Equations of $\overline{\mathrm{M}}_{0,n}$

Leonid Monin, Julie Rana

Abstract Following work of Keel and Tevelev, we give explicit equations for $\overline{\mathrm{M}}_{0,n}$ for $n = 5$ and $n = 6$. These equations realize $\overline{\mathrm{M}}_{0,n}$ as a subvariety of $\mathbb{P}^{(n-2)!-1}$, embedded by the complete log canonical linear system. We explain the connection to Kapranov's construction of $\overline{\mathrm{M}}_{0,n}$.

1 Introduction

Let $\mathrm{M}_{0,n}$ be the moduli space of n marked genus zero curves, and $\overline{\mathrm{M}}_{0,n}$ its Deligne-Mumford-Knudsen compactification [DM69]. These spaces were first studied in detail by Knudsen and Mumford [KM76, Knu83a, Knu83b].

In [Kap93b], Kapranov constructed $\overline{\mathrm{M}}_{0,n}$ as a Chow quotient of the Grassmannian $G(2,n)$, a construction which allowed him to present $\overline{\mathrm{M}}_{0,n}$ as a sequence of blowups of the projective space $\mathbb{P}^{n-3}$. Following this construction, Keel and Tevelev in [KT09] described an embedding of $\overline{\mathrm{M}}_{0,n}$ into the space of sections of particular characteristic classes on the moduli space. In particular, they prove existence of an embedding $\phi : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$. We obtain a list of equations satisfied by $\phi(\overline{\mathrm{M}}_{0,n})$ in the Cox ring of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$.

Lemma 1.1. *Consider the embedding* $\phi : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$. *Let* $w_0^{(i)}, \ldots, w_i^{(i)}$ *be homogeneous coordinates on the* i^{th} *component. With a choice of coordinates, the image of* ϕ *satisfies the* $\binom{n-1}{4}$ *equations given by the* 2×2 *minors of the matrices*

Leonid Monin
University of Toronto, 40 St. George Street, Toronto, Ontario, e-mail: lmonin@math.toronto.edu

Julie Rana
University of Minnesota, 206 Church St. SE, Minneapolis MN e-mail: jrana@umn.edu

$$\begin{bmatrix} w_0^{(i)}\left(w_0^{(j)}-w_{i+1}^{(j)}\right) & w_1^{(i)}\left(w_1^{(j)}-w_{i+1}^{(j)}\right) & \cdots & w_i^{(i)}\left(w_i^{(j)}-w_{i+1}^{(j)}\right) \\ w_0^{(j)} & w_1^{(j)} & \cdots & w_i^{(j)} \end{bmatrix}$$

for all $1 \leq i < j \leq n-3$.

Conjecture 1.2. Let I_n be the prime ideal in the Cox ring of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ that defines the embedding $\phi(\overline{\mathrm{M}}_{0,n})$. The equations of Lemma 1.1 define the embedding ϕ scheme-theoretically. The ideal is minimally generated by $\binom{n-1}{d+1}$ polynomials of degree d, for $d = 3,4,5,?,n-2$. The degree of I_n is $(2n-7)!!$, the number of trivalent phylogenetic trees on $n-1$ leaves. The lexicographic initial monomial ideals are square-free and Cohen-Macaulay.

We verified Conjecture 1.2 for $n = 5,6,7,8$ using *Macaulay2*. In particular, we have

Proposition 1.3. *For $n = 5,6,7,8$, let $I_{n,3}$ be the ideal in the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^{n-3}$ generated by the 2×2 minors of the matrices*

$$\begin{bmatrix} w_0^{(i)}\left(w_0^{(j)}-w_{i+1}^{(j)}\right) & w_1^{(i)}\left(w_1^{(j)}-w_{i+1}^{(j)}\right) & \cdots & w_i^{(i)}\left(w_i^{(j)}-w_{i+1}^{(j)}\right) \\ w_0^{(j)} & w_1^{(j)} & \cdots & w_i^{(j)} \end{bmatrix}$$

which defines the moduli space $\overline{\mathrm{M}}_{0,n}$ scheme-theoretically is given by the Let I_n be the saturation of $I_{n,3}$ with respect to the ideal $B = \cap_{i=1}^{n-3}(w_0^{(i)},\ldots,w_i^{(i)})$. Then

Proposition 1.4. *The ideal of the embedding of $\overline{\mathrm{M}}_{0,5}$ into $\mathbb{P}^5$ via the κ class is generated by the five equations*

$$x_0x_1 - x_0x_4 + x_2x_3 - x_1x_2,\; x_0x_4 - x_3x_4 + x_3x_5 - x_1x_5,$$

$$-x_1x_3 + x_0x_4,\; -x_2x_3 + x_0x_5,\; -x_2x_4 + x_1x_5.$$

Theorem 1.5. *Let I be the ideal in the multigraded ring $\mathbb{C}[x_0,x_1,y_0,y_1,y_2,z_0,z_1,z_2,z_3]$ generated by the five polynomials*

$$f_1 = y_1z_1z_2 - y_2z_1z_2 + y_2z_1z_3 - y_1z_2z_3$$

$$f_2 = y_0z_0z_2 - y_2z_0z_2 + y_2z_0z_3 - y_0z_2z_3$$

$$f_3 = y_0z_0z_1 - y_1z_0z_1 + y_1z_0z_3 - y_0z_1z_3$$

$$f_4 = x_0z_0z_1 - x_1z_0z_1 + x_1z_0z_2 - x_0z_1z_2$$

$$f_5 = x_0y_0y_1 - x_1y_0y_1 + x_1y_0y_2 - x_0y_1y_2.$$

Then the ideal of $\overline{\mathrm{M}}_{0,6}$ in the ring of sections of $\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^3}(1,1,1)$ is generated by the degree $(2,2,2)$ part of I. Equivalently, the embedding of $\overline{\mathrm{M}}_{0,n}$ in $\mathbb{P}^{23}$ defined by the κ class is generated by homogeneous polynomials of degree $(2,2,2)$ in I and the Segre relations.

Remark 1.6. Using the Chow quotient description of $\overline{\mathrm{M}}_{0,n}$ given by Kapranov [Kap93b], Gibney and Maclagan [GM11] provide equations for $\overline{\mathrm{M}}_{0,5} \subset \mathbb{P}^{21}$. Their work makes use of a combinatorial description of the ideal of relations of the invariant ring of $\overline{\mathrm{M}}_{0,n}$ presented in [HMSV09]. Since many of the listed equations in [GM11] are linear, they have effectively given an embedding of $\overline{\mathrm{M}}_{0,5}$ into $\mathbb{P}^5$. Indeed, using *Macaulay2*, we were able to eliminate variables in such a way that the resulting embedding is a nonsingular variety of dimension 2 given by five quadrics in $\mathbb{P}^5$. Beyond this, it is not yet clear how these equations relate to ours.

The paper is structured as follows. In Section 2, we give some background about the moduli spaces $\overline{\mathrm{M}}_{0,n}$. In Section 3, we describe in detail an embedding $\phi : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ which we will use in later sections to find equations of $\overline{\mathrm{M}}_{0,n}$. In particular, we give a parametrization of the map Φ away from the boundary of $\overline{\mathrm{M}}_{0,n}$ which extends to the full moduli space, provide a geometric interpretation of Φ in the cases $n = 4$ and $n = 5$, and give explicit equations satisfied by $\phi(\overline{\mathrm{M}}_{0,n})$ that are forced by the parametrization. In Sections 4 and 5, we state and prove the main theorems, using cohomological machinery developed in [KT09]. We conclude with Section 6 in which we describe some future directions.

2 Background on the moduli space $\overline{\mathrm{M}}_{0,n}$

In this section we give a brief introduction to the moduli space of pointed rational curves. For a concise introduction to moduli spaces and $\overline{\mathrm{M}}_{0,n}$, we recommend the lectures and lecture notes of Cavalieri [Cav16]. For those new to the theory of stable curves, we recommend in particular the book [HM98].

For $n \geq 3$, the moduli space $\mathrm{M}_{0,n}$ parametrizes ordered n-tuples of distinct points on $\mathbb{P}^1$. We say that two n-tuples $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ are equivalent if there exists a projective transformation $g \in \mathrm{PGL}(2, \mathbb{C})$ such that:

$$(q_1, \ldots, q_n) = (g(p_1), \ldots, g(p_n)).$$

Since a projective transformation can map three fixed points on $\mathbb{P}^1$ to any other three points and is uniquely determined by their image, the dimension of $\mathrm{M}_{0,n}$ equals $n-3$.

The space $\mathrm{M}_{0,n}$ is not compact. Intuitively, this is because the points p_i must all be distinct. There are a number of compactifications of $\mathrm{M}_{0,n}$, including those described by Losev-Manin [LM00] and Keel [Kee92]. But the first and most well-known is $\overline{\mathrm{M}}_{0,n}$, the Deligne-Mumford compactification described explicitly by Kapranov [Kap93a, Kap93b].

Remark 2.1. A general philosophy is that birational models of a given compactified moduli space should provide alternate compactifications which themselves have modular interpretations. From this perspective, describing the birational geometry of $\overline{\mathrm{M}}_{0,n}$ is not only interesting in its own right, but also provides a window into some of the most important research in moduli theory. As a first step in this direction, in [HM82], Harris and Mumford proved that the moduli spaces $\overline{\mathrm{M}}_{g,n}$ are of general type for large enough g. Thus, understanding the birational geometry of $\overline{\mathrm{M}}_{g,n}$ boils down to describing all ample divisors on the moduli space. A long-standing conjecture of Fulton and Faber, the so-called “F-conjecture”, describes the ample cone of $\overline{\mathrm{M}}_{g,n}$; see for example [GKM02] in which the F-conjecture is reduced to the genus 0 case. Notably, in [HK00], Hu and Keel conjectured that $\overline{\mathrm{M}}_{0,n}$ is a Mori Dream Space, a result that would have implied the F-conjecture. This was recently disproved for $n > 133$ by Castravet and Tevelev [CT15]; their techniques were quickly extended to $n > 13$ in [GK14]. Although it was written before this result, we recommend the excellent survey [FS13] for those interested in learning more about birational models and alternative compactifications of $\overline{\mathrm{M}}_{g,n}$.

The moduli space $\overline{\mathrm{M}}_{0,n}$ parametrizes *stable n-pointed rational curves*.

Definition 2.2. A stable rational n-pointed curve is a tuple $(C, p_1, \dots, p_n)$, where:

1. C is a connected curve of arithmetic genus 0 with at most simple nodal singularities;
2. $p_1, \dots, p_n$ are distinct nonsingular points on C;
3. each irreducible component of C has at least three special points (either marked points or nodes).

For the stable curve $(C, p_1, \dots, p_n)$ we define its dual graph by adding a vertex for each irreducible component of C, an edge between two vertices for each node between corresponding components, and a labeled half edge for each marked point adjacent to an appropriate vertex. For C to be arithmetic genus 0 the dual graph must be a tree.

The boundary $\overline{\mathrm{M}}_{0,n} \setminus \mathrm{M}_{0,n}$ is a normal crossing divisor with a natural stratification by the dual graphs. The codimension of the stratum $\delta(\Gamma)$ in $\overline{\mathrm{M}}_{0,n}$ corresponding to the dual graph Γ is one less than the number of vertices of Γ:

$$\mathrm{codim}(\delta(\Gamma)) = \#V(\Gamma) - 1.$$

Therefore, the divisorial components of the boundary of $\overline{\mathrm{M}}_{0,n}$ are given by dual graphs with only two vertices. These graphs correspond to divisions of the set $\{1, \dots, n\}$ into two subsets I and I^c, each of cardinality at least 2. Given such a dual graph, we denote the corresponding irreducible boundary divisor of $\overline{\mathrm{M}}_{0,n}$ by δ_I.

Remark 2.3. One can check that in the case of $\overline{\mathrm{M}}_{0,5}$, two divisors δ_I and δ_J intersect if and only if $I \subset J$, $I \subset J^c$, $J \subset I$, or $J \subset I^c$. In this case, let us define a graph whose vertices correspond to the irreducible boundary divisors in $\overline{\mathrm{M}}_{0,5}$. Two vertices are joined by an edge if the corresponding boundary divisors intersect (see Figure 1). This combinatorial object can be made more concrete using tropical geometry. Indeed, the Petersen graph shown in Figure 1 is the link of the tropical moduli space $M_{0,5}^{\mathrm{trop}}$. For a brief introduction to the moduli space $M_{g,n}^{\mathrm{trop}}$, we recommend the lectures and lecture notes of Melody Chan [Cha16]. For a thorough introduction to tropical geometry, see [MS15].

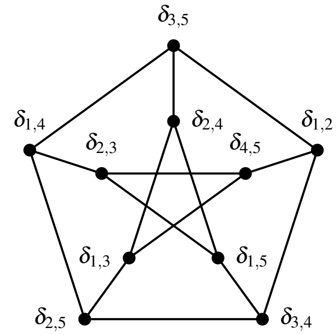

Fig. 1 Petersen graph describing the boundary complex of $\overline{\mathrm{M}}_{0,5}$.

3 The embedding $\overline{\mathrm{M}}_{0,n}$ in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$

In this section, we describe an embedding $\phi : \overline{\mathrm{M}}_{0,n} \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$. Section 3.1 contains a description of the map ϕ and a parametrization of ϕ on the interior of $\overline{\mathrm{M}}_{0,n}$ which extends to the full moduli space. In Section 3.2, we give a geometric interpretation of ϕ in the cases $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{0,5}$. This realizes the moduli space as a pencil of conics in $\mathbb{P}^1 \times \mathbb{P}^2$, the equation for which we give in Theorem 3.6. In Section 3.3, we use the parametrization to list a set of equations satisfied by $\phi(\overline{\mathrm{M}}_{0,n}) \subset \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$.

3.1 The embedding

The embedding ϕ depends on two well-studied maps from $\overline{\mathrm{M}}_{0,n}$, namely the forgetful map $\pi_n : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1}$ and Kapranov's map $\psi_n : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^{n-3}$.

The forgetful map $\pi_n : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1}$ is the morphism given by forgetting the last point of $[C, p_1, \dots p_n]$ and stabilizing the curve, i.e. contracting the components which have less than three special points and remembering the points of intersection.

To describe ψ_n, let $\mathbb{L}_i$ be the line bundle on $\overline{\mathrm{M}}_{0,n}$ whose fiber over a point $[C, p_1, \dots, p_n]$ is the cotangent space of $\mathbb{P}^1$ at p_i. Define $\psi_i = c_1(\mathbb{L}_i)$ to be the first Chern class of $\mathbb{L}_i$. The Kapranov map ψ_n is the rational map given by the linear system $|\psi_n|$. This map was first described in detail by Kapranov [Kap93b], who proved in particular that $\psi_n : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^{n-3}$.

The following theorem is the key to defining $\overline{\mathrm{M}}_{0,n}$ as a subvariety of a product of projective spaces.

Theorem 3.1. *[KT, Cor 2.7] The map* $\Phi = (\pi_n, \psi_n) : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$ *is a closed embedding.*

Corollary 3.2. *We have a closed embedding* $\phi : \overline{\mathrm{M}}_{0,n} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \dots \mathbb{P}^{n-3}$.

Proof. Apply Theorem 3.1 successively.

□

Let's describe the map ϕ of Corollary 3.2 explicitly. Since ϕ is a closed embedding, it is enough to describe it only on the smooth part $\mathrm{M}_{0,n}$ of $\overline{\mathrm{M}}_{0,n}$, which is an open, dense subset of $\overline{\mathrm{M}}_{0,n}$. To this end, consider the restriction of π_n to $\mathrm{M}_{0,n}$ and let $F = \pi_n^{-1}([\mathbb{P}^1, p_1, \ldots, p_{n-1}]) \simeq \mathbb{P}^1 \backslash \{p_1, \ldots, p_{n-1}\}$ be the fiber over a point in $\mathrm{M}_{0,n-1}$. We have the following description of the line bundle $\mathbb{L}_n$ over the fiber F.

Lemma 3.3. *[KT09] We have* $\mathbb{L}_n|_F = \omega_{\mathbb{P}^1}(p_1 + \cdots p_{n-1})$. *In particular,*

$$\psi_n|_F = K_{\mathbb{P}^1} + p_1 + \cdots + p_{n-1}.$$

Using Lemma 3.3, we can give a basis of $H^0(F, K_F + p_1 + \cdots + p_{n-1})$.

Lemma 3.4. *The vector space* $H^0(F, K_F + p_1 + \cdots + p_{n-1})$ *has dimension* $n-2$. *A basis is given by the one-forms*

$$\left\{ \frac{dx}{(x-p_1)(x-p_2)}, \ldots, \frac{dx}{(x-p_1)(x-p_{n-1})} \right\}.$$

Proof. Since the canonical class of $F \simeq \mathbb{P}^1$ is $K_F = -2[\mathrm{pt}]$, the dimension of the space of global sections follows from, for example, Riemann-Roch. Since the $n-2$ one-forms listed have two poles each at different pairs of points, they are linearly independent, and so form a basis.

□

Lemma 3.4 allows us to describe an explicit parametrization of the map $\psi_n|_F$: $F \hookrightarrow \mathbb{P}^{n-3}$, given in the following theorem.

Theorem 3.5. *Let*

$$F \simeq \mathbb{P}^1 \backslash \{p_1, \ldots, p_{n-1}\}$$

be the fiber over the point $[\mathbb{P}^1, p_1, \ldots, p_{n-1}] \in \mathrm{M}_{0,n-1}$. *Then in some coordinates*

$$\psi_n|_F : F \hookrightarrow \mathbb{P}^{n-3}$$

is given parametrically by

$$x \mapsto \left[\frac{p_1 - p_2}{x - p_2} : \cdots : \frac{p_1 - p_{n-1}}{x - p_{n-1}} \right]$$

Proof. Using a properly rescaled basis from Lemma 3.4, we can write the map parametrically as

$$x \mapsto \left[\frac{p_1 - p_2}{(x-p_1)(x-p_2)} : \ldots : \frac{p_1 - p_{n-1}}{(x-p_1)(x-p_{n-1})} \right]$$

Multiplying through by $x - p_1$ gives the result. Note that under such a parametrization, the points $p_2, \ldots, p_{n-1}$ in F map to the coordinate points on $\mathbb{P}^{n-3}$. The point $x = p_1 \in F$ maps to $[1 : \ldots : 1] \in \mathbb{P}^{n-3}$.

$\square$

By Corollary 3.2, we can extend the map ψ_n uniquely to all of $\overline{\mathrm{M}}_{0,n}$. We note that by our choice of basis, the points $p_1,\ldots,p_{n-1}\in\bar{F}$ map to points of $\mathbb{P}^{n-3}$ in general position. Thus, $\psi_n(\bar{F})$ is a degree $n-3$ curve in $\mathbb{P}^{n-3}$, i.e. a (generically) smooth rational normal curve passing through these $n-1$ fixed points in $\mathbb{P}^{n-3}$.

With this choice, we take the parameter x to be p_n, and thus the map

$$\phi:\overline{\mathrm{M}}_{0,n}\to\mathbb{P}^1\times\cdots\times\mathbb{P}^{n-3},$$

is given in the i^{th} component by

$$[C,p_1,\ldots,p_n]\mapsto\left[\frac{p_1-p_2}{p_{i+3}-p_2}:\cdots:\frac{p_1-p_{i+2}}{p_{i+3}-p_{i+2}}\right].$$

3.2 Geometric description of ϕ for $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{0,5}$

Let's describe the embedding $\phi:\overline{\mathrm{M}}_{0,n}\to\mathbb{P}^1\times\cdots\times\mathbb{P}^{n-3}$ in detail for $n=4$ and $n=5$. For $\overline{\mathrm{M}}_{0,4}$, away from p_2,p_3,p_4, the embedding from the previous section is given by

$$\phi:[\mathbb{P}^1,p_1,\ldots,p_4]\mapsto\left[\frac{p_1-p_2}{p_4-p_2}:\frac{p_1-p_3}{p_4-p_3}\right],$$

Thus, away from the boundary of $\overline{\mathrm{M}}_{0,4}$, we see that the map ϕ is an isomorphism mapping a 4-tuple of distinct points on $\mathbb{P}^1$ to their cross-ratio.

The boundary of $\overline{\mathrm{M}}_{0,4}$ consists of three points: $\delta_{1,2}$, $\delta_{1,3}$, and $\delta_{1,4}$ (see Fig. 2). By taking limits $p_1\to p_2$, $p_1\to p_3$, and $p_1\to p_4$, respectively, we see that these boundary points map under ϕ to the points $[0:1],[1:0]$, and $[1:1]$ respectively.

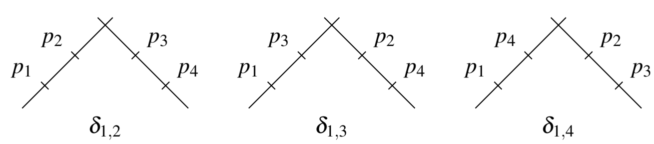

Fig. 2 The boundary divisors of $\overline{\mathrm{M}}_{0,4}$.

In the case of $\overline{\mathrm{M}}_{0,5}$ the embedding from the previous section has the form

$$\phi:\overline{\mathrm{M}}_{0,5}\to\mathbb{P}^1\times\mathbb{P}^2\simeq\overline{\mathrm{M}}_{0,4}\times\mathbb{P}^2.$$

$$[\mathbb{P}^1,p_1,\ldots,p_5]\mapsto\left(\left[\frac{p_1-p_2}{p_4-p_2}:\frac{p_1-p_3}{p_4-p_3}\right],\left[\frac{p_1-p_2}{p_5-p_2}:\frac{p_1-p_3}{p_5-p_3}:\frac{p_1-p_4}{p_5-p_4}\right]\right).$$

Note also that the forgetful map π_5 is given by restricting to $\overline{M}_{0,5}$ the projection of $\overline{M}_{0,4} \times \mathbb{P}^2$ onto the first factor. Thus, the fiber of this projection over any point in $M_{0,4} \simeq \mathbb{P}^1 \setminus \{[0:1],[1:0],[1:1]\}$ is a smooth conic passing through four fixed points $[1:0:0]$,$[0:1:0]$,$[0:0:1]$, and $[1:1:1]$ in $\mathbb{P}^2$. The fibers over the other points $[0:1],[1:0],[1:1]$ are precisely the three singular conics passing through these four fixed points. In particular, the boundary divisors of $\overline{M}_{0,5}$ take a nice form in this description: six are the components of three singular conics, and the remaining four are the fibers $\mathbb{P}^1 \times \{[1:0:0]\}$, $\mathbb{P}^1 \times \{[0:1:0]\}$, $\mathbb{P}^1 \times \{[0:0:1]\}$, and $\mathbb{P}^1 \times \{[1:1:1]\}$ (see Fig. 3).

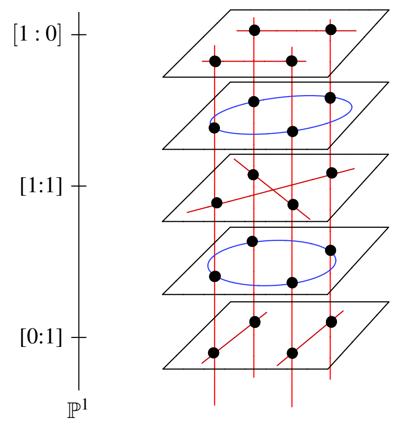

Fig. 3 The embedding $\phi : \overline{M}_{0,5} \to \mathbb{P}^1 \times \mathbb{P}^2$. The red lines correspond to boundary divisors.

Since $\overline{M}_{0,5}$ is the pencil of conics in $\mathbb{P}^2$ passing through these given four points, we can easily write down an embedding of $\overline{M}_{0,5}$ in $\mathbb{P}^1 \times \mathbb{P}^2$. It is given by the equation

$$x_0 y_1 (y_0 - y_2) - x_1 y_0 (y_1 - y_2),$$

where the x_i are coordinates on $\mathbb{P}^1$ and the y_i are coordinates on $\mathbb{P}^2$.

From this picture, we can also see that the map $\psi : \overline{M}_{0,5} \to \mathbb{P}^2$ is a blowdown map. Indeed, since there is unique conic passing through any five points in $\mathbb{P}^2$, the map $\psi : \overline{M}_{0,5} \to \mathbb{P}^2$ is 1-to-1 outside of the points $[1:0:0],[0:1:0],[0:0:1]$, and $[1:1:1]$. And the fibers over these points are all isomorphic to $\mathbb{P}^1$.

With this description, we have

Theorem 3.6. *With the choice of coordinates above, the ideal which defines the moduli space* $\overline{M}_{0,5}$ *as a projective variety in* $\mathbb{P}^1_{[x_0:x_1]} \times \mathbb{P}^2_{[y_0:y_1:y_2]}$ *is generated by*

$$x_0 y_1 (y_0 - y_2) - x_1 y_0 (y_1 - y_2).$$

Proof. This is a direct computation consisting of checking that the variety described by this ideal in the multigraded ring $\mathbb{C}[x_0,x_1,y_0,y_1,y_2]$ is two-dimensional, smooth, and irreducible. We verified this using *Macaulay2*. □

For the following corollary, we define the κ class on $\overline{\mathrm{M}}_{0,n}$. Let $\sigma : U \to \overline{\mathrm{M}}_{0,n}$ be the universal curve over $\overline{\mathrm{M}}_{0,n}$ and let ω be the relative dualizing sheaf. Then the κ class on $\overline{\mathrm{M}}_{0,n}$ is by definition the pushforward of the first Chern class of ω, i.e. $\kappa = \sigma_*(c_1(\omega))$.

Now let $K_{\overline{\mathrm{M}}_{0,n}}$ be the canonical class on $\overline{\mathrm{M}}_{0,n}$ and δ_I the classes of the boundary divisors. In [KT09], Keel and Tevelev prove that $\kappa \sim K_{\overline{\mathrm{M}}_{0,n}} + \sum \delta_I$. It follows that the κ class is very ample. They go on to prove that the composition of the embedding ϕ with the Segre embedding $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3} \hookrightarrow \mathbb{P}^{(n-2)!-1}$ is precisely the embedding of $\overline{\mathrm{M}}_{0,n}$ via the κ class. Thus, we have the following

Corollary 3.7. *The ideal of the embedding of $\overline{\mathrm{M}}_{0,5}$ into $\mathbb{P}^5$ via the κ class is generated by the five quadrics*

$$x_0x_1 - x_0x_4 + x_2x_3 - x_1x_2,\; x_0x_4 - x_3x_4 + x_3x_5 - x_1x_5,$$

$$-x_1x_3 + x_0x_4,\; -x_2x_3 + x_0x_5,\; -x_2x_4 + x_1x_5$$

Proof. The first two equations are given by first multiplying the equation of Theorem 3.6 by x_0 and x_1 to obtain equations homogeneous of the same degree in x_i and y_j, then mapping into $\mathbb{P}^5$ by the Segre embedding. The final three are the Segre relations. By [KT09], the resulting map from $\overline{\mathrm{M}}_{0,5} \to \mathbb{P}^5$ is given by the κ class. □

3.3 Known equations

We give a list of equations contained in the ideal of $\overline{\mathrm{M}}_{0,n}$ as a subvariety of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$. Let $w_0^{(i)}, \ldots, w_i^{(i)}$ be homogeneous coordinates on the i^{th} component of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$. Following Theorem 3.5, the embedding $\phi : \overline{\mathrm{M}}_{0,n} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ is given in coordinates by

$$w_j^{(i)} = \frac{p_{1,j+2}}{p_{i+3,j+2}},$$

where $p_{i,j} = p_i - p_j$.

Lemma 3.8. *The image of ϕ in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ satisfies the $\binom{n-1}{4}$ equations given by the 2×2 minors of the matrices*

$$\begin{bmatrix} w_0^{(i)}\left(w_0^{(j)} - w_{i+1}^{(j)}\right) & w_1^{(i)}\left(w_1^{(j)} - w_{i+1}^{(j)}\right) & \cdots & w_i^{(i)}\left(w_i^{(j)} - w_{i+1}^{(j)}\right) \\ w_0^{(j)} & w_1^{(j)} & \cdots & w_i^{(j)} \end{bmatrix}$$

for all $1 \le i < j \le n-3$.

Proof. The proof is direct calculation. Choose the r^{th} and s^{th} columns. We show that

$$\left(w_r^{(j)} - w_{i+1}^{(j)}\right) w_s^{(j)} w_r^{(i)} = \left(w_s^{(j)} - w_{i+1}^{(j)}\right) w_r^{(j)} w_s^{(i)},$$

Indeed, the following equalities hold:

$$\begin{aligned} w_r^{(j)} - w_{i+1}^{(j)} &= \frac{p_{1,r+2}}{p_{j+3,r+2}} - \frac{p_{1,i+3}}{p_{j+3,r+3}} \\ &= \frac{p_{1,r+2}p_{j+3,i+3} - p_{1,i+3}p_{j+3,r+2}}{p_{j+3,r+2}p_{j+3,i+3}} \\ &= \frac{p_{1,j+3}p_{r+2,i+3}}{p_{j+3,r+2}p_{j+3,i+3}}. \end{aligned}$$

So we have:

$$\begin{aligned} \left(w_r^{(j)} - w_{i+1}^{(j)}\right) w_s^{(j)} w_r^{(i)} &= \frac{p_{1,j+3}p_{r+2,i+3}}{p_{j+3,r+2}p_{j+3,r+2}} \cdot \frac{p_{1,s+2}}{p_{j+3,s+2}} \cdot \frac{p_{1,r+2}}{p_{i+3,r+2}} \\ &= -\frac{p_{1,j+3}p_{1,s+2}p_{1,r+2}}{p_{j+3,r+2}p_{j+3,s+2}p_{j+3,i+3}}. \end{aligned}$$

A similar calculation gives that $\left(w_s^{(j)} - w_{i+1}^{(j)}\right) w_r^{(j)} w_s^{(i)}$ boils down to the same expression.

We can see that there are $\binom{n-1}{4}$ of these equations as follows. For each $i \in \{1,\dots,n-4\}$, we obtain $n-3-i$ matrices, each with $i+1$ columns. Thus, the number of 2×2 minors is given by

$$\sum_{i=1}^{n-4}(n-3-i)\binom{i+1}{2}.$$

Using the well-known expressions for $\sum i$, $\sum i^2$ and $\sum i^3$, one quickly checks that this is equal to $\binom{n-1}{4}$.

□

Conjecture 3.9. The equations in Lemma 3.8 define the embedding ϕ scheme-theoretically.

Conjecture 3.10. The number of polynomials of degree d in a reduced Groebner basis of the ideal of $\phi(\overline{\mathrm{M}}_{0,n})$ is $\binom{n-1}{d+1}$.

Computations in *Macaulay2* support Conjecture 3.10 for $4 \le n \le 8$. In particular, upon observation of the equations in degree d, we expect that these polynomials may be counted as follows.

Let G be a reduced Groebner basis of I_n, the ideal of $\phi\overline{\mathrm{M}}_{0,n}$, and let G_d be the subset of G consisting of polynomials of degree d. For a fixed degree d, fix a projective space $\mathbb{P}^i$ and choose two variables $w_{j_1}^{(i)}, w_{j_2}^{(i)}$. For each choice of $d-2$ of the remaining projective spaces $\mathbb{P}^{i+1},\dots,\mathbb{P}^{n-3}$, we claim that there is a unique polynomial in G_d with degree 1 all variables other than the last occurring projective space, in which the polynomial has degree 2.

Explicit example (with $n = 7$, $d = 4$). Also a comment about how any polynomial of degree d in the ideal I_n (for $d < n-2$ seems to have the same structure as the unique polynomial of degree d in I_{d+2}.

If the polynomials of degree d can be counted as above, then Conjecture 3.10 will be true, upon application of the following

Lemma 3.11.

$$\sum_{i=1}^{n-3} \binom{n-3-i}{d-2}\binom{i+1}{2} = \binom{n-1}{d+1}$$

Proof. We rewrite the left hand side as a hypergeometric series and apply the Chu-Vandermonde Identity, see e.g. [Roy87]. Changing to index of the left hand side to $k = i-1$ and letting C_k be the k^{th} term in this series, we have

$$\frac{C_{k+1}}{C_k} = \frac{(k+d+2-n)(k+3)}{(n-k-4)(k+1)}.$$

Thus, using the notation of [Roy87], the left hand side of the above equation can be written as

$$\sum_{k=0}^{n-4} \binom{n-4-k}{d-2}\binom{k+2}{2} = \binom{n-4}{d-2} {}_2F_1\begin{pmatrix} -(n-d-2) & 3 \\ & 4-n \end{pmatrix}.$$

The notation...

The Chu-Vandermonde Identitiy [Roy87, Equation (2.7)] gives

$$\begin{aligned}
\binom{n-4}{d-2} {}_2F_1\begin{pmatrix} -(n-d-2) & 3 \\ & 4-n \end{pmatrix} &= \binom{n-4}{d-2}\frac{(1-n)(2-n)\cdots(-(d+2))}{(4-n)(5-n)\cdots(-(d-1))} \\
&= \binom{n-4}{d-2}\frac{(n-1)(n-2)\cdots(d+2)}{(n-4)(n-5)\cdots(d-1)} \\
&= \binom{n-4}{d-2}\frac{(n-1)!}{(d+1)!}\frac{(d-2)!}{(n-4)!} \\
&= \frac{(n-1)(n-2)(n-3)(n-4)!}{(d+1)(d)(d-1)(d-2)!(n-4-d+2)!} \\
&= \binom{n-1}{d+1}.
\end{aligned}$$

4 Tools to compute the number of equations for $\overline{\mathrm{M}}_{0,n}$ in $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$

In this section, we use cohomological techniques to prove that the equations of Lemma 3.8 generate the respective ideals in the homogeneous ring of sections of $\mathscr{O}(1,\dots,1)$. At first, it appears we need to compute the number of equations in every multidegree. However, if we are interested only in the ideal of $\overline{\mathrm{M}}_{0,n}$ after the Segre embedding $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3} \to \mathbb{P}^{(n-2)!-1}$, the theorem below allows us to check only one multidegree.

Theorem 4.1. *[KT09] The ideal of $\Phi(\overline{\mathrm{M}}_{0,n}) \subset \mathbb{P}^{(n-2)!-1}$ is generated by quadrics. Equivalently the ideal of $\phi(\overline{\mathrm{M}}_{0,n})$ in the homogeneous ring of sections of $\mathscr{O}(1,\dots,1)$ on $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ is generated by polynomials of multidegree $(2,\dots,2)$.*

This section will be organized as follows. Section 4.1 contains a description of tools we use from [KT09], concluding with Lemma 4.4 which describes the number of equations in the ideal of the embedding of $\overline{\mathrm{M}}_{0,n}$ in $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$ of given bidegree. In section 4.2 we illustrate the use of these tools in the case $n = 5$.

4.1 Consequence of a resolution of the structure sheaf of $\overline{\mathrm{M}}_{0,n}$

Let V_{ψ_n} be the vector bundle on $\overline{\mathrm{M}}_{0,n}$ defined by the exact sequence

$$0 \to V_{\psi_n} \to H^0(\overline{\mathrm{M}}_{0,n}, \psi_5) \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \to \psi_n \to 0. \tag{1}$$

Given $\mathscr{F}$ and $\mathscr{G}$ vector bundles on projective varieties X and Y, respectively, we let σ_i be projection of $X \times Y$ onto the i^{th} coordinate. We define $\mathscr{F} \boxtimes \mathscr{G}$ to be the vector bundle $\sigma_1^* \mathscr{F} \otimes \sigma_2^* \mathscr{G}$ on $X \times Y$.

Let $\Phi = (\pi_n, \psi_n) : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$. By [KT09], we have the following resolution of the structure sheaf of $\Phi(\overline{\mathrm{M}}_{0,n})$ in $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$:

$$0 \to \mathscr{M}_n^{n-4} \boxtimes \mathscr{O}(3-n) \to \cdots \to \mathscr{M}_n^1 \boxtimes \mathscr{O}(-2) \to \mathscr{O}_{\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}} \to \Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \to 0, \tag{2}$$

where $\mathscr{M}_n^p = R^1 \pi_* (\wedge^{p+1} V_{\psi_n})$.

This exact sequence of sheaves is the main tool we will use to compute the number of the equations of a certain multidegree. Indeed, by tensoring (2) with the correct line bundles one can compute the expected number of equations for $\overline{\mathrm{M}}_{0,n} \subset \overline{\mathrm{M}}_{0,n} \times \mathbb{P}^{n-3}$ in particular degrees.

To gain control of the sheaves in (2), we have

Theorem 4.2. *[KT09] There exists a vector bundle Q on $\overline{\mathrm{M}}_{0,n}$ and exact sequences*

$$0 \to \pi^* \mathscr{M}_{n-1}^p \to \mathscr{M}_n^p \to Q \to 0 \tag{3}$$

$$0 \to V_{\psi_n} \to Q \to \mathscr{M}_n^{p-1} \to 0 \tag{4}$$

Remark 4.3. The vector bundle Q is defined explicitly in [KT09], but since we won't use it in our computations, we omit it to simplify the exposition.

Theorem 4.2 together with the resolution of the structure sheaf of $\Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}}$ given in Equation (2) allow us to determine the number of equations of $\overline{\mathrm{M}}_{0,n}$ in $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$ of bidegree $(a,2)$, using the exact sequence given in the following lemma.

Lemma 4.4. *For all integers $n \geq 5$ and $a > 0$, the ideal defining $\overline{\mathrm{M}}_{0,n}$ as a subvariety of $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$ contains exactly $h^0(\overline{\mathrm{M}}_{0,n-1}, \mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a})$ linearly independent equations of bidegree $(a,2)$. Additionally, we have the short exact sequence*

$$0 \to H^0(\overline{\mathrm{M}}_{0,n-1}, \pi^* \mathscr{M}_{n-2}^1 \otimes \kappa_{n-1}^{\otimes a}) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \to$$

$$\to H^0(\overline{\mathrm{M}}_{0,n-1}, V_{\psi_{n-2}} \otimes \kappa_{n-1}^{\otimes a}) \to 0.$$

Proof. We use the resolution of the structure sheaf of $\Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}}$ given by the exact sequence (2). Tensoring this with $\kappa_{n-1}^{\otimes a} \boxtimes \mathscr{O}(2)$, we obtain an exact sequence on $\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}$, which by Theorem 4.1 is the number of linearly independent equations of the given degree.

$$0 \to (\mathscr{M}_{n-1}^{n-4} \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(1-n) \to (\mathscr{M}_{n-1}^{n-3} \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(2-n) \to \cdots$$

$$\to (\mathscr{M}_{n-2}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(-1) \to (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O} \to \kappa_{n-1}^{\otimes a} \boxtimes \mathscr{O}(2) \to$$

$$\to \Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \otimes (\kappa^{\otimes a} \boxtimes \mathscr{O}(2)) \to 0.$$

By the Künneth Formula, and since $\mathscr{O}(k)$ is acyclic for $1-n \leq k \leq -1$, we have

$$H^i(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^{k+2n-5} \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}(k)) = 0$$

for all $1-n \leq k \leq -1$ and $i \geq 0$. Moreover, by Lemma 6.5 of [KT09], we have

$$H^1(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}) = 0.$$

Thus, we obtain a short exact sequence in cohomology:

$$0 \to H^0(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O}) \to H^0(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, \kappa_{n-1}^{\otimes a} \boxtimes \mathscr{O}(2)) \xrightarrow{\tau}$$

$$\xrightarrow{\tau} H^0(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^{n-3}, \Phi_* \mathscr{O}_{\overline{\mathrm{M}}_{0,n}} \otimes (\kappa^{\otimes a} \boxtimes \mathscr{O}(2))) \to 0.$$

In particular, the number of equations in this ideal of bidegree $(a,2)$ is given by the dimension of the kernel of τ: $H^0(\overline{\mathrm{M}}_{0,n-1} \times \mathbb{P}^3, (\mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a}) \boxtimes \mathscr{O})$. By the Künneth Formula, this vector space is isomorphic to $H^0(\overline{\mathrm{M}}_{0,n-1}, \mathscr{M}_{n-1}^1 \otimes \kappa_{n-1}^{\otimes a})$.

Using the short exact sequences (3) and (4), and noting that $\mathscr{M}_n^0 = 0$ for all n, we have the short exact sequence

$$0 \to \pi^*\mathscr{M}^1_{n-2} \to \mathscr{M}^1_{n-1} \to V_{\psi_{n-2}} \to 0.$$

We tensor with $\kappa_{n-1}^{\otimes a}$. By Lemma 6.5 in[KT09], we have

$$H^i(\overline{\mathrm{M}}_{0,n-1}, \pi^*\mathscr{M}^1_{n-2} \otimes \kappa_{n-1}^{\otimes a}) = 0$$

for $i > 0$. Taking cohomology gives the short exact sequence.

□

4.2 *The precise number of equations for* $\overline{\mathrm{M}}_{0,5}$

For a reality check, recall that in Theorem 3.6 we found that there is one equation of bidegree $(1,2)$ defining $\overline{\mathrm{M}}_{0,5}$ as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^2 = \overline{\mathrm{M}}_{0,4} \times \mathbb{P}^2$. Furthermore, Lemma 3.8 gives precisely one equation that $\overline{\mathrm{M}}_{0,5}$ satisfies after embedding into $\mathbb{P}^1_{[x_0:x_1]} \times \mathbb{P}^2_{[y_0:y_1:y_2]}$, namely

$$x_0y_1(y_0 - y_2) - x_1y_0(y_1 - y_2),$$

which indeed coincides with the equation found in Theorem 3.6. On the other hand, Theorem 4.4 tells us that there are $h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}^1_4 \otimes \kappa_4)$ equations of bidegree $(1,2)$ defining $\overline{\mathrm{M}}_{0,5}$ as a subvariety of $\overline{\mathrm{M}}_{0,4} \times \mathbb{P}^2$. Let's compute the dimension of this vector space using Lemma 4.4 and the exact sequences of sheaves described in the previous section.

Theorem 4.5. *We have* $h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}^1_4 \otimes \kappa_4) = 1$ *and* $h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}^1_4 \otimes \kappa_4^{\otimes 2}) = 2$.

Proof. Under the isomorphism $\overline{\mathrm{M}}_{0,4} \simeq \mathbb{P}^1$, we have $\kappa_4 = \mathscr{O}_{\mathbb{P}^1}(1)$. We show that on $\overline{\mathrm{M}}_{0,4}$, we have $\mathscr{M}^1_4 = \psi_4^{-1} = \mathscr{O}_{\mathbb{P}^1}(1)$.

Using the exact sequences (3) and (4), we note that since $\mathscr{M}^1_3 = 0$, the exact sequence (3) becomes

$$0 \to 0 \to \mathscr{M}^1_4 \to Q \to 0$$

and tells us that $\mathscr{M}^1_4 \simeq Q$. As $\mathscr{M}^0_n = 0$ for any n the sequence (4) boils down to

$$0 \to V_{\psi_4} \to Q \to 0 \to 0$$

and therefore $\mathscr{M}^1_4 \simeq Q \simeq V_{\psi_4}$. We can determine the bundle V_{ψ_4} by analyzing the exact sequence (1). On $\overline{\mathrm{M}}_{0,4}$ this is

$$0 \to V_{\psi_4} \to H^0(\overline{\mathrm{M}}_{0,4}, \psi_4) \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,4}} \to \psi_4 \to 0.$$

Taking determinants gives the following equality of line bundles

$$\mathscr{O}_{\overline{\mathrm{M}}_{0,4}} = \det\left(H^0(\overline{\mathrm{M}}_{0,4}, \psi_4) \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,4}}\right) = \det(V_{\psi_4}) \otimes \det(\psi_4) = V_{\psi_4} \otimes \psi_4,$$

In particular, $\mathscr{M}_4^1 \simeq V_{\psi_4}$ is a line bundle dual to ψ_4. To bring the calculation to the conclusion we recall that $\overline{\mathrm{M}}_{0,4} \simeq \mathbb{P}^1$ and $\psi_4 \simeq \mathscr{O}_{\mathbb{P}^1}(1)$, so the dimension of $H^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(2))$ is given by

$$h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(2)) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1) \otimes \mathscr{O}_{\mathbb{P}^1}(2)) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) = 2.$$

Similarly we compute the dimension of $H^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(1))$ to be

$$h^0(\overline{\mathrm{M}}_{0,4}, \mathscr{M}_4^1 \otimes \mathscr{O}_{\mathbb{P}^1}(1)) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-1) \otimes \mathscr{O}_{\mathbb{P}^1}(1)) = h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 1.$$

□

5 The precise number of equations for $\overline{\mathrm{M}}_{0,6}$

In this section we study the ideal of $\overline{\mathrm{M}}_{0,6}$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. In 5.1 we give a list of explicit equations satisfied by the embedding of $\overline{\mathrm{M}}_{0,6}$ into $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. We continue on to formulate the main result of this section, Theorem 5.1. In 5.2 we apply the cohomological techniques developed in Section 4.1 to compute the dimensions of particular degrees of the ideal of $\overline{\mathrm{M}}_{0,6}$ and continue on to prove Theorem 5.1.

5.1 List of equations

In the case $n = 6$, we obtain from Lemma 3.8 five equations satisfied by $\overline{\mathrm{M}}_{0,6}$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. These are

$$f_1 = y_1z_1z_2 - y_2z_1z_2 + y_2z_1z_3 - y_1z_2z_3$$
$$f_2 = y_0z_0z_2 - y_2z_0z_2 + y_2z_0z_3 - y_0z_2z_3$$
$$f_3 = y_0z_0z_1 - y_1z_0z_1 + y_1z_0z_3 - y_0z_1z_3$$
$$f_4 = x_0z_0z_1 - x_1z_0z_1 + x_1z_0z_2 - x_0z_1z_2$$
$$f_5 = x_0y_0y_1 - x_1y_0y_1 + x_1y_0y_2 - x_0y_1y_2.$$

The main result of this section is

Theorem 5.1. *Let I be the ideal generated by the equations $f_1, \ldots, f_5$ and let*

$$\phi : \overline{\mathrm{M}}_{0,6} \to \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$$

be the embedding from Corollary 3.2. Then the ideal of $\phi(\overline{\mathrm{M}}_{0,6})$ in the homogeneous ring of sections of $\mathscr{O}(1, \ldots, 1)$ is generated by polynomials of tridegree $(2,2,2)$ from the ideal I. Equivalently, the embedding $\Phi(\overline{\mathrm{M}}_{0,n})$ in $\mathbb{P}^{23}$ defined by the κ class *is generated by homogeneous polynomials of degree* $(2,2,2)$ *in I and the Segre relations.*

Note that I is not an ideal of $\phi(\overline{\mathrm{M}}_{0,6})$ in the ring of multihomogeneous functions. Indeed, using *Macaulay2* we were able to find one more equation for $\overline{\mathrm{M}}_{0,6}$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. This is

$$f = x_0y_0z_1z_2 - x_0y_2z_1z_2 - x_0y_0z_1z_3 + x_1y_0z_1z_3 + x_0y_2z_1z_3 - x_1y_0z_2z_3.$$

We checked by hand that this polynomial vanishes under the parametrization of Theorem 3.5.

Let J be the ideal generated by the adding polynomial f to the ideal I.

Proposition 5.2. *The ideal I is properly contained in the ideal J, but the parts of homogeneous degrees* $(2,2,2)$ *of ideals I and J coincide.*

Proof. Let $I_{(a,b,c)}$, respectively $J_{(a,b,c)}$, be the vector space of polynomials of tridegree (a,b,c) in I, respectively J. By multiplying the polynomials $f_1,\dots,f_5$ by all monomials of the correct bidegree, one can obtain a basis of the vector space $I_{(a,b,c)}$. Computing the dimension of $I_{(a,b,c)}$ then becomes a question of determining which of the resulting polynomials are redundant. Because these computations quickly become unwieldy, we used *Macaulay2* to show that

$$\dim I_{(1,1,2)} = 9 < \dim J_{(1,1,2)} = 10.$$

But

$$\dim I_{(2,2,2)} = \dim J_{(2,2,2)} = 55,$$

which finishes the proof.

□

Remark 5.3. As we will see, the second part of Proposition 5.2 implies that the entire homogeneous parts of the ideals I and J coincide. Corollary 5.7 in the next section shows that these ideals contain the correct number of homogeneous equations of tridegree $(2,2,2)$.

Remark 5.4. Using *Macaulay2*, we found that the ideal J is prime, but contains a 1-dimensional projective singular locus. This leads us to suspect that although we have the correct equations in degree $(2,2,2)$, we are missing some equations in nonhomogeneous degrees. Still, as we will see below, these equations are enough to give an embedding into $\mathbb{P}^{23}$.

5.2 The proof of Theorem 5.1

We begin with two lemmas.

Lemma 5.5. *We have* $h^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) = 24$ *and* $h^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5) = 11$.

Proof. We have on $\overline{\mathrm{M}}_{0,5}$ the short exact sequence

$$0 \to V_{\psi_4} \to \mathbb{C}^3 \otimes \mathscr{O}_{\overline{\mathrm{M}}_{0,5}} \to \psi_4 \to 0.$$

We tensor with $\kappa_5^{\otimes 2}$. Taking the long exact sequence in cohomology, noting that $H^i(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) = 0$ for $i > 0$ (by Lemma 6.5 in [KT09]), we obtain the short exact sequence

$$0 \to H^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5^{\otimes 2}) \to 0.$$

We prove that the dimensions of the middle and last terms have dimensions 48 and 24, respectively.

For the middle term, we note that any global section of $\mathbb{C}^3 \otimes \kappa_5^{\otimes 2}$ is of the form $\alpha \otimes \beta$ where $\alpha \in \mathbb{C}^3$ and $\beta \in H^0(\overline{\mathrm{M}}_{0,5}, \kappa_5^{\otimes 2})$. The κ class is given by $\kappa_5 = K_{\overline{\mathrm{M}}_{0,5}} + \sum \delta_I$ where the sum is taken over all boundary divisors δ_I of $\overline{\mathrm{M}}_{0,5}$. Using that $\overline{\mathrm{M}}_{0,5} \simeq \mathrm{Bl}_{q_1,\ldots,q_4}\mathbb{P}^2$, we let $\sigma : \mathrm{Bl}_{q_1,\ldots,q_4}\mathbb{P}^2 \to \mathbb{P}^2$ be the blowup map, $E_1, \ldots, E_4$ the exceptional divisors, and $L_{i,j} \subset \mathrm{Bl}_4\mathbb{P}^2$ the proper transform of the line passing through q_i and q_j on $\mathbb{P}^2$. We have the linear equivalence

$$\kappa_5 \sim (\sigma^*(K_{\mathbb{P}^2}) + \sum_{i=1}^{4} E_i) + (\sum_{i=1}^{4} E_i + \sum_{1 \le i < j \le 4} L_{i,j}).$$

Since $L_{i,j} \sim \sigma^* H - E_i - E_j$, where H is the class of a hyperplane section on $\mathbb{P}^2$, we can write $\kappa_5^{\otimes 2}$ in terms of E_i, $L_{i,j}$ as

$$(\kappa_5)^{\otimes 2} \sim \sigma^*(6H) - 2\sum_{i=1}^{4} E_i.$$

In particular, this gives an isomorphism of vector spaces

$$H^0(\overline{\mathrm{M}}_{0,5}, \kappa_5^{\otimes 2}) \simeq H^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(6H) \bigotimes_{i=1}^{4} \mathscr{I}_{q_i}^{\otimes 2}\right),$$

where $\mathscr{I}_q$ denotes the skyscraper sheaf at q. By explicitly writing out equations, one can check that the conditions that a curve of degree 6 on $\mathbb{P}^2$ pass through four points in general position, all four of which are nodes, are linearly independent. Thus, this latter vector space has dimension $\binom{8}{2} - 12 = 16$, and so the dimension of $H^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5^{\otimes 2})$ is given by

$$h^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5^{\otimes 2}) = 3 \cdot 16 = 48.$$

Repeating the above argument with $\kappa_5^{\otimes 2}$ replaced by κ_5, we compute the dimension of $H^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5)$ to be

$$h^0(\overline{\mathrm{M}}_{0,5}, \mathbb{C}^3 \otimes \kappa_5) = 3\left(\binom{5}{2} - 4\right) = 18.$$

Next, using that $\psi_4 \sim \delta_{1,2} + \delta_{3,5} + \delta_{4,5} \sim H$ gives the linear equivalence

$$\psi_4 \otimes \kappa_5^{\otimes 2} \sim \sigma^*(7H) - 2\sum_{i=1}^{4} E_i.$$

With this, we compute the dimension of $H^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5^{\otimes 2})$ to be

$$h^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5^{\otimes 2}) = h^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(7)\bigotimes_{i=1}^{4} \mathscr{I}_{q_i}^{\otimes 2}\right) = \binom{9}{2} - 12 = 24.$$

Finally, we repeat the above computation with $\kappa_5^{\otimes 2}$ replaced by κ_5, and obtain that the dimension of $H^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5)$ is

$$h^0(\overline{\mathrm{M}}_{0,5}, \psi_4 \otimes \kappa_5) = h^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4)\bigotimes_{i=1}^{4} \mathscr{I}_{q_i}\right) = \binom{6}{2} - 4 = 11.$$

□

Lemma 5.6. *We have* $h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}) = 11$ *and* $h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5) = 3$.

Proof. We note first that $\mathscr{M}_4^1 = \psi_4^{-1}$ on $\overline{\mathrm{M}}_{0,4}$, so $\pi^*\mathscr{M}_4^1 = \pi^*(\psi_4^{-1})$. On $\overline{\mathrm{M}}_{0,5} \simeq \mathbb{P}^1$, the ψ_4 class is given by have $\psi_4 = \mathscr{O}(-1)$. In particular, we can write $\pi^*(\mathscr{M}_4^1)$ in terms of the generators of $\mathrm{Pic}(\overline{\mathrm{M}}_{0,5}) = \mathrm{Pic}(\mathrm{Bl}_{q_1,\dots,q_4}\mathbb{P}^2)$ as

$$\pi^*\mathscr{M}_4^1 = \pi^*(\mathscr{O}(1)) = -2H + \sum_{i=1}^{4} E_i.$$

This allows us to write the class $\pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}$ as follows:

$$\pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2} \sim \sigma^*(-2H) + \sum_{i=1}^{4} E_i + \sigma^*(6H) - 2\sum_{i=1}^{4} E_i = \sigma^*(4H) - \sum_{i=1}^{4} E_i.$$

Together with the push-pull formula, we then compute the dimension of space of sections $H^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2})$ to be

$$h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}) = h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(4)\bigotimes_{i=1}^{4} \mathscr{I}_{q_i}) = \binom{6}{2} - 4 = 11.$$

Similarly, we obtain that the dimension of $H^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5)$ is given by

$$h^0(\overline{\mathrm{M}}_{0,5}, \pi^*\mathscr{M}_4^1 \otimes \kappa_5) = h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) = 3.$$

□

Applying lemmas 5.5 and 5.6 to the short exact sequences of Lemma 4.4 gives the following

Corollary 5.7. *The number of equations for $\overline{\mathrm{M}}_{0,6}$ in the line bundle $\kappa_5^{\otimes 2} \boxtimes \mathscr{O}(2)$ on $\overline{\mathrm{M}}_{0,5} \times \mathbb{P}^3$ is equal to $h^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5^{\otimes 2}) = 35$.*

The number of equations for $\overline{\mathrm{M}}_{0,6}$ in the line bundle $\kappa_5 \boxtimes \mathscr{O}(2)$ on $\overline{\mathrm{M}}_{0,5} \times \mathbb{P}^3$ is equal to $h^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5) = 10$.

Proof. For $n = 6$ and $a = 1, 2$, Lemma 4.4 gives the short exact sequences

$$0 \to H^0(\overline{\mathrm{M}}_{0,5}, \pi^* \mathscr{M}_4^1 \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5^{\otimes 2}) \to H^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5^{\otimes 2}) \to 0$$

and

$$0 \to H^0(\overline{\mathrm{M}}_{0,5}, \pi^* \mathscr{M}_4^1 \otimes \kappa_5) \to H^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5) \to H^0(\overline{\mathrm{M}}_{0,5}, V_{\psi_4} \otimes \kappa_5) \to 0.$$

The result follows from these short exact sequences and Lemmas 5.5 and 5.6.

□

Finally, we conclude with the proof of the main theorem.

Proof (Proof of Theorem 5.1). We note that the number of equations of $\overline{\mathrm{M}}_{0,5}$ of bidegree $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ is equal to zero. Therefore the number equations for $\overline{\mathrm{M}}_{0,6}$ of tridegree $(1,1,2)$ coincides with the number of equations for $\overline{\mathrm{M}}_{0,6}$ in the line bundle $\kappa_5 \boxtimes \mathscr{O}(2)$ on $\overline{\mathrm{M}}_{0,5} \times \mathbb{P}^3$.

Since there are two equations for $\overline{\mathrm{M}}_{0,5}$ of bidegree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, both of which must be homogenized, we see that there are 20 equations of tridegree $(2,2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. Therefore by Corollary 5.7, the number of equations for $\overline{\mathrm{M}}_{0,6}$ of tridegree $(2,2,2)$ is equal to

$$h^0(\overline{\mathrm{M}}_{0,5}, \mathscr{M}_5^1 \otimes \kappa_5^{\otimes 2}) + 20 = 55.$$

and the number of equations for $\overline{\mathrm{M}}_{0,6}$ of tridegree $(1,1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$ is equal to 10. This, together with the *Macaulay2* computations in the proof of Proposition 5.2 completes the proof of Theorem 5.1.

□

6 Future directions

Let $I \subset \mathscr{O}_{\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}}(1, \ldots, 1)$ be the ideal generated by the equations given in Lemma 3.8. In the case $\overline{\mathrm{M}}_{0,6}$ we proved that these equations generate the ideal of $\overline{\mathrm{M}}_{0,6}$ in the ring of sections of $\mathscr{O}(1,1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^3$. This implies that the equations listed in Lemma 3.8, together with the Segre relations, generate the ideal of $\overline{\mathrm{M}}_{0,6}$ in $\mathbb{P}^{23}$. Does the list from Lemma 3.8 generate the ideal of $\overline{\mathrm{M}}_{0,n}$ in $\mathbb{P}^{(n-2)!-1}$ for any n? To answer this question one might compute the dimension of the homogenous part of multi-degree $(2, \ldots, 2)$ of the corresponding ideal and compare it with the dimension of the space of sections of the corresponding bundle.

Also, one can ask for a list of equations generating the ideal of $\overline{\mathrm{M}}_{0,n}$ in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ itself. How many generators do we need, and in which degrees? Even in the case of $\overline{\mathrm{M}}_{0,6}$ this remains unknown.

Finally, we would like to study the tropicalization of the embedding of $\overline{\mathrm{M}}_{0,n}$ defined by the equations we found. Having an explicitly written ideal of $\overline{\mathrm{M}}_{0,n}$ in $\mathbb{P}^{(n-2)!-1}$ or in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^{n-3}$ one can compute the tropical fan of the intersection of the $\overline{\mathrm{M}}_{0,n}$ with the torus. This would help to understand the geometry of the particular embedding parametrized in Theorem 3.5.

Acknowledgements This article was initiated during the Apprenticeship Weeks (22 August-2 September 2016), led by Bernd Sturmfels, as part of the Combinatorial Algebraic Geometry Semester at the Fields Institute. We thank Bernd Sturmfels, Christine Berkesch Zamaere, Renzo Cavalieri, Diane Maclagan, Steffen Marcus, Vic Reiner, and Jenia Tevelev. The second author was partially supported by a scholarship from the Clay Math Institute.

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