Eigenforms of hyperelliptic curves with many automorphisms

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Abstract

Given a pair of translation surfaces it is very difficult to determine whether they are supported on the same algebraic curve. In fact, there are very few examples of such pairs. In this note we present infinitely many examples of finite collections of translation surfaces supported on the same algebraic curve. The underlying curves are hyperelliptic curves with many automorphisms, and for each curve the corresponding collection of 1-forms is an eigenbasis for an automorphism of maximal order.

Contents

1   Introduction

Let $X \in \mathcal{M}_g$ be an algebraic curve and let $f \in \mathrm{Aut}(X)$ be its automorphism. Let $\Omega(X)$ be a g-dimensional space of holomorphic 1-forms on X. An automorphism $f \in \mathrm{Aut}(X)$ induces an action $f^* : \Omega(X) \to \Omega(X)$ via pullback. A non-zero holomorphic 1-form $\omega \in \Omega(X)$ is called an *eigenform of f* if it is an eigenvector of f^*.

Following the notation of [R70] and [W97] we say that an algebraic curve X has *many automorphisms* if it does not belong to a non-trivial family of curves with the same group of automorphisms. All hyperelliptic curves with many automorphisms are listed in [MP17, Table 1]. The list consists of three infinite families spread across all genera and sporadic examples. In this note we focus on the three infinite families.

A pair (X, ω), where $X \in \mathcal{M}_g$ is an algebraic curve, or a Riemann surface, of genus g and ω is a non-zero holomorphic 1-form on X, is called an *Abelian differential*. The set of zeroes of ω is denoted by $Z(\omega)$. For any Abelian differential (X, ω) the integration of ω produces a *translation structure*, an atlas of complex charts on $X \setminus Z(\omega)$ with parallel translations $z \mapsto z + c$ as transition functions. A neighborhood of a conical point possesses a singular flat structure obtained by pulling back flat metric on a disk via the covering map $z \mapsto z^k$. Conversely, every translation structure defines an Abelian differential as pullback of complex structure from $\mathbb{C}$ and the holomorphic 1-form dz on it. Abelian differentials are also called *translation surfaces*. In fact, every Abelian differential can be represented as a collection of polygons with pairs of parallel sides identified by translation (see Figure 1). For more background on translation surfaces see [Z06].
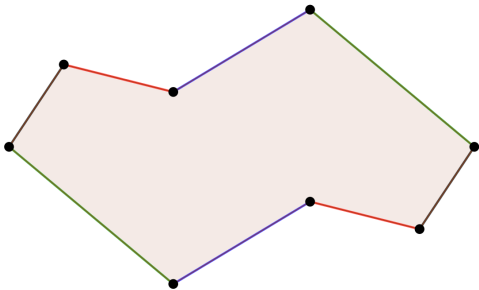
Figure 1: An example of a translation surface: the edges of the polygon of the same color are identified by parallel translations.

This gives a convenient way of visualizing holomorphic 1-forms on algebraic curves. However it is almost never possible to determine the underlying algebraic curve from the polygonal picture. In particular, there are very few examples of pairs of polygonal representations of holomorphic 1-forms ω and ω' supported on the same algebraic curve X. We use eigenforms to provide such examples. In this note we construct translation surfaces representing eigenbases on hyperelliptic curves with many automorphisms:

Theorem 1.1. *Let S_N^k and H_N^k be polygonal representations of translation surfaces defined in Section 3. Then for any $k = 1, \ldots, g$:*

1. *S_{2g+2}^k represents an eigenform $\omega_k = x^{k-1}dx/y$ on the hyperelliptic curve $\mathcal{A}_g : y^2 = x^{2g+2} - 1$;*

2. *S_{2g+1}^k represents an eigenform $\omega_k = x^{k-1}dx/y$ on the hyperelliptic curve $\mathcal{P}_g : y^2 = x^{2g+1} - 1$;*

3. *H_{2g}^k represents an eigenform $\omega_k = x^{k-1}dx/y$ on the hyperelliptic curve $\mathcal{D}_g : y^2 = x(x^{2g} - 1)$.*

Acknowledgements

2 Hyperelliptic curves with many automorphisms

In this section we describe three families of hyperelliptic curves with many automorphisms and their eigenforms. The content of this section is summarized in Table 1. Consider the following curves in the affine plane:

$$\mathcal{A}_g : y^2 = x^{2g+2} - 1$$
$$\mathcal{P}_g : y^2 = x^{2g+1} - 1$$
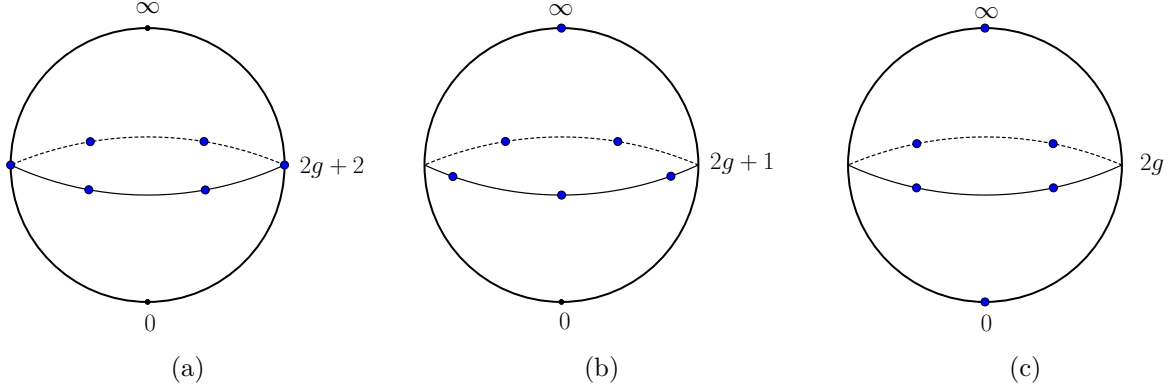$$\mathcal{D}_g : y^2 = x(x^{2g} - 1)$$

Figure 2: The branching loci of hyperelliptic curves (a) $\mathcal{A}_g$, (b) $\mathcal{P}_g$ and (c) $\mathcal{D}_g$, via hyperelliptic covers to $\mathbb{P}^1$.

The homogenization of these equations define algebraic curves in $\mathbb{P}^2$. Curve $\mathcal{D}_g$ is smooth, however $\mathcal{A}_g$ and $\mathcal{P}_g$ have singularities at $[0:0:1]$, and one needs to consider normalizations of these curves.

Remark 2.1. Apart from these three families there are only finitely many hyperelliptic curves with many automorphisms (see [MP17]).

The hyperelliptic involutions of $\mathcal{A}_g, \mathcal{P}_g, \mathcal{D}_g$ are given by $\eta : (x, y) \mapsto (x, -y)$. For any hyperelliptic curve $X \in \mathcal{M}_g$ the hyperelliptic cover $\pi_\eta : X \to \mathbb{P}^1$ of degree two is branched over $2g + 2$ points on $\mathbb{P}^1$. These points uniquely determine the curve X (see Figure 2). We describe the three above families in terms of their hyperelliptic covers and exhibit their automorphisms of locally maximal order. Let ξ_k be a primitive k-th root of unity.

Apolar curve. The hyperelliptic curve $\mathcal{A}_g$ is a double cover over $\mathbb{P}^1$ branched over $(2g + 2)$-th roots of unity (see Figure 2a). It has an automorphism of order $2g + 2$ given by:

$$f_{\mathcal{A}_g} : (x, y) \mapsto (\xi_{2g+2}x, y).$$

We will call $\mathcal{A}_g$ an *apolar curve*.

Polar curve. The hyperelliptic curve $\mathcal{P}_g$ is a double cover over $\mathbb{P}^1$ branched over ∞ and $(2g+1)$-st roots of unity (see Figure 2b). An automorphism of order $2g + 1$ is given by:

$$f_{\mathcal{P}_g} : (x, y) \mapsto (\xi_{2g+1}x, y).$$

We will call $\mathcal{P}_g$ a *polar curve*.

Dipolar curve. The hyperelliptic curve $\mathcal{D}_g$ is a double cover over $\mathbb{P}^1$ branched over $0, \infty$ and $2g$-th roots of unity (see Figure 2c). An automorphism of order $4g$ is given by:

$$f_{\mathcal{D}_g} : (x, y) \mapsto (\xi_{2g}x, \xi_{4g}y).$$

We will call $\mathcal{D}_g$ a *dipolar curve*.

Eigenbasis. For any $g > 1$ the forms:

$$\omega_k = x^{k-1}dx/y, \ \ k = 1, \ldots, g$$

define a basis of 1-forms on $\mathcal{A}_g$, $\mathcal{P}_g$ and $\mathcal{D}_g$. The induced action of $f_{\mathcal{A}_g}, f_{\mathcal{P}_g}$ and $f_{\mathcal{D}_g}$ on these forms is given by:

$$f_{\mathcal{A}_g}^*(\omega_k) = \xi_{2g+2}^k \cdot \omega_k \tag{1}$$

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$$f_{\mathcal{P}_g}^*(\omega_k) = \xi_{2g+1}^k \cdot \omega_k \tag{2}$$

$$f_{\mathcal{D}_g}^*(\omega_k) = \xi_{4g}^{2k-1} \cdot \omega_k \tag{3}$$

Therefore $\{\omega_k \mid k = 1, \ldots, g\}$ is an eigenbasis for the automorphisms $f_{\mathcal{A}_g}$, $f_{\mathcal{P}_g}$ and $f_{\mathcal{D}_g}$. Next we find the multiplicities of the zeroes of these eigenforms, in other words determine the strata of Abelian differentials that contain $(\mathcal{A}_g, \omega_k)$, $(\mathcal{P}_g, \omega_k)$ and $(\mathcal{D}_g, \omega_k)$.

Strata. The eigenform ω_k on $\mathcal{A}_g$ has two zeroes of order $k - 1$ at the preimages of $0 \in \mathbb{P}^1$ and two zeroes of order $g - k$ at the preimages of ∞. Indeed, the hyperelliptic cover is unbranched over 0 and ∞. Therefore for $(x = 0, y = \pm i)$ one can use x as a coordinate and $x^{k-1}dx/y$ has zero of order $k - 1$ at each of these points. The only other zeroes of ω_k are in the fiber over ∞, they are switched by hyperelliptic involution, and therefore they have the same order. Considering that the sum of orders of all zeroes of ω_k is $2g - 2$, each of the zeroes over ∞ has order $g - k$. Therefore, $(\mathcal{A}_g, \omega_k) \in \mathcal{H}(k-1, k-1, g-k, g-k)$.

The eigenform ω_k on $\mathcal{P}_g$ has a zero of order $2g - 2k$ over $\infty \in \mathbb{P}^1$ and two zeroes of order $k - 1$ over $0 \in \mathbb{P}^1$. Indeed, the hyperelliptic cover is unbranched over 0 and hence ω_k has two zeroes of orders $k - 1$ at the preimages. At the same time the cover has only one preimage over ∞, and since the sum of orders of all zeroes of ω_k is $2g - 2$, this preimage is a zero of order $2g - 2k$. Therefore $(\mathcal{P}_g, \omega_k) \in \mathcal{H}(k-1, k-1, 2g-2k)$.

Similarly, $(\mathcal{D}_g, \omega_k) \in \mathcal{H}(2k-2, 2g-2k)$.

	Equation	Automorphism f	Eigenform ω_k	Stratum	$f^*(\omega_k)$
$\mathcal{A}_g$	$y^2 = x^{2g+2} - 1$	$(x, y) \mapsto (\xi_{2g+2}x, y)$	$\dfrac{x^{k-1}dx}{y}$	$\mathcal{H}((k-1)^2, (g-k)^2)$	$\xi_{2g+2}^k \cdot \omega_k$
$\mathcal{P}_g$	$y^2 = x^{2g+1} - 1$	$(x, y) \mapsto (\xi_{2g+1}x, y)$	$\dfrac{x^{k-1}dx}{y}$	$\mathcal{H}((k-1)^2, 2g-2k)$	$\xi_{2g+1}^k \cdot \omega_k$
$\mathcal{D}_g$	$y^2 = x(x^{2g} - 1)$	$(x, y) \mapsto (\xi_{2g}x, \xi_{4g}y)$	$\dfrac{x^{k-1}dx}{y}$	$\mathcal{H}(2k-2, 2g-2k)$	$\xi_{4g}^{2k-1} \cdot \omega_k$

Table 1: Hyperelliptic curves with many automorphisms and their eigenforms.

3 Translation surfaces and eigenforms

In this section we construct polygonal representations of translation surfaces corresponding to all eigenforms of hyperelliptic curves $\mathcal{A}_g, \mathcal{P}_g$ and $\mathcal{D}_g$. The content of this section is summarized in Table 4.

Spingon. For any integers $N > 1$, $1 \le k < (N+1)/2$ we define a spingon S_N^k as follows. First, consider a rhombus $R_0 = OA_0B_0C_0$ with an angle $2\pi k/N$ at the vertex O. Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by $2\pi k/N$ with the center at O. The rotation r generates a cyclic subgroup of isometries

of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}/N\mathbb{Z}$. For any $i \in \mathbb{Z}/N\mathbb{Z}$ define $R_i = OA_iB_iC_i$ to be an image of R_0 under r^i. Now identify the following pairs of parallel sides by parallel translations:

$$OC_i \sim OA_{i+1} \text{ and } A_iB_i \sim C_{i+1}B_{i+1}, \text{ for all } i \in \mathbb{Z}/N\mathbb{Z}.$$

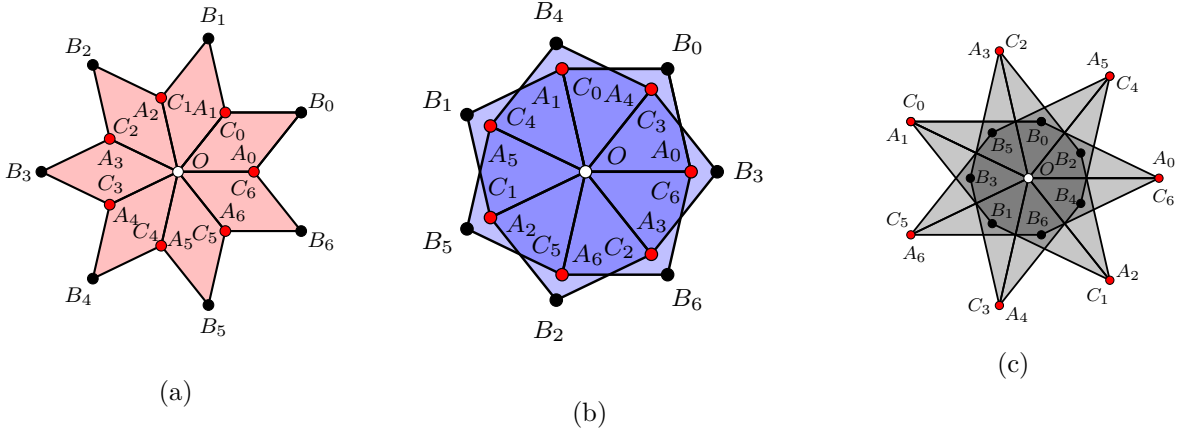The resulting polygon with the above identifications of parallel sides is a translation surface that we will call a *spingon* and will denote by S_N^k (see Figure 3).

Figure 3: The spingons (a) S_7^1, (b) S_7^2 and (c) S_7^3.

Half-spingon. For any integers $N > 1$ and $1 \le k \le N/2$ we define half-spingon H_N^k as follows. Start with the rhombus $P_0 = OA_0B_0C_0$ with an angle $\pi(2k-1)/N$ at the vertex O. A union of images of P_0 under $N-1$ successive rotations around O by an angle $\pi(2k-1)/N$ is a polygon that consists of the rhombi P_i. This time, because $2k-1$ is odd, the sides OA_0 and OC_{N-1} are given by the opposite vectors. The identifications are:

$$OA_0 \sim B_{N-1}A_{N-1}; \ C_0B_0 \sim C_{N-1}O;$$
$$OC_i \sim OA_{i+1}, \text{ for } i = 0, \ldots, N-2;$$
$$A_iB_I \sim C_{i+1}B_{i+1}, \text{ for } i = 0, \ldots, N-2.$$

The resulting polygon with these identifications is a translation surface that we will call a *half-spingon* and will denote it by H_N^k. Now we are ready to prove Theorem 1.1. **add Figure for half-spingon.**

Proof of Theorem 1.1. **Apolar curves $\mathcal{A}_g$.** Let $g > 1$ be any integer. We will show that S_{2g+2}^k represents an eigenform $\omega_k = x^{k-1}dx/y$ on the apolar hyperelliptic curve $\mathcal{A}_g$.

Note that all vertices B_i are identified into one that we denote by B. Since the number of rhombi $2g + 2$ is even, all A_i and C_i are identified into two other vertices that we denote by A and C. Hence there are four vertices: O, A, B and C.

We first identify which stratum does translation surface S_{2g+2}^k belongs. The total angle around O and B is $2\pi k$, and around A and C it is $(2g+2)(\pi - 2\pi k/(2g+2)) = 2\pi(g-k+1)$. Hence S_{2g+2}^k belongs to $\mathcal{H}(k-1, k-1, g-k, g-k)$. In particular, the genus of the underlying algebraic curve of S_{2g+2}^k is indeed g.

Now let us show that the underlying Riemann surface of S_{2g+2}^k is hyperelliptic. The hyperelliptic involution is given by rotating each rhombus $R_i = OA_iB_iC_i$ around its center by $180°$. Because of

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the side identifications this is a well-defined map from S_{2g+2}^k to itself. It has $2g + 2$ fixed points: one at the center of each of the $2g + 2$ rhombi. This implies that S_{2g+2}^k is hyperelliptic.

The automorphism $f_{\mathcal{A}_g}$ of order $2g + 2$ is given by rotating S_{2g+2}^k around O by the angle $2\pi k/(2g + 2)$ sending each R_i to R_{i+1}. Since hyperelliptic curve of genus g with an automorphism of order $2g + 2$ is unique we conclude that S_{2g+2}^k is supported on the apolar curve $\mathcal{A}_g$. Finally $S_{2g+2}^k \in \mathcal{H}(k-1, k-1, g-k, g-k)$ is preserved up to a scalar by f and we conclude that $S_{2g+2}^k = (\mathcal{A}_g, \omega_k)$. Note that the eigenvalue of the automorphism $f_{\mathcal{A}_g}$ is ξ_{2g+2}^k, which agrees with (1).

Polar curves $\mathcal{P}_g$. Next we will show that S_{2g+1}^k represents an eigenform $\omega_k = x^{k-1}dx/y$ on the polar hyperelliptic curve $\mathcal{P}_g$.

In this case all vertices B_i are identified into one that we denote by B, and all A_i and C_i are identified into another that we denote by A. Hence there are three vertices: O, A and B with total angles $2\pi k$, $2(2g+1)(\pi - 2\pi k/(2g+1)) = 2\pi(2g - 2k + 1)$ and $2\pi k$ respectively. Hence S_{2g+1}^k belongs to $\mathcal{H}(k-1, k-1, 2g-2k)$. In particular, the genus of the underlying algebraic curve of S_{2g+1}^k is indeed g.

Similarly the hyperelliptic involution is given by rotating each rhombus $R_i = OA_iB_iC_i$ around its center by $180°$. It has $2g + 2$ fixed points: the point A and the center of each of the $2g + 1$ rhombi. This implies that S_{2g+1}^k is hyperelliptic.

The automorphism $f_{\mathcal{P}_g}$ of order $2g+1$ is given by the rotation around O by $2\pi k/(2g+1)$. Since the hyperelliptic curve of genus g with an automorphism of order $2g + 1$ is unique, the 1-form of $S_{2g+1}^k \in \mathcal{H}(k-1, k-1, 2g-2k)$ is preserved by $f_{\mathcal{P}_g}$ up to scalar, we conclude that $S_{2g+1}^k = (\mathcal{P}_g, \omega_k)$. Again, the eigenvalue of the automorphism $f_{\mathcal{P}_g}$ is ξ_{2g+1}^k, which agrees with (2).

Dipolar curves $\mathcal{D}_g$. Next we will show that H_{2g}^k represents an eigenform $\omega_k = x^{k-1}dx/y$ on the polar hyperelliptic curve $\mathcal{D}_g$.

The vertices B_i and O are identified into one that we denote by O, and the vertices A_i and C_i are identified into another that we denote by A. Hence there are two vertices: O and A with total angles $2\pi(2k - 1)$ and $2 \cdot 2g(\pi - 2\pi(2k-1)/4g) = 2\pi(2g - 2k + 1)$ respectively. Hence H_{2g}^k belongs to $\mathcal{H}(2k-2, 2g-2k)$. In particular, the genus of the underlying algebraic curve of H_{2g}^k is indeed g.

The hyperelliptic involution is given by rotating each rhombus $P_i = OA_iB_iC_i$ around its center by $180°$. It has $2g + 2$ fixed points: the points O and A, and a point at the center of each of the $2g$ rhombi. This implies that H_{2g}^k is hyperelliptic.

The automorphism $f_{\mathcal{D}_g}$ of order $4g$ is given by sending each R_i to R_{i+1} via rotation around O by $\pi(2k - 1)/2g$. Since the hyperelliptic curve of genus g with an automorphism of order $4g$ is unique, the 1-form of $H_{2g}^k \in \mathcal{H}(2k-2, 2g-2k)$ is preserved by $f_{\mathcal{D}_g}$ up to scalar, we conclude that $H_{2g}^k = (\mathcal{D}_g, \omega_k)$. Finally, the eigenvalue of the automorphism $f_{\mathcal{D}_g}$ is ξ_{4g}^{2k-1}, which agrees with (2). $\square$

Remark 3.1. Note that apolar and dipolar curves admit an extra automorphism $\iota : (x, y) \mapsto (1/x, iy/x^{g+1})$. Notice that ι^2 is the hyperelliptic involution. Since ι commutes with hyperelliptic involution it descends to the sphere and acts there by switching the North and the South poles. The action of ι on eigenforms is given by $\iota^*(\omega_k) = i \cdot \omega_{g-k+1}$. Therefore translation surfaces represented by S_{2g+2}^k and S_{2g+2}^{g-k+1} (similarly, H_{2g}^k and $H_{2g}^{2(g-k+1)}$) are the same up to rotation. This can be observed in the construction of spingons (and half-spingons) by switching the center of rotation from O to A.

Familiar examples. The collection of described eigenforms contains familiar families of examples of symmetric translation surfaces like regular n-gons and double n-gons. Below we give a more specific correspondence.

Recall that a regular n-gon is a translation surface obtained by identifying opposite sides of a regular n-gon via parallel translations, and a double n-gon is a translation surface obtained by

taking a copy of a regular n-gon rotated by π and identifying the sides with their rotational images via parallel translations. Below we identify which eigenforms and spingons represent regular and double n-gons.

Proposition 3.2. *The following identifications hold:*

1. *Double $(2g+2)$-gon coincides with the eigenforms $(\mathcal{A}_g, \omega_1) = (\mathcal{A}_g, \omega_g)$ also represented by the spingons $S_{2g+2}^1 = S_{2g+2}^g$.*

2. *Regular $(4g+2)$-gon coincides with the eigenform $(\mathcal{P}_g, \omega_g)$ also represented by the spingon S_{2g+1}^g.*

3. *Double $(2g+1)$-gon coincides with the eigenform $(\mathcal{P}_g, \omega_1)$ also represented by the spingon S_{2g+1}^1.*

4. *Regular $4g$-gon coincides with the eigenforms $(\mathcal{D}_g, \omega_1) = (\mathcal{D}_g, \omega_g)$ also represented by the half-spingons $H_{2g}^1 = H_{2g}^g$.*

As an example, on Figures 4 and 5 we demonstrate the necessary regluings to go from regular and double n-gons to spingons. We leave the proof to the reader.
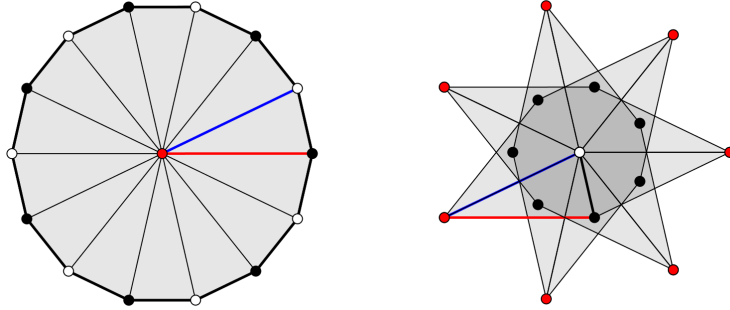
Figure 4: Regular 14-gon is the same as the spingon S_7^3.
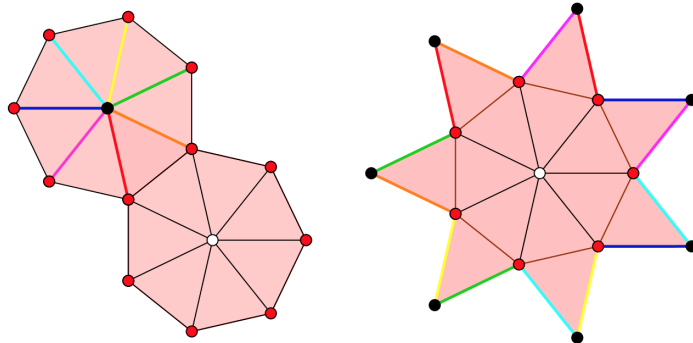
Figure 5: Double 7-gon is the same as the spingon S_7^1.

Examples in low genus. In Table 2 and 3 we give simpler polygonal representations of all eigenforms on hyperelliptic curves $\mathcal{A}_g$, $\mathcal{P}_g$ and $\mathcal{D}_g$ in genus 2 and 3 respectively. Note that when

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identifications of parallel edges are ambiguous or not obvious, the identified edges are labeled by the same color. One can observe that a lot of these eigenforms come from regular n-gons and double n-gons, but not all of them, for example, $(\mathcal{P}_3, \omega_2)$ is new. Another interesting observation is that $(\mathcal{A}_3, \omega_2)$ and $(\mathcal{D}_3, \omega_2)$ are particular examples of *square-tiled cyclic covers* introduced in [FMZ11]. In general, $(\mathcal{A}_{2k-1}, \omega_k)$ and $(\mathcal{D}_{2k-1}, \omega_k)$ describe a series of eigenforms that also happen to be square-tiled cyclic covers $y^{2k} = x^{2k-1}(x-1)(x+1)$ and $y^{2k-1} = x^{2k-2}(x-1)(x+1)$. **verify that**.
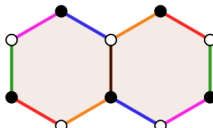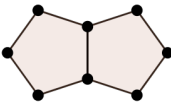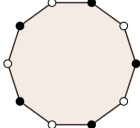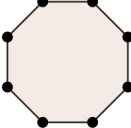
	ω_1	ω_2
$\mathcal{A}_2$	$S_6^1 = S_6^2$	
$\mathcal{P}_2$	S_5^1	S_5^2
$\mathcal{D}_2$	$H_4^1 = H_4^2$	

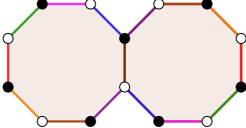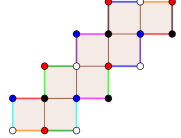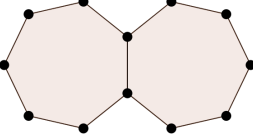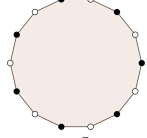Table 2: Eigenforms on hyperelliptic curves with many automorphisms in genus 2.
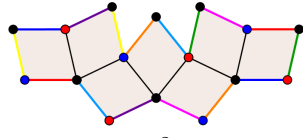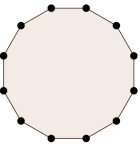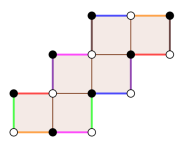
	ω_1	ω_3	ω_2
$\mathcal{A}_3$	$S_8^1 = S_8^3$		S_8^2
$\mathcal{P}_3$	S_7^1	S_7^3	S_7^2
$\mathcal{D}_3$	$H_6^1 = H_6^3$		H_6^2

Table 3: Eigenforms on hyperelliptic curves with many automorphisms in genus 3.

	Polygon	Rhombus angle $\angle O$	Conical points
$(\mathcal{A}_g, \omega_k)$	S_{2g+2}^k	$\dfrac{2\pi k}{2g+2}$	$O : 2\pi k, B : 2\pi k$ $A : 2\pi(g-k+1)$ $C : 2\pi(g-k+1)$
$(\mathcal{P}_g, \omega_k)$	S_{2g+1}^k	$\dfrac{2\pi k}{2g+1}$	$O : 2\pi k, B : 2\pi k$ $A = C : 2\pi(2g-2k+1)$
$(\mathcal{D}_g, \omega_k)$	H_{2g}^k	$\dfrac{\pi(2k-1)}{2g}$	$O = B : 2\pi(2k-1)$ $A = C : 2\pi(2g-2k+1)$

Table 4: Eigenforms are translation surfaces obtained by identifying parallel sides of polygons composed of equal rhombi $R_i = OA_iB_iC_i$.

4 Triangulations and rational billiards

The triangulation of each of the rhombi $R_i = OA_iB_iC_i$ obtained by adding the diagonals gives triangulations of spingons and half-spingons. Therefore each eigenform ω_k on $\mathcal{A}_g$, $\mathcal{P}_g$ and $\mathcal{D}_g$ defines a triangulation of the underlying Riemann surface. In this section we investigate these triangulations.
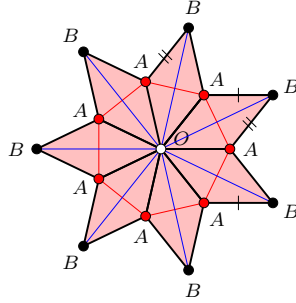

Figure 6: Triangulation $\mathcal{T}$ of $\mathcal{P}_3$ given by the eigenform ω_1.

We begin by setting some notation. Let X be $\mathcal{A}_g$, $\mathcal{P}_g$ or $\mathcal{D}_g$ and N be $2g+2$, $2g+1$ or $2g$ respectively. Let ω_k be the eigenform $x^{k-1}dx/y$. Recall that $\eta : X \to X$ denotes the hyperelliptic involution and $\pi_\eta : X \to \mathbb{P}^1$ is the hyperelliptic cover given by $(x, y) \mapsto x$. The content of this section is summarized by the following proposition:

Proposition 4.1. *Consider the triangulation $\mathcal{T}$ of X given by the subdivision of each rhombus $R_i = OA_iB_iC_i$ of (X, ω_k) by its diagonals. Then:*

1) the triangulation $\mathcal{T}$ maps under the hyperelliptic cover to the triangulation of the sphere $\mathbb{P}^1$, which in x-coordinate has the following vertices: O at 0, A at ∞ and W_i's at all N-th roots

of unity, and the edges belonging to the radial rays passing through all $2N$-th roots of unity (see Figure 7b);

2) *the triangulation $\mathcal{T}$ does not depend on the choice of the eigenform ω_k;*

3) *the translation surface (X, ω_k) is an unfolding of the triangular billiard table with angles:*

$$\mathcal{A}_g : \left(\frac{\pi}{2}, \frac{\pi k}{2g+2}, \frac{\pi(g-k+1)}{2g+2} \right) \tag{4}$$

$$\mathcal{P}_g : \left(\frac{\pi}{2}, \frac{\pi k}{2g+1}, \frac{\pi(2g-2k+1)}{4g+2} \right) \tag{5}$$

$$\mathcal{D}_g : \left(\frac{\pi}{2}, \frac{\pi(2k-1)}{4g}, \frac{\pi(2g-2k+1)}{4g} \right). \tag{6}$$
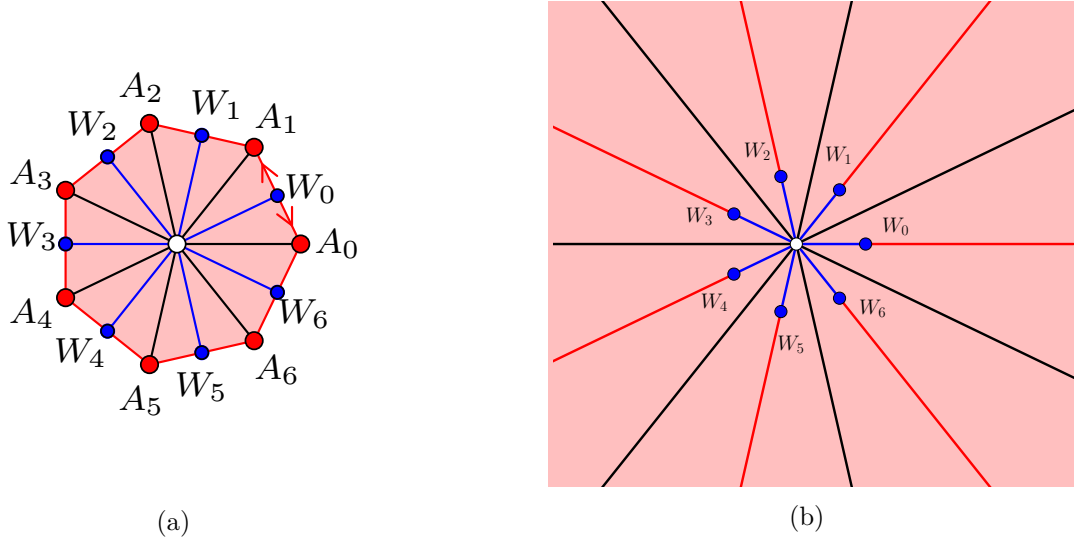

Figure 7: The image of the triangulation $\mathcal{T}$ under the hyperelliptic cover on $\mathbb{P}^1$ (a) in flat coordinate, (b) in x-coordinate.

Proof. We begin by determining the vertices of the triangulation of the sphere in x-coordinate. Let $O \in \mathbb{P}^1$ denote the image of O and $B \in X$, and let $A \in \mathbb{P}^1$ denote the image of A and $C \in X$. The automorphism $f : X \to X$ (see Table 1) has only two orbits of length less than its order: $\{O, B\}$ and $\{A, C\}$. At the same time f descends under π_η to the rotation map $\rho : \mathbb{P}^1 \to \mathbb{P}^1$, given by $x \mapsto \xi_N \cdot x$, whose only two orbits of length less than N are $\{0\}$ and $\{\infty\}$. Therefore $\{0, \infty\} = \{O, A\} \subset \mathbb{P}^1$, and we choose to put O at 0 and A at ∞. Note that for apolar and dipolar curves this choice is due to an extra automorphism of X, which is explained Remark 3.1, and for polar curves this is the only correct choice, since $\pi_\eta : X \to \mathbb{P}^1$ has two preimages over zero. The remaining vertices W_i are the images of the Weierstrass points at the intersections of diagonals of the rhombi, therefore they are the N-th roots of unity.

Now that we have determined the vertices of the triangulation on the sphere, it remains to determine its edges. For this we need to take a further quotient and follow what happens with the triangulation in flat metric.

The flat metric on (X, ω_k) descends under π_η to the flat metric on the sphere given by replacing each rhombus $OA_iB_iC_i$ with a triangle OA_iC_i and folding each side A_iC_i in half (see Figure 7a). We omit some identifications on the picture: each W_iA_i is identified with W_iA_{i+1} by π-rotation around W_i. We will denote the resulting flat metric by $(\mathbb{P}^1, q_k)$, since it is given by some meromorphic quadratic differential q_k.

Furthermore, the quotient of $(\mathbb{P}^1, q_k)$ under the rotation map ρ is an orbifold with a flat metric obtained by identifying sides of two identical right angle triangles (see Figure 8a). The angles can be calculated from the rhombus angle (see Table 4), they are given in (4), (5) and (6). In x-coordinate the vertices O, W and A of the triangles are $0, 1$ and ∞ respectively.
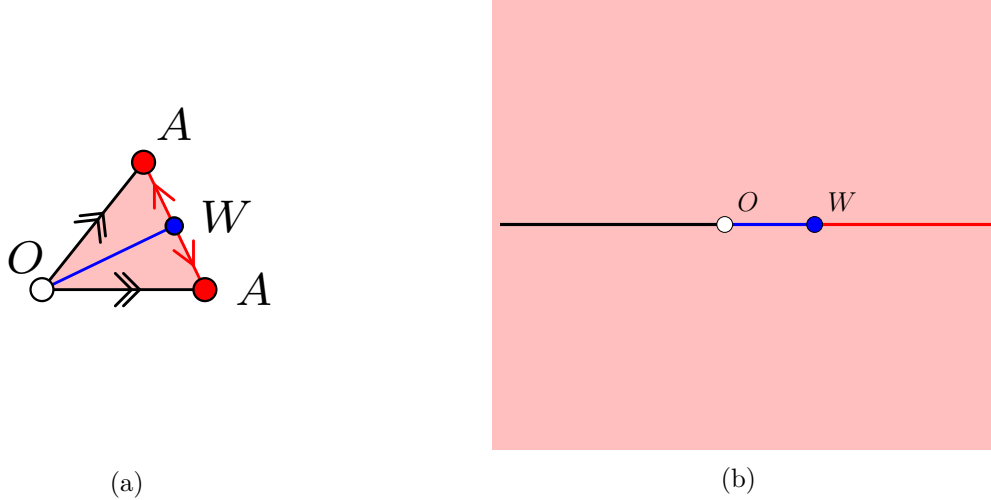

Figure 8: The image of the triangulation $\mathcal{T}$ under the hyperelliptic projection on $\mathbb{P}^1$ (a) in flat coordinate, (b) in x-coordinate.

Now note that there is an anti-holomorphic map given by reflection in OW line which swaps two triangles and fixes the vertices and the edges the triangulation. Such anti-holomorphic map on $\mathbb{P}^1$ is unique and given by $z \mapsto \bar{z}$. Therefore the edges of the triangulation are real line intervals: $AO = (-\infty, 0), OW = (0, 1)$ and $WA = (1, \infty)$ (see Figure 8b). Lifting these edges via the rotation quotient covering map one obtains the triangulation of the sphere as on Figure 7b. This proves the first part of the proposition, and the second part is its immediate corollary.

Finally, for the third part it remains to notice that the unfolding procedure for the discussed right angle triangle follows the pattern of the composed covers from above. $\qquad\square$

References

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