariant linear series on spherical homogeneous spaces we provide a strategy for computing discrete invariants of such generic non-empty set $Z_{\mathbf{s}}$. For equivariant linear series on the torus $(\mathbb{C}^*)^n$ this strategy provides explicit calculations and generalizes the theory of Newton polyhedra.

Contents

1 Introduction

Let X be an irreducible algebraic variety over $\mathbb{C}$ and let $\mathcal{E} = (E_1, \dots, E_k)$ be a collection of base-point free linear series on X. That is, $E_i \subset H^0(X, \mathcal{L}_i)$ is a finite dimensional subspace of the space of regular sections of globally generated line bundles $\mathcal{L}_i$, such that there are no points $x \in X$ with $s(x) = 0$ for any $s \in E_i$.

A collection of linear series $\mathcal{E}$ defines systems of equations on X of the form

$$s_1 = \cdots = s_k = 0, \tag{1}$$

where $s_i \in E_i$. A collection $\mathcal{E}$ is called *overdetermined* if system (1) does not have any roots on X for the generic choice of $\mathbf{s} = (s_1, \dots, s_k) \in \mathbf{E} = E_1 \times \dots \times E_k$. If generic system (1) has a solution we will say that $\mathcal{E}$ is generically solvable. Here, and everywhere in this paper, by saying that some property is satisfied by a generic point of an irreducible algebraic variety Y we mean that there exists a Zariski closed subset $Z \subset Y$ such that for any $y \in Y \setminus Z$ this property is satisfied.

In this paper we study overdetermined collections of linear series and their generic non-empty zero sets. For a collection $\mathcal{E} = (E_1, \dots, E_k)$ we define the consistency variety $R_{\mathcal{E}} \subset \mathbf{E}$ to be the closure of the set of all systems $\mathbf{s}$ which have at least one common root on X. A consistency variety $R_{\mathcal{E}}$ is an irreducible variety which is a generalization of the resultant hypersurface. We study $R_{\mathcal{E}}$ in Section 2.

One of the main goals of this paper is to find discrete invariants of the zero set of a generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$. This goal is motivated by the study of families of complete intersections. A motivating example for us is a system of equations defining the singular locus of an algebraic hypersurface. Let H be a hypersurface defined by a polynomial f on $\mathbb{C}^n$. The singular locus of H is given by the conditions $f = df = 0$ which could be viewed as $n+1$ algebraic equations in n variables. Since this system is overdetermined , the generic hypersurface is smooth. However, for a generic 1-parameter family of hypersurfaces H_t defined by polynomials f_t one would see singular hypersurfaces. In this setting, questions about the topology of the singular loci appearing in the generic families are translated to questions about the topology of generic non-empty zero sets of overdetermined linear systems.

Previous results. There are lots of results on the zero sets of a system of equations defined by the generic member of a linear series. First such results are provided by the theory of Newton polyhedra which work with equivariant linear series on $(\mathbb{C}^*)^n$. With a Laurent polynomial $f = \sum a_i x^{k_i}$ one can associate its Newton polyhedron $\Delta(f) \subset \mathbb{R}^n$, which is the convex hull of vectors k_i with $a_i \neq 0$. For a fixed polytope Δ let E_Δ be a finite dimensional vector space of Laurent polynomials f such that $\Delta(f) = \Delta$. Newton polyhedra theory allows one to find discrete invariants of the zero set $Z_{\mathbf{s}}$ of the system (1) where s_i is a generic member of E_{Δ_i} in terms of combinatorics of polytopes $\Delta_1, \dots, \Delta_k$. An example of such a result is the celebrated Bernstein-Kouchnirenko-Khovanskii Theorem (see [B75]).

Theorem (BKK Theorem)**.** *Let $s_1, \dots, s_n$ be generic Laurent polynomials with $\Delta(s_i) = \Delta_i$. Then all solutions of the system $s_1 = \dots = s_n = 0$ in $(\mathbb{C}^*)^n$ are non-degenerate and the number solutions is*

$$n! Vol(\Delta_1, \dots, \Delta_n),$$

where Vol is the mixed volume.

For more examples of results of Newton polyhedra theory see [Kho78, Kho88, DKho87].

Newton polyhedra theory has generalizations to other classes of algebraic varieties such as spherical homogeneous spaces G/H with a collection of G-invariant linear systems. The first result in this direction was a generalization of the BKK Theorem and was obtained by Brion and Kazarnovskii in [Bri89, Kaz87]. For more results see for example [Kir06, Kir07, KKho16]. The role of the Newton polytope in these results is played by the Newton-Okounkov polytope, which is a polytope fibered over the moment polytope with string polytopes as fibers.

Even more generally, in [KKho12] and [LM09] Newton polyhedra theory was generalized to the theory of Newton-Okounkov bodies. For a linear series E on an irreducible algebraic variety X one can associate a convex body $\Delta(E)$ called the Newton-Okounkov body in such a way that the number of roots of a generic system

$$s_1 = \ldots = s_n = 0$$

with $s_i \in E_i$ can be expressed in terms of volumes of Newton-Okounkov bodies $\Delta(E_i)$.

All these results work for a generic system. That is, as before, in the space of systems $\mathbf{E} = E_1 \times \ldots \times E_k$ there exists a Zariski closed subset Z such that for any system $\mathbf{s} \in \mathbf{E} \setminus \mathbf{Z}$ discrete invariants of the zero set $Z_{\mathbf{s}}$ are the same and can be computed combinatorially. In particular, for overdetermined systems all the answers provided by these results are trivial.

Studying a generic solvable system given by an overdetermined collection of linear series is a particular case of studying of non-generic systems. The study of non-generic systems is usually hard. Even in the case of the BKK Theorem such results are quite technical and recent ([Pin]). However, if a collection of linear systems is overdetermined , there are lots of cases where questions about the topology of generic non-empty zero sets could be answered rather explicitly in terms of combinatorics.

Structure of the paper and formulation of the results. In Section 2, for an overdetermined collection of linear series, we define the consistency variety $R_{\mathcal{E}} \subset \mathbf{E}$ which is the closure of the set of all solvable systems and prove that it is irreducible. Basic geometric properties of $R_{\mathcal{E}}$ are studied in Theorems 2.4 and 2.5. In Subsection 2.4 we consider the case when $codim(R_{\mathcal{E}}) = 1$ and define the resultant polynomial of a collection $\mathcal{E}$. The resultant of a collection of linear series is a generalization of the $\mathcal{L}$-resultant defined in [GKZ08]. We prove that all the basic properties of $\mathcal{L}$-resultant are also satisfied by the resultant of a collection of linear series.

Section 3 is devoted to studying the generic non-empty zero set of an overdetermined collection of linear systems $\mathcal{E}$ on an irreducible variety X. One of the main results of this section is Theorem 3.1 which expresses a generic non-empty zero set of system (1), defined by $\mathcal{E}$, as the generic zero set of another linear series which is generically solvable. This allows one to use classical results described in the previous subsection to find the topology of the generic non-empty zero set in number of examples.

In Section 4 we study G-equivariant linear series on homogeneous G-spaces. Spherical homogeneous spaces are of special interest for us. We apply Theorem 3.1 to obtain Theorem 4.2 which provides a strategy for computing discrete invariants of a generic non-empty zero set of an overdetermined linear series on a spherical homogeneous space. An example of an application of Theorem 4.2 is given in Subsection 4.3.

In Section 5 we study overdetermined linear systems on the torus $(\mathbb{C}^*)^n$. In this case the strategy provided by Theorem 4.2 can be made absolutely explicit. In particular, one can explicitly express invariants of generic non-empty zero sets in terms of combinatorics of Newton polyhedra. Theorem 5.2 is an explicit version of Theorem 4.2 in this case. Theorem 5.3, which generalizes BKK-theorem, is an example of how one can use Theorem 5.2 together with results of Newton polyhedra theory to compute discrete invariants of generic non-empty zero sets. The results of Section 5 appeared previously in [Mon17] and are included in this paper for completeness of exposition.

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2 Consistency variety and resultant of a collection of linear series on a variety

In this section we define the consistency variety of a collection of linear series and describe its main properties.

2.1 Background

Let X be an irreducible complex algebraic variety. Let $\mathcal{L}_1, \ldots \mathcal{L}_k$ be globally generated line bundles on X. For $i = 1, \ldots, k$, let $E_k \subset H^0(X, \mathcal{L}_i)$ be finite-dimensional, base points free linear series. Let $\mathbf{E}$ denote the k-fold product $E_1 \times \cdots \times E_k$.

Definition 1. The *incidence variety* $\widetilde{R}_{\mathbf{E}} \subset X \times \mathbf{E}$ is defined as:

$$\widetilde{R}_{\mathbf{E}} = \{(p, (s_1, \ldots, s_k)) \in X \times \mathbf{E} \mid f_1(p) = \ldots = f_k(p) = 0\}.$$

Let $\pi_1 : X \times \mathbf{E} \to X$, $\pi_2 : X \times \mathbf{E} \to \mathbf{E}$ be natural projections to the first and the second factors of the product.

Definition 2. The *consistency variety* $R_{\mathbf{E}} \subset \Omega_A$ is the closure of the image of $\widetilde{R}_L$ under the projection π_2.

Theorem 2.1. *The incidence variety $\widetilde{R}_{\mathbf{E}} \subset X \times L$ and the consistency variety $R_{\mathbf{E}}$ are irreducible algebraic varieties.*

Proof. Since $E_1, \ldots E_k$ are base points free, the preimage $\pi_1^{-1}(p) \subset \widetilde{R}_{\mathbf{E}}$ of any point $p \in X$ is defined by k independent linear equations on elements of $\mathbf{E}$. Therefore, the projection π_1 restricted to $\widetilde{R}_{\mathbf{E}}$:

$$\pi_1 : \widetilde{R}_{\mathbf{E}} \to X$$

forms a vector bundle of rank $\dim(L) - k$ and in particular is irreducible.

The set of concistency systems $R_{\mathbf{E}} = \pi_2(\widetilde{R}_{\mathbf{E}})$ is the image of an irreducible algebraic variety $\widetilde{R}_{\mathbf{E}}$ under the algebraic map π_2, so it is irreducible constructible set. Hence it's closure is an irreducible algebraic variety. □

For two linear systems E_i, E_j, let their product E_iE_j be a vector subspace of $H^0(X, \mathcal{L}_i \otimes \mathcal{L}_j)$ generated by all the elements of the form $f \otimes g$ with $f \in E_i, g \in E_j$. For any $J \subset \{1, \ldots, k\}$, by E_J we will denote the product $\prod_{j \in J} E_j$.

To a base points free linear system E, one can associate a morphism $\Phi_E : X \to \mathbb{P}(E^*)$ called a *Kodaira map*. It is defined as follows: for a point $x \in X$ its image $\Phi_E(x) \in \mathbb{P}(E^*)$ is the hyperplane $E_x \in E$ consisting of all the sections $g \in E$ which vanish at x. We will denote by Y_E the image of the Kodaira map Φ_E and by τ_E the dimension of Y_E. For a collection of linear series $E_1, \ldots, E_k$ and $J \subset \{1, \ldots, k\}$ we will write Φ_J, Y_J and τ_J for Φ_{E_J}, Y_{E_J} and τ_{E_J} respectively.

Definition 3. For a collection of linear series $E_1 \ldots, E_k$, the defect of a subcollection $J \subset \{1, \ldots, k\}$ is defined as

$$\operatorname{def}(E_J) = \tau_J - |J|.$$

The following theorem of Kaveh and Khovanskii gives a condition on a collection of linear series to be generically solvable in terms of defects.

Theorem 2.2 ([KKho16] Theorems 2.14 and 2.19)**.** *The generic system of equations $s_1 = \ldots = s_k = 0$ with $f_i \in E_i$ is solvable if and only if $\operatorname{def}(E_J) \geq 0$ for any $J \subset \{1, \ldots, k\}$.*

In other words, Theorem 2.2 states that the codimension of the consistency variety is equal to 0 in $\mathbf{E}$ if and only if $\operatorname{def}(E_J) \geq 0$ for any $J \subset \{1, \ldots, k\}$. In Subsection 2.3 we will generalize this result by finding the codimension of R_E in terms of defects of subcollections.

2.2 The defect of vector subspaces and essential subcollections

In this subsection we will introduce a combinatorial version of the defect and relate it to the one given in Definition 3. Let k be any field, $\mathcal{V} = (V_1, \ldots, V_k)$ be a collection of vector subspaces of a vector space $W \cong k^n$ (in this paper we will always work with $k = \mathbb{C}$ or $\mathbb{R}$). For $J \subset \{1, \ldots, k\}$ let V_J be the Minkowski sum $\sum_{j \in J} V_j$ and $\pi_J : W \to W/V_J$ be the natural projection.

Definition 4. For a collection of vector subspaces $\mathcal{V} = (V_1, \ldots, V_k)$ of W we define
i) the defect of a subcollection $J \subset \{1, \ldots, k\}$ by $\operatorname{def}(J) = \dim(V_J) - |J|$;
ii) the minimal defect $\mathrm{d}(\mathcal{V})$ of a collection V to be the minimal defect of $J \subset \{1, \ldots, k\}$.
iii) an essential subcollection to be a subcollection J so that $\operatorname{def}(J) = d(\mathcal{V})$ and $\operatorname{def}(I) > \operatorname{def}(J)$ for any proper subset I of J.

The essential subcollections has proved to be useful in studying systems of equations which are generically inconsistent and their resultants. The above definition is related to the definition of an essential subcollection given in [St94]: two definitions coincide if $\mathrm{d}(\mathcal{V}) = -1$, but are different in general. The following theorem provides combinatorial tools to work with collection of vector subspaces.

Theorem 2.3 ([Mon17] Section 3). *Let $\mathcal{V} = (V_1, \dots, V_k)$ be a collection of vector subspaces of W with $\mathrm{d}(\mathcal{V}) \le 0$, then:*
i) an essential subcollection exists and is unique.
ii) if J is the unique essential subcollection of $\mathcal{V}$, then $\mathrm{d}(\pi_J(J^c)) = 0$.
iii) if J is the essential subcollection there exists a subcollection $I \subset J$ of size dim(V_J) with $\mathrm{d}(I) = 0$.

Parts *i*) and *ii*) of Theorem 2.3 are still true in the case $\mathrm{d}(\mathcal{V}) > 0$, the unique essential subcollection in this case is the empty subcollection. A subcollection I from part *iii*) of Theorem 2.3 is almost never unique.

To relate the combinatorial version of defect to the geometric version defined in Definition 3 we introduce a collection of distributions (and codistributions) on X related to a collection of linear systems $E_1, \dots, E_k$. Let X and $E_1, \dots, E_k$ be as before, denote further by X^{sing} the singular locus of X, by Y_J^{sing} the singular locus of $Y_J = \Phi_J(X)$.

Let also $\Sigma_J^c \subset X \setminus \left(X^{sing} \cup \Phi_J^{-1}(Y_J^{sing})\right)$ be the set of all critical points of Φ_J. Finally, let $B_J = X^{sing} \cup \Phi_J^{-1}(Y_J^{sing}) \cup \Sigma_J^c$ and let $U \subset X$ be a Zariski open subset defined by:

$$U = X \setminus \bigcup_J B_J.$$

So we get that U is a smooth algebraic variety and the restriction of Φ_J to U is a regular map for any $J \subset \{1, \dots, k\}$.

Definition 5. Let $a \in U$ and $\widetilde{F}_J(a)$ be the subspace of the tangent space T_aU defined by the linear equations $dg_a = 0$ for all $g \in E_J$. Let also $\widetilde{F}_J^\vee(a) \subset T_a^*U$ be the annihilator of $\widetilde{F}_J(a)$. Then
(1) $\widetilde{F}_J$ is an $(n - \tau_J)$-dimensional distribution on the Zariski open set $U \subset X$ defined by the collection of subspaces $\widetilde{F}_J(a)$.
(2) $\widetilde{F}_J^\vee$ is a τ_J-dimensional codistribution on the Zariski open set $U \subset X$ defined by the collection of subspaces $\widetilde{F}_J^\vee(a)$.

The next lemma is a corollary of the Implicit Function Theorem.

Lemma 2.1. *The foliation $\widetilde{F}_J$ on U is completely integrable. Its leaves are connected components of the fibers the Kodaira map $\Phi_J : U \to Y_J$.*

Fibers of the Kodaira maps Φ_J can be described in terms of systems of equations defined by collection of lynear series $E_1, \dots, E_k$.

Lemma 2.2 ([KKho16] Lemma 2.11). *For $a, b \in X$ we have $\Phi_J(a) = \Phi_J(b)$ if and only if for every $i \in J$ the sets $\{g_i \in E_i | g_i(a) = 0\}$ and $\{g_i \in E_i | g_i(b) = 0\}$ coincide.*

Corollary 2.1. *Let $U \subset X$ be as before, then for any $a \in U$ one has:*

$$F_{E_J}(a) = \bigcap_{i \in J} F_i(a) \quad F_{E_J}^\vee(a) = \sum_{i \in J} F_i^\vee(a)$$

Proof. Indeed, by Lemma 2.1 $F_{E_J}(a)$ is a tangent space to the fiber of Φ_J at a point $a \in U \setminus \Sigma_c$. But by Lemma 2.2 the fiber of Φ_J through the point a is equal to the intersection of fibres of Φ_{E_i}, with $i \in J$ passing through the same point. □

The following Proposition 2.1 relates two definitions of the defect and is the main result of this subsection. Proposition 2.1 will allow us to apply combinatorial results of Theorem 2.3 to the geometric version of defect.

Proposition 2.1. *Let $U \subset X$ be as before, then the defect* $\mathrm{def}(E_J)$ *of linear system E_J (as in Definition 3) is equal to the defect of a collection of vector subspaces $(F^\vee_{E_i}(a))_{i\in J}$ of T^*U_a (as in Definition 4) for any $a \in U$.*

Proof. By construction, $F^\vee_{E_J}$ is τ_J-dimensional codistribution on U, so $\dim(F^\vee_{E_J}(a)) = \tau_J$ for any $a \in U$. Therefore, by Corollary 2.1 we have

$$\begin{aligned}\mathrm{def}(E_J) = \tau_J - |J| &= \dim(F^\vee_{E_J}(a)) - |J| = \dim(F^\vee_{E_J}(a)) - |J| \\ &= \dim\left(\sum_{i\in J} F^\vee_i(a)\right) - |J| = \mathrm{def}(F^\vee_{E_i}(a))_{i\in J}.\end{aligned}$$

□

2.3 Properties of the consistency variety

In this subsection we will investigate basic properties of the consistency variety. One of the main results of this section is the following theorem which computes the codimension of the consistency variety in terms of defects.

Theorem 2.4. *Let $\mathcal{E} = (E_1, \ldots E_k)$ be a collection of base points free linear systems on a quasi projective irreducible variety X. Then the codimension of the consistency variety $R_{\mathbf{E}}$ is equal to $-\mathrm{d}(\mathcal{E})$ where $\mathrm{d}(\mathcal{E})$ is the minimal possible defect* $\mathrm{def}(E_J)$ *for $J \subset \{1, \ldots, k\}$.*

We say that a collection of linear series $\mathcal{E} = (E_1, \ldots, E_k)$ on X is *injective* if linear series from $\mathcal{E}$ separate points of X. In other words, $\mathcal{E}$ is injective if the product of Kodaira maps $\prod_{i=1}^k \Phi_{E_i}$ is injective on X. Note that by Lemma 2.2 the product of Kodaira maps $\prod_{i=1}^k \Phi_{E_i}$ is injective if and only if the Kodaira map Φ_E for $E = \prod_{i=1}^k E_i$ is such. Therefore, equivalently $\mathcal{E}$ is injective if the Kodaira map Φ_E is injective.

Any collection of linear series $\mathcal{E}$ on X could be reduced to an injective collection $\widetilde{\mathcal{E}}$ such that zero sets of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are related in an easy way. In order to do so, let us describe the zero set $Z_{\mathbf{s}}$ of a system of equations $s_1 = \ldots = s_k = 0$ with $s_i \in E_i$ in terms of Kodaira maps Φ_{E_i}.

For the product of projective spaces $\mathbb{P}_{\mathbf{E}} = \mathbb{P}(E_1^*) \times \ldots \times \mathbb{P}(E_k^*)$ let $p_i : \mathbb{P}_{\mathbf{E}} \to \mathbb{P}(E_i^*)$ be the natural projection on the i-th factor. Each function $s_i \in E_i$ defines a hyperplane H_{s_i} on $\mathbb{P}(E_i^*)$, with slight abuse of notation let us denote its preimage under p_i by the same letter. Let $\Phi_{\mathbf{E}} : X \to \mathbb{P}_{\mathbf{E}}$ be the product of Kodaira maps and $Y_{\mathbf{E}} = \Phi_{\mathbf{E}}(X)$ be its image. In this notation the zero set $Z_{\mathbf{s}}$ is given by

$$Z_{\mathbf{s}} = \Phi_{\mathbf{E}}^{-1}\left(Y_{\mathbf{E}} \cap \bigcap_{i=1}^k H_{s_i}\right).$$

Therefore for any collection of linear series $\mathcal{E} = (E_1, \ldots, E_k)$ on X one can associate an injective collection $\widetilde{\mathcal{E}}$ on $Y_{\mathbf{E}}$, where $\widetilde{\mathcal{E}}$ is the restriction of $E_1, \ldots, E_k$ to $Y_{\mathbf{E}}$ such that

$$Z_{\mathbf{s}} = \Phi_{\mathbf{E}}^{-1}(Z_{\widetilde{\mathbf{s}}}).$$

Proposition 2.2. *Let X be an irreducible algebraic variety over $\mathbb{C}$ and $\mathcal{E} = (E_1, \ldots, E_k)$ be an overdetermined collection of linear systems on X, let J be the essential subcollection of $\mathcal{E}$. Let also X_a be a fiber of Φ_J passing through a point $a \in X$. Then for generic $a \in X$, the restriction of the collection $\mathcal{E}_{J^c} = (E_i)_{i\notin J}$ on X_a is generically solvable.*

Proof. Here one can take any $a \in U$, with $U = X \setminus \bigcup_J B_J$ as before. For $J = \{i_1, \ldots, i_s\}$ and any point $a \in U$ the defect $\mathrm{def}(J)$ can be computed as

$$\mathrm{def}(J) = \mathrm{def}(\widetilde{F}^\vee_{i_1}, \ldots, \widetilde{F}^\vee_{i_s})_a.$$

So it is enough to show that the minimal defect of the restriction of $\mathbf{E}_{J^c}$ on X_a is nonnegative. But the codistribution $\widetilde{F}^\vee$ of the restriction of $\mathbf{E}_{J^c}$ on X_a is given by $\pi_J(F_{J^c}^\vee)$, where

$$\pi_J : T^*U \to T^*(X_a \cap U) \cong T^*U/F_J^\vee,$$

is the natural projection. By the second part of Theorem 2.3 we know that $d(\pi_J(J^c)) = 0$. Therefore by Theorem 2.2 the collection $\mathbf{E}_{J^c}$ on X_a is generically solvable. □

Proof of Theorem 2.4. Without loss of generality we can assume that $\mathcal{E}$ is injective. Indeed, $\widetilde{\mathcal{E}}$ is solvable in codimension r if and only if $\mathcal{E}$ is solvable in codimension r.

By Proposition 2.2 we can assume that the essential subcollection of $E_1, \ldots, E_k$ is the collection itself. We will call such collections of linear series *essential*. For an essential collection, by Theorem 2.3 there exists a solvable subcollection of size $\tau_{\mathbf{E}}$, assume it is equal to $J = \{1, \ldots, r\}$. Then the dimension $\tau_{\mathbf{E}}$ is equal to the dimension τ_J. Therefore the generic solution linear conditions from J on $\Phi_J(X)$ is a union of finitely many points.

Since linear systems E_i's are base points free, the condition on the sections from $E_{r+1}, \ldots, E_k$ to vanish at any of these points is union of $k - r = -\operatorname{def}(\mathbf{E})$ clearly independent linear conditions, which finishes the proof of the theorem in this case. □

We finish this section with another corollary of Proposition 2.2 which reduces the study of resultant subvarieties to the study of resultant subvarieties of essential collections of linear systems.

Theorem 2.5. *Let X be a complex irreducible quasi-projective algebraic variety and $\mathcal{E} = (E_1, \ldots, E_k)$ be overdetermined collection of linear systems on X with the essential subcollection J. Then the consistency variety $R_{\mathcal{E}}$ does not depend on E_i with $i \notin J$. In other words:*

$$R_{\mathcal{E}} = p^{-1}(R_{\mathcal{E}_J}), \text{ where } p : \mathbf{E} \to \mathbf{E}_J = \prod_{i \in J} E_i$$

is the natural projection.

Proof. By Proposition 2.2 there exists a Zariski open subset W of $R_{\mathcal{E}_J}$ such that for any $\mathbf{s} \in W$ the collection of linear systems J^c restricted to a zero set of $\mathbf{s}$ is generically solvable. Therefore if $V \subset R_{\mathcal{E}}$ is a set of solvable systems $\mathbf{s}$ such that $p(\mathbf{s}) \in U$ one has $\operatorname{codim}(V) = \operatorname{codim}(R_{\mathcal{E}}) = -d(\mathcal{E})$. Since $R_{\mathbf{E}}$ is an irreducible variety it coincides with the closure of V, which finishes the proof. □

2.4 Resultant of a collection of linear series on a variety

In this subsection we will define resultant of a collection of linear series and translate results of previous subsection to the language of resultants. The notion of resultant defined here is a generalization of $\mathcal{L}$-resultant defined in [GKZ08], and most of the results are analogous to the results on $\mathcal{L}$-resultants in [GKZ08].

Let $\mathcal{E} = (E_1, \ldots, E_{n+1})$ be a collection of linear systems on an irreducible variety X of dimension n. Assume also, that the codimension of the consistency variety $R_{\mathcal{E}}$ is equal 1.

Lemma 2.3. *Let $X, \mathcal{E}$ be as before, then there exists an Zariski open subset $U \subset R_{\mathcal{E}}$ so that for any $\mathbf{s} = (s_1, \ldots, s_{n+1}) \in U$, the zero set $Z_{\mathbf{s}}$ of the system $s_1 = \ldots = s_{n+1} = 0$ on X is finite. Moreover, the cordiality of $Z_{\mathbf{s}}$ is the same for any $\mathbf{s} \in U$.*

Proof. Let π_1, π_2 be restrictions of two natural projections from $X \times \mathbf{E}$ to $X, \mathbf{E}$ respectively to a incidence variety $\widetilde{R}_{\mathcal{E}}$. For a system $\mathbf{s} \in R_{\mathcal{E}}$ the zero set $Z_{\mathbf{s}}$ is given by $\pi_1(\pi_2^{-1}(\mathbf{s}))$, in particular if $\pi_2^{-1}(\mathbf{s})$ is finite of cordiality k such is $Z_{\mathbf{s}}$. Easy dimension counting shows that $\dim \widetilde{R}_{\mathcal{E}} = \dim R_{\mathcal{E}}$, so for the generic $\mathbf{s} \in R_{\mathcal{E}}$ the preimage $\pi_2^{-1}(\mathbf{s})$ is finite, with fixed cordiality. □

Note that the conditions that codim$R_{\mathcal{E}} = 1$ and generic non-empty zero set is finite forces the number of linear series to be $n+1$.

Definition 6. Let $X, \mathcal{E}$ be as before, then the resultant $Res_{\mathcal{E}}$ is a polynomial which defines the hypersurface $R_{\mathcal{E}}$ with multiplicity equal to the cordiality of the generic non-empty zero set $Z_{\mathbf{s}}$ [1]. Since R_E is irreducible such polynomial is well defined up to multiplicative constant. For $(s_1, \dots, s_{n+1}) \in \mathbf{E}$ by $Res_{\mathcal{E}}(s_1, \dots, s_{n+1})$ we will denote the value of resultant on the tuple $s_1, \dots, s_{n+1}$.

The next theorem is an immediate corollary of Theorems 2.4 and 2.5. This theorem was proved by Sturmfels in [St94] for equivariant linear systems on an algebraic torus. In that setting resultant of a collection of linear series on a variety is usually called *sparse resultant*.

Theorem 2.6. *The consistency variety $R_{\mathcal{E}}$ of a collection of linear systems $\mathcal{E} = (E_1, \dots, E_{n+1})$ has codimension 1 if and only if* $\mathrm{d}(E_1, \dots, E_{n+1}) = -1$. *Moreover, if J is essential subcollection of $\mathcal{E}$, then the resultant $Res_{\mathcal{E}}$ depends only on equations from E_i with $i \in J$.*

It was shown by Kaveh and Khovanskii (see for example [KKho12]) that for a collection $E_1, \dots, E_n$ of linear series on an irreducible variety X the number of roots of a system $s_1 = \dots = s_n = 0$ is constant for the generic $s_i \in E_i$. The generic number of roots of a system $s_1 = \dots = s_n = 0$ is called *the intersection index* of $E_1, \dots, E_n$ and is denoted by $[E_1, \dots, E_n]$.

Theorem 2.7. *The resultant $Res_{\mathcal{E}}$ is a quasihomogeneous polynomial with degree in the i-th entry equal to the intersection index $[E_1, \dots, \hat{E}_i, \dots, E_{n+1}]$. In particular, if J is essential subcollection of E and $i \notin J$ the degree in the i-th entry is 0.*

Proof. The resultant $Res_{\mathcal{E}}$ is a quasihomogeneous polynomial since system $s_1 = \dots = s_n = 0$ has a root on X, if and only if system $\lambda_1 s_1 = \dots = \lambda_n s_n = 0$ for any $\lambda_i \in \mathbb{C}^*$. To find the degree of $Res_{\mathcal{E}}$ in the i-th entry consider

$$Res_{\mathcal{E}}(s_1, \dots, s_i + \lambda s_i', \dots, s_{n+1})$$

as a polynomial of λ for the fixed generic choice of $s_1, \dots, s_i, s_i', \dots, s_{n+1}$. It is easy to see that the number of roots of $Res_{\mathcal{E}}(s_1, \dots, s_i + \lambda s_i', \dots, s_{n+1})$ counting with multiplicities is equal to the number of common roots of $s_1 = \dots = \hat{s}_i = \dots = s_{n+1} = 0$, so the degree of $Res_{\mathcal{E}}$ in the i-th entry is $[E_1, \dots, \hat{E}_i, \dots, E_{n+1}]$. □

3 Generic non-empty zero set and reduction theorem

In this section we first study generic non-empty zero sets. In particular, we show in Theorem 3.1 that a generic non-empty zero set given by an overdetermined collection of linear series can be also defined as a generic zero set of generically solvable collection. Then we define a notion of equivalence of two collections of linear systems. Informally speaking, two collections of linear systems are equivalent if they have the same generic nonempty zero sets. We show that every generically inconsistent collection of linear series is equivalent to a collection of minimal defect -1.

3.1 Zero sets of essential collection of linear systems

First, we will study *essential* collections of linear series i.e. collections $\mathcal{E} = (E_1, \dots, E_k)$ such that $\mathrm{def}(J) > \mathrm{d}(\mathcal{E})$ for any $J \subset \{1, \dots, k\}$.

Let $Y \subset \mathbb{P}_{\mathbf{E}} = \mathbb{P}(E_1^*) \times \dots \times \mathbb{P}(E_k^*)$ be an irreducible variety of dimension d. For a subset $J \subset \{1, \dots, k\}$ denote by $\mathbb{P}_J$ the product $\prod_{i \in J} \mathbb{P}(E_i)$ and by π_J the natural projection $\mathbb{P}(\mathbf{E}^*) \to \mathbb{P}_J$. With slight abuse of notation let us denote the restriction of this projection on Y also by π_J and by Y_J its image. Assume also, that $\mathcal{E} = (E_1, \dots, E_k)$ is an essential collection of linear systems on Y with a solvable subcollection J of size d which exists by Theorem 2.3 and Proposition 2.1.

[1] In some places the resultant is defined as unique up to constant irreducible polynomial defining $R_{\mathcal{E}}$, but the definition provided here seems more natural. See [D'AS15] for details.

Lemma 3.1. *In situation as above for the generic pair of points $x_1, x_2 \in Y_J$ the sets $F_{x_i} = \pi_{J^c}(\pi_J^{-1}(x_i))$, for $i = 1, 2$ are disjoint.*

Proof. The condition on sets F_{x_1} and F_{x_2} to be disjoint is open in the space of pairs, so since Y is irreducible it is enough to show that there exists at least one pair x_1, x_2 with $F_{x_1} \cap F_{x_2} = \varnothing$.

Assume otherwise, then for a given point $x_0 \in Y_J$ there exists an preimage $y_0 \in \pi_J^{-1}(x_0)$ and an open set $U \in Y_J$ such that for any $x \in U$ there exists $y \in \pi_J^{-1}(x)$ with $\pi_{J^c}(y) = \pi_{J^c}(y_0)$. So there exists a section $s : U \to Y$ of π_J defined by $s(x) = y$ with a property that $\pi_{J^c} \circ s$ is a constant map on U. But since π_J is a finite morphism, the image $s(U)$ is an Zariski open in Y, and, therefore π_{J^c} is constant on Y, which contradicts the essentiallity of $E_1, \ldots, E_k$. □

The main result of this subsection is the following proposition.

Proposition 3.1. *Let $E_1, \ldots, E_k$ be an essential collection of linear systems on a quasi-projective irreducible variety X. Then for the generic point $\mathbf{s} \in R_{\mathbf{E}}$, the zero set $Z_{\mathbf{s}}$ of a system given by $\mathbf{s}$ is a unique fiber of the Kodaira map Φ_E.*

Proof. As in subsection 2.3 we can assume that $\mathcal{E}$ is injective by replacing X with $Y_{\mathbf{E}} = \Phi_{\mathbf{E}}(X) \subset \mathbb{P}_{\mathbf{E}} = \prod_i E_i^*$. Note that the collection $\mathcal{E}$ restricted to $Y_{\mathbf{E}}$ is still essential.

Therefore, it is enough to show that for an irreducible variety $Y_{\mathbf{E}} \subset \mathbb{P}_{\mathbf{E}}$ so that $\mathcal{E}$ is an essential collection on $Y_{\mathbf{E}}$, and for the generic choice of $\mathbf{s} \in R_{\mathcal{E}}$ the intersection $Y \cap H_{s_1} \cap \ldots \cap H_{s_k}$ is a point.

Let $J \subset \{1, \ldots, k\}$ be solvable subcollection of size $\tau_{\mathbf{E}}$, J^c be its complement and let π_J, π_{J^c} be two natural projections restricted to $Y_{\mathbf{E}}$:

$$\begin{array}{ccc} Y_{\mathbf{E}} & \xrightarrow{\;p_{J^c}\;} & \mathbb{P}_{J^c} = \prod_{i \notin J} \mathbb{P}(E_i^*) \\ \Big\downarrow{\scriptstyle p_J} & & \\ \mathbb{P}_J = \prod_{i \in J} \mathbb{P}(E_i^*) & & \end{array}$$

The generic intersection $Y_J \cap (\bigcap_{i \in J} H_{s_i})$ is nonempty and finite, and hence of the same cardinality. It is enough to show that for for generic choice of s_i's with $i \in J$ there are no two points x, y in $Y_J \cap (\bigcap_{i \in J} H_{s_i})$ with $p_{J^c}(p_J^{-1}(x)) \cap p_{J^c}(p_J^{-1}(y)) \neq \varnothing$. Indeed, in such a case any two points in the finite intersection

$$Y_{\mathbf{E}} \cap \bigcap_{i \in J} H_{s_i} = \pi_J^{-1}\left(Y_J \cap \bigcap_{i \in J} H_{s_i}\right)$$

would be separated by generic hyperplanes H_{s_i}'s with $i \notin J$.

Let the cardinality of the generic intersection $Y_J \cap (\bigcap_{i \in J} H_{s_i})$ be equal to r, we will show that for the generic r-tuple of points $x_1, \ldots, x_r$ the sets $F_{x_i} = \pi_{J^c}(\pi_J^{-1}(x_i))$, for $i = 1, \ldots, r$ are mutually disjoint. Since this condition is open in the space of tuples $x_1, \ldots, x_r$ and Y_J is irreducible it is enough to show that there exist at least one tuple with such property.

Assume otherwise, that for any tuple $x_1, \ldots, x_r$ the sets F_{x_i}, for $i = 1, \ldots, r$ are not mutually disjoint. This is only possible if for any pair of point x_1, x_2 the sets F_{x_1}, F_{x_2} are not disjoint, but this contradicts Lemma 3.1 since $Y_{\mathbf{E}}$ satisfy its conditions. □

3.2 Generic non-empty zero set

In this subsection we study the generic non-empty zero set of a system of equations $s_1 = \ldots = s_k = 0$ with $s_i \in E_i$. First let us summarize results of the last two sections on the generic non-empty zero set $Z_{\mathbf{s}}$.

Theorem 3.1. *Let $\mathcal{E}$ be an overdetermined collection of linear series on an irreducible variety X, with the essential subcollection J. Then for the generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$, the zero set $Z_{\mathbf{s}}$ is the generic zero set of the collection $\mathcal{E}_{J^c} = (E_i)_{i \notin J}$ restricted to a fiber of a Kodaira map Φ_J.*

Proof. By Theorem 2.5 and Proposition 3.1 the zero set of a generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$ is the zero set of a generic system $\mathcal{E}_{J^c} = (E_i)_{i \notin J}$ restricted to a unique fiber of the Kodaira map Φ_J. Moreover, such a restriction is generically solvable by Proposition 2.2. □

Theorem 3.1 expresses the zero set of a system generic in the space of solvable systems defined by the collection $\mathcal{E}$ as the zero set of the system which is generic in the space of *all systems* defined by collection $\mathcal{E}_{J^c}$ restricted to a fiber of Φ_J. We will use this result in coming sections to reduce questions about topology of generic non-empty zero set to questions about topology of generic zero set.

Proposition 3.2. *Let $f : X \to Y$ be a morphism between two algebraic varieties with X - smooth. Then, the generic fiber is smooth.*

Proof. Since the condition is local in Y we can assume Y to be affine. By Noether normalization lemma there exists a finite morphism $g : Y \to \mathbb{C}^k$, with $k = \dim Y$. By Bertini theorem the generic fiber of the composition $g \circ f : X \to \mathbb{C}^k$ is smooth. But since the generic fiber of $g \circ f$ is a finite union of disjoint fibers of f, the generic fiber of f is also smooth. □

Theorem 3.2. *Let X be smooth algebraic variety and let $E_1, \dots, E_k$ be base points free linear systems on X. Then for generic solvable k-tuple $\mathbf{s} = (s_1, \dots, s_k) \in R_E$, the zero set $Z_{\mathbf{s}}$ is smooth. Moreover, the arithmetic genus of $Z_{\mathbf{s}}$ is constant for a generic choice of $\mathbf{s}$.*

Proof. Let π_1, π_2 be two natural projections from $X \times \mathbf{E}$ to $X, \mathbf{E}$ respectively. Denote by π_1, π_2 also their restrictions to a incidence variety $\widetilde{R}_E$. For a system $\mathbf{s} \in R_E$ the zero set $Z_{\mathbf{s}}$ is given by $\pi_1(\pi_2^{-1}(\mathbf{s}))$, in particular is isomorphic to $\pi_2^{-1}(\mathbf{s})$. But since $\widetilde{R}_E$ is smooth ($\widetilde{R}_E$ is a vector bundle over X), the fiber of π_2 over the generic point $\mathbf{s} \in R_E$ is smooth by Proposition 3.2, and therefore such is the generic zero set $Z_{\mathbf{s}}$.

For the second part notice that any algebraic morphism to an irreducible variety is flat over Zariski open subset. So for some Zariski open $U \subset R_E$ the projection $\pi_2^{-1}(U) \to U$ is a flat family. The statement then follows from a fact that arithmetic genus is constant in flat families. □

3.3 Reduction theorem

In this subsection we will formulate and prove Reduction theorem. First we define what does it mean for two collections of linear systems to be equivalent.

Definition 7. Two collections $E_1, \dots, E_k$ and $W_1, \dots, W_l$ of linear systems on a quasi-projective irreducible variety X are called equivalent if there exist Zariski open subsets $U \subset R_{\mathbf{E}}$ and $V \subset R_{\mathbf{W}}$ such that for any $u \in U$ $(v \in V)$ there exists $v \in V$ $(u \in U)$ such that the zero sets X_u and X_v coincide.

Theorem 3.3 (Reduction theorem)**.** *Any collection $\mathcal{E} = (E_1, \dots, E_k)$ of generically nonsolvable linear series is equivalent to some collection $\mathcal{W}$ of minimal defect -1. Moreover, if $\mathrm{d}(\mathcal{E}) = -d$ and $\mathcal{E}_J = (E_1, \dots, E_r)$ is an essential subcollection of $\mathcal{E}$, then $\mathcal{W}$ can be defined as*

$$W_1 = \ldots = W_{r-d+1} = E_J, \quad W_{r-d+2} = E_{r+1}, \dots, W_{k-d+1} = E_k.$$

Proof. First note that collections $\mathcal{E}_J = (E_1, \dots, E_r)$ and $\mathcal{W}_K = (W_1, \dots, W_{r-d+1})$ are equivalent collections which are the essential subcollections of $\mathcal{E}$ and $\mathcal{W}$ respectively. Indeed, since both collections are essential, by Proposition 3.1 they are equivalent if and only if generic fibres of their Kodaira maps coincide. But this follows directly from the fact that $W_1 = \ldots = W_{r-d+1} = E_J = E_1 \cdot \ldots \cdot E_r$.

Therefore, we have two collections $\mathcal{E}$ and $\mathcal{W}$ with equivalent essential subcollections $J = (E_1, \dots, E_r)$ and $K = (W_1, \dots, W_{r-d+1})$ and coinciding complements:

$$\mathcal{E}_{J^c} = (E_{r+1}, \dots, E_k) = (W_{r-d+2}, \dots, W_{k-d+1}) = \mathcal{W}_{K^c}.$$

But any two such collections are equivalent since the zero set $Z_{\mathbf{u}}$ of a generic system is a solution of the system compliment to the essential subsystem restricted to the zero set of the essential subsystem. □

4 Equivariant linear systems on homogeneous varieties

This section is devoted to the study of G-invariant linear systems on a complex variety X with a transitive G-action. First, we work with general homogeneous space and prove Theorem 4.1 which reduces the study of generic non-empty zero sets of overdetermined systems to the study of generic complete intersections.

We apply then this result to obtain Theorem 4.2 which together with results of [KKho16] provides a strategy for computation of discrete invariants of generic non-empty zero set of a system of equations associated to a collection of overdetermined linear series on a spherical homogeneous space.

4.1 Linear systems on homogeneous varieties

In this subsection we will study some general results on linear systems on homogeneous varieties. The main result of this subsection is a reduction of an overdetermined collection of linear series to several isomorphic generically solvable collections.

Let G be a connected algebraic group, and $X = G/H$ be a G-homogeneous space. Let us denote by $x_0 \in X$ the class of identity element $e \cdot H \in G/H$. Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be globally generated G-linearized line bundles on G/H. For each $i = 1, \dots, k$ let E_i be a nonzero G-invariant linear system for $\mathcal{L}_i$ i.e. E_i is a finite dimensional G-invariant subspace of $H^0(X, \mathcal{L}_i)$.

Each E_J is G-invariant and hence it is base points free on G/H. Thus the Kodaira map Φ_J is defined on the whole G/H. Since $E_1, \dots, E_k$ are G-invariant, E_J^* is a linear representation of G for any $J \subset \{1, \dots, k\}$ and, therefore, there is a natural action of G on $\mathbb{P}(E_J^*)$. It is easy to see that the Kodaira map Φ_J is equivariant for this action. Therefore, the image $\Phi_J(X)$ is a quasi-projective homogeneous G-variety isomorphic to $G/(G_{\Phi_J(x_0)})$, where $G_{\Phi_J(x_0)}$ is a stabilizer of $\Phi_J(x_0) \in \mathbb{P}(E_J^*)$. For $J \subset \{1, \dots, k\}$ we will denote the stabilizer $G_{\Phi_J(x_0)}$ by $\Gamma_J \subset G$.

Definition 8. Two collections of G-invariant linear systems $\mathcal{E} = (E_1, \dots, E_k)$, $\mathcal{E}' = (E_1', \dots E_k')$ on homogeneous spaces X, X' respectively are isomorphic if there exists an G equivariant isomorphism $f : X \to X'$ and an isomorphism of G-linearized line bundles $\phi_i : \mathcal{L}_i \to f^*\mathcal{L}_i'$ for any $i = 1, \dots, k$ such that $\phi_i^* \circ f^*(E_i') = E_i$.

Proposition 4.1. *Let $\Phi_J : X \to \mathbb{P}(E_J^*)$ and Γ_J be as before then:*
(i) for any $y \in \Phi_J(X)$ the fiber $F_y := \Phi_J^{-1}(y)$ has a structure of Γ_J-variety;
(ii) any fiber F_y is isomorphic to $F_{y_0} \cong \Gamma_J/H$ as Γ_J-variety;
(iii) for any G equivariant linear system V on X and any point $y \in \Phi_J(X)$ the restriction $E|_{F_y}$ is Γ_J-invariant. Moreover, a pair $F_y, V|_{F_y}$ is isomorphic to the pair $F_{y_0}, V|_{F_{y_0}}$.

Proof. For the parts (i) and (ii) let G_y be the stabilizer of a point $y \in \Phi_J(X)$, then the fiber F_y is an homogeneous G_y variety. But also G_y is conjugated to Γ_J, i.e. $G_y = g\Gamma_J g^{-1}$ for some $g \in G$. Then the equivariant isomorphism between F_{y_0} and F_y (which also will define Γ_J on F_y) can be defined by

$$\Gamma_J \times F_{y_0} \to G_y \times F_y, \quad (\gamma, p) \mapsto (g\gamma g^{-1}, gp).$$

The part (iii) follows directly from the construction above and the fact that linear system V is G-invariant. □

By Proposition 4.1, any fiber of the Kodaira map associated to a G-invariant linear series is a homogeneous variety and in particular is smooth. Therefore, irreducible components of fibers coincide with connected components. Next proposition is a more precise version of Proposition 4.1, which deals with connected components of fibers of the Kodaira map.

Proposition 4.2. *Let $\Phi_J : X \to \mathbb{P}(E_J^*)$ and Γ_J be as before then:*
(i) connected components of fibres of Φ_J have a structure of Γ_J^0-variety;
(ii) any two connected components of any two fibres are isomorphic as Γ_J^0-varieties and, in particular are isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$, where Γ_J^0 is the connected component of identity in Γ_J;
(iii) for any G equivariant linear system V on X and two connected components C_1, C_2 of any two fibres, the restrictions $E|_{C_1}, E|_{C_2}$ are Γ_J^0-invariant. Moreover, pairs $C_1, V|_{C_1}$ and $C_2, V|_{C_2}$ are isomorphic.

Proof. Most of the proof is absolutely analogous to the proof of Proposition 4.1. The only statement which needs clarification is that any connected component of a fiber is isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$. By Proposition 4.1 it is enough to check this for a connected component of a given fiber, say F_{y_0}. The rest easily follows from the fact that $F_{y_0} = \Gamma_J/H$. □

For a subcollection $J \subset \{1, \dots, k\}$ we will call the number of connected components of a fiber of the Kodaira map $\Phi_J : G/H \to \mathbb{P}(E_J^*)$ *the index* of J and denote it by $ind(J)$. One can describe $ind(J)$ in a group theoretic way, this description clarifies the term "index".

Connected components of identity Γ_J^0, H^0 are normal subgroups of groups Γ_J, H respectively. There exists a natural homomorphism between $i : H/H^0 \to \Gamma_J/\Gamma_J^0$ induced by the inclusion of $H \subset \Gamma_J$. The homomorphism i is well-defined since H^0 is a subgroup of Γ_J^0. It is easy to see that connected components of Γ_J/H and hence of any other fiber of Φ_J are in one to one correspondence with elements of the coset set

$$(\Gamma_J/\Gamma_J^0)\Big/ i(H/H^0),$$

i.e. $ind(J)$ is equal to the index of $i(H/H^0)$ in Γ_J/Γ_J^0.

Theorem 4.1. *Let $X = G/H$ be a homogeneous space and let $E_1, \dots, E_k$ be G - invariant linear systems on X, with the essential subcollection $J \subset \{1, \dots, k\}$. Then the zero set $X_{\mathbf{s}}$ for generic $\mathbf{s} \in R_{\mathcal{E}}$ is a disjoint union of $ind(J)$ subvarieties $Y_1, \dots, Y_{ind(J)}$. Moreover, for any $i = 1, \dots, ind(J)$ Y_i is a generic zero set of a collection of linear series isomorphic to $\mathcal{E}_{J^c} = (E_i)_{i \notin J}$ restricted to $\Gamma_J^0/(\Gamma_J^0 \cap H)$.*

Proof. By Theorem 3.1 the zero set $Z_{\mathbf{s}}$ of a generic solvable system given by $\mathbf{s} \in R_{\mathcal{E}}$ is the zero set of a generic system $\mathcal{E}_{J^c}$ restricted to the fiber of the Kodaira map Φ_J. But by Proposition 4.2 connected components with the restrictions of the collection $\mathcal{E}_{J^c}$ to them are isomorphic. □

4.2 Linear series in spherical varieties

In this subsection we will study G-invariant linear series on spherical varieties. A homogeneous space G/H of a reductive group G is called *spherical* if some (and hence any) Borel subgroup $B \subset G$ has an open dense orbit in G/H. Starting from this point a group G will assumed to be reductive and a homogeneous space G/H will assumed to be spherical.

Any G-invariant linear series V on G/H is a representation of a reductive group G, therefore it is a direct sum of irreducible representations:

$$V = \bigoplus_{\lambda} V_\lambda.$$

It is well known that the decomposition above is multiplicity free, that is each irreducible representation appears at most once in it. Indeed, let V_λ and V_λ' be two different irreducible representations with the same highest weight appearing in decomposition of V, and let s and s' be highest weight vectors in V_λ and V_λ' respectively. In particular, both s, s' are B-eigensections of weight λ of some G-linearised line bundle $\mathcal{L}$. Therefore the ratio s/s' is a B-invariant rational function on G/H. Since X has an open B-orbit we conclude that s/s' is constant, so $V_\lambda = V_\lambda'$. The set of weights appearing in the decomposition of V is called *G-spectrum* of V and is denoted by $\mathrm{Spec}_G(V)$. The pair of a G-linearized line bundle $\mathcal{L}$ and a finite subset A of $\mathrm{Spec}_G(H^0(X, \mathcal{L}))$ determines uniquely a G-invariant linear series on X.

The main result of this subsection is Theorem 4.2 which realizes a generic non-empty zero set defined by overdetermined collection of linear series, as a zero set defined by generically solvable collection. In [KKho16], for a collection of linear series $\mathcal{E} = (E_1, \dots, E_k)$ and for a generic choice of $\mathbf{s} \in \mathbf{E}$ some discrete invariants of the zero set $Z_{\mathbf{s}}$ were computed in terms of combinatorics of the Newton-Okounkov polytope. The Newton-Okounkov polytope is constructed as a polytope fibered over moment polytope with string polytopes as fibers. The construction of Newton-Okounkov polytope depends only on G-spectra of linear series E^k (for more details see [KKho16]). These results together with Theorem 4.2 provide a strategy for computing discrete invariants of generic non-empty zero set defined by overdetermined collection of linear series.

Lemma 4.1. *Let H be a spherical subgroup of a reductive group G, let K be a connected reductive subgroup of G which contains H (i.e $H \subset K \subset G$). Then H^0 is a spherical subgroup of K.*

Proof. First notice that if H is spherical subgroup of G then such is H^0 so without loss of generality we can assume that $H = H^0$.

Now consider a point $x \in K/H \subset G/H$, there exists a Borel subgroup B of G such that $B \cdot x$ is a dense in G/H. Let U be the intersection of the dense orbit $B \cdot x$ with K/H, in other words $U = (B \cap K) \cdot x$. Note that $(B \cap K)$ is solvable and therefore $(B \cap K)^0$ is contained in some Borel subgroup B_K of K. From the other hand, since $B \cdot x$ is open in G/H, U is open and dense in K/H, and since K/H is irreducible the orbit $(B \cap K)^0 \cdot x$ is dense in K/H. We conclude that the orbit $B_K \cdot x$ is dense in K/H as B_K contain $(B \cap K)^0$. □

Proposition 4.3. *Let $E_1, \dots, E_k$ be G-invariant linear systems on a spherical homogeneous space G/H and Γ_J be a reductive group for some $J \subset \{1, \dots, k\}$. Then the connected components of fibers of the Kodaira map Φ_J are isomorphic spherical homogeneous spaces. Moreover, for any G-invariant linear series V, all restrictions of V to connected components of fibers are isomorphic.*

Proof. By part (ii) of Proposition 4.2 connected components of the fibers of the Kodaira map Φ_J are homogeneous spaces which are isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$, and since Γ_J^0 is reductive by Lemma 4.1 they are also spherical. The last statement of the above proposition is identical to part (iii) of Proposition 4.2. □

Theorem 4.2. *Let $\mathcal{E} = (E_1, \dots, E_k)$ and G/H be as before. Let also J be the essential subcollection of $\mathcal{E}$ such that Γ_J is a reductive group. Then for the generic system $\mathbf{s} \in R_{\mathbf{E}}$ the zero set $Z_{\mathbf{s}}$ is a disjoint union of $ind(J)$ varieties $Y_1, \dots, Y_{ind(J)}$. Moreover, subvarieties Y_j's are defined by isomorphic Γ_J^0-invariant collections of linear series on the spherical variety $\Gamma_J^0/(\Gamma_J^0 \cap H)$.*

Proof. By Theorem 4.1 the zero set $Z_{\mathbf{s}}$ of a generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$ is a union of $ind(J)$ subvarieties $Y_1, \dots, Y_{ind(J)}$, with Y_j's defined by isomorphic Γ_J^0-invariant collections of linear series on connected components of Kodaira map Φ_J. But connected components of Kodaira map Φ_J are spherical and isomorphic to $\Gamma_J^0/(\Gamma_J^0 \cap H)$ by Proposition 4.3. □

Corollary 4.1. *In situation as above, the arithmetic genus $g(Z_{\mathbf{s}})$ of the generic non-empty zero set $Z_{\mathbf{s}}$ could be computed as*

$$g(Z_{\mathbf{s}}) = ind(J) g(Y_i),$$

for any $i = 1, \dots, ind(J)$, and can be computed in terms combinatorics of Newton-Okounkov polytopes. If in addition all linear series from collection $\mathcal{E}_{J^c}$ restricted to $\Gamma_J^0/(\Gamma_J^0 \cap H)$ are injective, mixed Hodge numbers $h^{p,0}(Z_{\mathbf{s}})$ of the generic non-empty zero set $Z_{\mathbf{s}}$ can be computed as

$$h^{p,0}(Z_{\mathbf{s}}) = ind(J) h^{p,0}(Y_i),$$

for any $i = 1, \dots, ind(J)$, and can be computed in terms combinatorics of Newton-Okounkov polytopes.

Proof. Corollary follows immediately from Theorem 4.2 and Theorems 1, 2 of [KKho16]. □

Theorem 4.2 involves condition on Γ_J to be reductive, the following theorem provides a geometric criterion for a subgroup of a reductive group to be reductive.

Theorem 4.3 ([Tim11])**.** *Let G be a reductive algebraic group then a closed subgroup H of G is reductive if and only if the coset space G/H is an affine algebraic variety.*

Remark 4.1. The condition on Γ_J to be reductive is quite restrictive. However, for any triple $H \subset K \subset G$ where H is a spherical subgroup of G and K is reductive one can realize K as Γ_J for some collection of G-invariant linear series on G/H. Indeed, let $\pi : G/H \to G/K$ be natural projection and let $\mathcal{E} = (E_1, \dots, E_k)$ be an injective (i.e. such that E_J is very ample) essential collection of G-invariant linear system on G/K. Then the pullback collection $\pi^*\mathcal{E} = (\pi^*E_1, \dots, \pi^*E_k)$ is an essential collection of linear series on G/H with $\Gamma_J = K$. It will be interesting to classify all such triples $H \subset K \subset G$. Some of the results in this direction could be found in [Hof18].

4.3 Example

In this subsection we will give a concrete example of an application of Theorem 4.2. We will work with homogeneous space GL_n/U. Let $\mathcal{E} = (E_1, E_2, E_3)$ be a collection of linear series defined as

$$E_1 = E_2 = Span(c, \det^k), \quad E_3,$$

where c is a constant function, $\det^k(g \cdot U) = \det(g)^k$, and E_3 is a very ample linear series.

The minimal defect of $\mathcal{E}$ is -1, the essential subcollection is $J = \{1, 2\}$, and $\Gamma_J = \mathrm{SL}_n[k]$, where

$$\mathrm{SL}_n[k] = \left\{ g \in \mathrm{GL}_n \,|\, det(g)^k = 1 \right\}.$$

Therefore, a connected component of a fiber of the Kodaira map Φ_J is isomorphic to SL_n/U and the number of connected components of a fiber is $ind(J) = k$.

It follows by Theorem 4.2 that the generic non-empty zero set $Z_{\mathbf{s}}$ of a system $s_1 = s_2 = s_3 = 0$ is a union of k subvarieties $Y_1, \ldots, Y_k$ such that each of Y_i is a hypersurface in SL_n/U cut out by a generic section $s \in E_3|_{\mathrm{SL}_n/U}$. In particular geometric genus and mixed Hodge numbers $h^{p,0}$ of $Z_{\mathbf{s}}$ could be computed using results of [KKho16].

5 Linear systems on Algebraic Torus

In this section we study equivariant linear series on $(\mathbb{C}^*)^n$. The results of this section were previously published in [Mon17] and included here for completeness of exposition. We will start with some notations and definitions. With a Laurent polynomial f in n variables one can associate its support $supp(f) \subset \mathbb{Z}^n$ which is the set of exponents of monomials having non-zero coefficient in f and its Newton polyhedra $\Delta(f) \subset \mathbb{R}^n$ which is the convex hull of $supp(f)$ in $\mathbb{R}^n$. For a finite set $A \subset \mathbb{Z}^n$, let E_A be a vector space of Laurent polynomials with support in A. It is easy to see that any equivariant linear series on $(\mathbb{C}^*)^n$ is of the form E_A for some A and that A is a $(\mathbb{C}^*)^n$-spectrum of E_A.

Any connected algebraic subgroup of $(\mathbb{C}^*)^n$ is an algebraic torus, in particular such is group $Gamma^0_J$. Therefore, Theorem 4.2 has a much nicer form in the case of equivariant linear series on $(\mathbb{C}^*)^n$.

Theorem 5.1. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ be a collection of finite subsets of $\mathbb{Z}^n$ and $E_i = E_{A_i}$ be corresponding linear systems on $(\mathbb{C}^*)^n$. Let J the essential subcollection of $E_1, \ldots, E_k$. Then for the generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$ the zero set $Z_{\mathbf{s}}$ is a disjoint union of $ind(J)$ subvarieties each of which is defined by a generic system given by isomorphic collections of linear systems on algebraic torus.*

The second difference between toric and spherical cases is that Theorem 5.1 could be formulated much more concretely. This will allow us to find discrete invariants of a generic non-empty zero set explicitly. We give an example of such result in Theorem 5.3. To state a concrete version of Theorem 5.1 we would need more notations. For a collection $\mathcal{A} = (A_1, \ldots, A_k)$ of finite subsets of $\mathbb{Z}^n$ and subcollection J let

- A_J be the Minkowski sum $\sum_{i \in J} A_i$;
- $L(J)$ be a vector subspace of $\mathbb{R}^n$ parallel tor the affine span of A_J and $\pi_J : \mathbb{R}^n \to \mathbb{R}^n/L(J)$ be the natural projection;
- $\Lambda(J) = L(J) \cap \mathbb{Z}^n$ the lattice of integral points in $L(J)$;
- G_J the group generated by all the differences of the form $(a - b)$ with $a, b \in A_i$ for any $i \in J$;
- $ind(G_J)$ the index of G_J in $\Lambda(J)$.

Theorem 5.2. *Let $\mathcal{A} = (A_1, \dots, A_k)$ be a collection of finite subsets of $\mathbb{Z}^n$ and $E_i = E_{A_i}$ be corresponding linear systems on $(\mathbb{C}^*)^n$, let also J the essential subcollection of $E_1, \dots, E_k$. Then*
(i) The defect of a collection of linear series $\operatorname{def}(J)$ *(in the sense of Definition 3) is equal to the defect of the vector subspaces* $\operatorname{def}(L(A_i))_{i\in J}$ *(in the sense of Definition 4);*
(ii) The number $ind(J)$ of connected components of a fiber of Kodaira map Φ_J is equal to the index $ind(G_J)$;
(iii) for the generic solvable system $\mathbf{s} \in R_{\mathcal{E}}$ the zero set $Z_{\mathbf{s}}$ is a disjoint union of $ind(J)$ subvarieties each of which is given by a generic system with the same supports $(\pi_J(\mathcal{A}_i))_{i\notin J}$.

Proof. For the proof see Section 4 of [Mon17] and in particular proof of Theorem 18. □

The following Theorem is an example of the application of Theorem 5.2.

Theorem 5.3 ([Mon17] Theorem 20)**.** *Let $\mathcal{A}_1, \dots, \mathcal{A}_{n+k} \subset \mathbb{Z}^n$ be such that $d(\mathcal{A}) = -k$ and J be the unique essential subcollection. Then the zero set $Y_{\mathbf{f}}$ of the generic consistent system has dimension 0, and the number of points in $Y_{\mathbf{f}}$ is equal to*

$$(n - \#J + k)! \cdot ind(J) \cdot Vol(\pi_J(\Delta_i)_{i\notin J}),$$

where Δ_i is the convex hull of $\mathcal{A}_i$ and Vol is the mixed volume on $\mathbb{R}^n/L(J)$ normalized with respect to the lattice $\mathbb{Z}^n/\Lambda(J)$.

If $k = 0$ this theorem coincides with the BKK Theorem. In the case $k = 1$ the generic number of solution appears as the corresponding degree of sparse resultants and was computed in [D'AS15]. In a similar fascion, Theorem 5.2 could be applied to the computation of any other discrete invariants which can be computed by means of Newton polyhedra theory.

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