Introduction to Differential Geometry

Manifolds, vector fields, and differential forms

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Preface

This book is intented as a modern introduction to Differential Geometry, at a level accessible to advanced undergraduate students. Earlier versions of this text have been used as lecture notes for a third year course in Differential Geometry at the University of Toronto, taught by the second author, and later tried out by his colleagues.

As the subtitle of this book indicates, we take 'Differential Geometry' to mean the theory of *manifolds*. Over the past few decades, manifolds have become increasingly important in many branches of mathematics and physics. There is an enormous amount of literature on the subject, and many outstanding textbooks. However, most of these references are pitched at a graduate or postgraduate level, and are not suited for an undergraduate course. It is this gap that this book aims to address.

Accordingly, the book will rely on a minimal set of prerequisites. The required background material is typically covered in the first two or three years of university: a solid grounding in linear algebra and multivariable calculus, and ideally a course on ordinary differential equations. We will not require knowledge of abstract algebra or point set topology, but rather develop some of the necessary notions *on the fly*, and only in the generality needed here.

A few words about the philosophy of this book. First of all, we believe that it is important to develop *intuition* for the concepts introduced, to get a feel for the subject. To a large extent, this means *visualization*, but this does not always involve drawing pictures of curves and surfaces in two or three dimensions. For example, it is not difficult to get an understanding of the projective plane intrinsically, and also to 'visualize' the projective plane, but it is relatively hard to depict the projective plane as a surface (with self-intersections) in 3-space. For surfaces such as the Klein bottle or the 2-torus, such depictions are easier, but even in those cases they are not always helpful, and can even be a little misleading.

While we are trying to adopt a 'hands-on' approach to the theory of manifolds whenever possible, we believe that a certain level of *abstraction* cannot and should not be avoided. By analogy, when students are exposed to general vector spaces in Linear Algebra, the concept may seem rather abstract at first. But it usually does not take long to absorb these ideas, and gain familiarity with them even if one cannot always draw pictures. Likewise, not all of differential geometry is accounted for by drawing pictures, and what may seem abstract initially will seem perfectly natural with some practice and experience.

Another principle that we aim to follow is to provide good *motivation* for all concepts, rather than just impose them. Some subtleties or technical points have emerged through the long development of the theory, but they exist for reasons, and we feel it is important to expose those reasons.

Finally, we consider it important to offer some *practice* with the theory. Sprinkled throughout the text are questions, indicated by a feather symbol, that are meant to engage the student, and encourage active learning. Often, these are 'review questions' or 'quick questions' which an instructor might pose to students to stimulate class participation. In other cases they amount to routine calculations, which are more instructive if the student attempts to do them on his or her own. In other cases, the student is encouraged to try out a new idea or concept before moving on. Answers to these questions are provided at the end of the book.

Each chapter concludes with 'problems', which are designed as homework assignments. These include simpler problems whose goal is to reinforce the material, but also a large number of rather challenging problems. Most of these problems have been tried out on students at the University of Toronto, and have been revised to make them as clear and interesting as possible. We are grateful to Boris Khesin for letting us include some of his homework problems, and we suggest his wonderful article [10] with Serge Tabachnikov, titled 'Fun problems in geometry and beyond', for further readings along these lines.

The book contains significantly more material than can possibly be covered in a one-semester course, and the instructor will have to be selective. One possible approach is to treat the theory of vector fields in full detail, ending with chapter 6, and leave out the material on differential forms. On the other hand, if it is desired to cover differential forms as well, it is will be necessary to take shortcuts with some of the earlier material, for example by spending less time on normal forms for constant rank maps, flows of vector fields, or Frobenius' theorem.

Acknowledgements. As already indicated, earlier versions of this book have been used as a textbook at University of Toronto for several years. We thank the students participating in these courses for numerous helpful comments and excellent questions, improving the readability of this text. We thank Marco Gualtieri and Boris Khesin at University of Toronto, as well as Chenchang Zhu at Göttingen University, for pointing out errors and many suggestions.

Contents

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Introduction

1.1 A very short history

In the words of S.S. Chern, "*the fundamental objects of study in differential geometry are manifolds.*" [4, Page 332]. Roughly, an n-dimensional manifold is a mathematical object that "locally" looks like $\mathbb{R}^n$. The theory of manifolds has a long and complicated history. For centuries, manifolds have been studied extrinsically, as subsets of Euclidean spaces, given for example as level sets of equations. In this context, it is not always easy to separate the properties of a manifold from the choice of an embedding: a famous discovery in this context is Carl Friedrich Gauss' *Theorem Egregium* from 1828, proving that the Gauss curvature of embedded surfaces depends only on the choice of a metric on the surface itself . The term 'manifold' goes back to the 1851 thesis [15] of Bernhard Riemann, *"Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"* ("foundations of a general theory of functions of a complex variable") and his 1854 *Habilitation* address [16] *"Über die Hypothesen, welche der Geometrie zugrunde liegen"* ("on the assumptions underlying geometry").

However, in neither reference did Riemann attempt to give a precise definition of the concept. This was done subsequently through the work of many authors, including Riemann himself. See e.g. [17] for the long list of names involved. Henri Poincaré, in his 1895 work *analysis situs* [14], introduced the idea of a *manifold atlas*.

A rigorous axiomatic definition of manifolds was given by Oswald Veblen and J.H.C. Whitehead [20] only in 1931. We will see below that the concept of a manifold is really not all that complicated; and in hindsight it may come as a surprise that it took so long to evolve. Initially, the concept may have been regarded as simply a change of perspective – describing manifolds intrinsically from the outset, rather than extrinsically, as regular level sets of functions on Euclidean space.
Developments in physics played a major role in supporting this new perspective. In Albert Einstein's theory of General Relativity from 1916, space-time is regarded as a 4-dimensional manifold with no distinguished coordinates (not even a distinguished separation into space and time directions); a local observer may want to introduce local *xyzt* coordinates to perform measurements, but all physically meaningful quantities must admit formulations that are 'manifestly coordinate-independent'. At the same time, it would seem unnatural to try to embed the 4-dimensional curved space-time continuum into some higher-dimensional flat space, in the absence of any physical significance for the additional dimensions. For the various vector-valued functions appearing in the theory, such as electromagnetic fields, one is led to ask about their 'natural' formulation consistent with their transformation properties under local coordinate changes. The theory of differential forms, introduced in its modern form by Elie Cartan in 1899, and the associated coordinate-free notions of differentiation and integration become inevitable at this stage. Many years later, *gauge theory* once again emphasized coordinate-free formulations, and provided physics-based motivations for more elaborate constructions such as fiber bundles and connections.
Since the late 1940s and early 1950s, differential geometry and the theory of manifolds have become part of the basic education of any mathematician or theoretical physicist, with applications in other areas of science such as engineering and economics. There are many sub-branches, such as complex geometry, Riemannian geometry, and symplectic geometry, which further subdivide into sub-sub-branches. It continues to thrive as an active area of research, with exciting new results and deep open questions.

1.2 The concept of manifolds: Informal discussion

To repeat, an n-dimensional manifold is something that "locally" looks like $\mathbb{R}^n$. The prototype of a manifold is the surface of planet Earth:

It is (roughly) a 2-dimensional sphere, but we use local charts to depict it as subsets of 2-dimensional Euclidean spaces. Note that such a chart will always give a somewhat distorted picture of the planet; the distances on the sphere are never quite correct, and either the areas or the angles (or both) are wrong. For example, in the standard maps of the world, Greenland always appears much bigger than it really is. (Do you know how its area compares to that of India?)

To describe the entire planet, one uses an atlas with a collection of such charts, such that every point on the planet is depicted in at least one such chart.
This idea will be used to give an 'intrinsic' definition of manifolds, as essentially a collection of charts glued together in a consistent way. One then proceeds to develop analysis on such manifolds, for example a theory of integration and differentiation, by working in charts. The task is then to understand the change of coordinates as one leaves the domain of one chart and enters the domain of another.

1.3 Manifolds in Euclidean space

In multivariable calculus, you may have encountered manifolds as solution sets of equations. For example, the solution set $S \subseteq \mathbb{R}^3$ of an equation of the form

$f(x,y,z) = a$ defines a smooth surface in $\mathbb{R}^3$, provided the gradient of f is non-vanishing at all points of S. We call such a value of f a *regular value*, and hence $S = f^{-1}(a)$ a *regular level set**. Similarly, the joint solution set $C \subseteq \mathbb{R}^3$ of two equations

$$f(x,y,z) = a, \quad g(x,y,z) = b$$

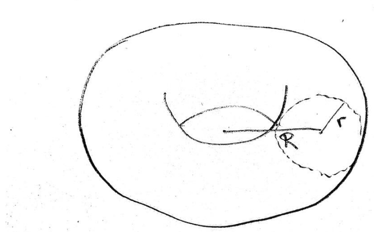
defines a smooth curve in $\mathbb{R}^3$, provided (a,b) is a regular value† of (f,g) in the sense that the gradients of f and g are linearly independent at all points of C. A familiar example of a manifold is the 2-dimensional sphere S^2, conveniently described as a level surface inside $\mathbb{R}^3$:

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}.$$

There are many ways of introducing local coordinates on the 2-sphere: For example, one can use spherical polar coordinates, cylindrical coordinates, stereographic projections, or orthogonal projections onto the coordinate planes. We will discuss some of these coordinates below. More generally‡, one has the n-dimensional sphere S^n inside $\mathbb{R}^{n+1}$,

$$S^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \,|\, (x^0)^2 + \cdots + (x^n)^2 = 1\}.$$

The 0-sphere S^0 consists of two points, the 1-sphere S^1 is the *unit circle*. Another example is the *2-torus*, T^2. It is often depicted as a surface of revolution: Given real numbers r, R with $0 < r < R$, take a circle of radius r in the $x-z$ plane, with center at $(R,0)$, and rotate about the z-axis.



* Let us also take this opportunity to remind the reader of certain ambiguities of notation. Given a function $f : X \to Y$ and any subset $B \subseteq Y$, we have teh pre-image defined by $f^{-1}(B) = \{x \in X \,|\, f(x) \in B\}$. It is common to write $f^{-1}(a)$ for $f^{-1}(\{a\})$, so that the former is a subset of the domain of f. If it happens that f is bijective, one also has the inverse function $f^{-1} : Y \to X$ defined by $f^{-1}(y)$ is the unique $x \in X$ with $f(x) = y$ (that is, if f is given by the rule $x \mapsto y$, then f^{-1} is given by the rule $y \mapsto x$), so that the former is an element of the domain of f. One has to rely on context to distinguish between the usage of f^{-1} as the preimage and as the inverse function.

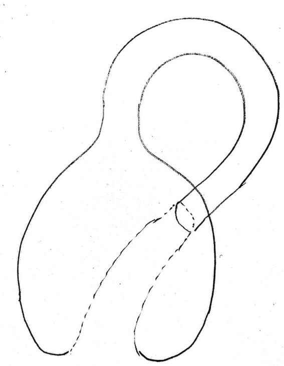
† Here $(\cdot,\cdot)$ denotes an ordered pair. Context will dictate where $(\cdot,\cdot)$ should be interpreted as an ordered pair, or as an open interval.

‡ Following common practice, we adopt the superscript notation for indices, so that a point in say $\mathbb{R}^4$ is written as $x = (x^1, x^2, x^3, x^4)$.

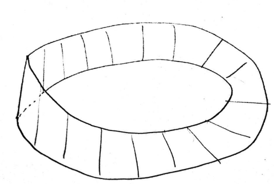
The resulting surface is given by an equation,

$$T^2 = \{(x,y,z) | \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2\}. \tag{1.1}$$

Not all surfaces can be realized as 'embedded' in $\mathbb{R}^3$; for non-orientable surfaces one needs to allow for self-intersections. This type of realization is referred to as an *immersion*: We don't allow edges or corners, but we do allow that different parts of the surface pass through each other. An example is the *Klein bottle*



The Klein bottle is an example of a *non-orientable surface*: It has only one side. A simpler example of a non-orientable surface is the *open Möbius strip*

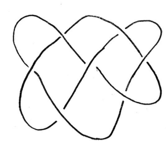


here *open* means that we are excluding the boundary. (Note that only at interior points the Möbius strip looks like $\mathbb{R}^2$, while at boundary points it looks like a half space $\{(x,y) \in \mathbb{R}^2 | \ x \geq 0\}$). In fact, one way of seeing that the Klein bottle is non-orientable is to show that it contains a Möbius strip – see Problem **??**. Note that a surface given as a regular level set $f^{-1}(0)$ of a function f is necessarily orientable: For any such surface one has one side where f is positive, and another side where f is negative.

1.4 Intrinsic descriptions of manifolds

In this book, we will mostly avoid concrete embeddings of manifolds into any $\mathbb{R}^N$. Here, the term 'embedding' is used in an intuitive sense, for example as the realization as the level set of some equations. (Later we will give a precise definition.) There are a number of reasons for why we prefer developing an 'intrinsic' theory of manifolds.

a) Embeddings of simple manifolds in Euclidean space can look quite complicated. The following one-dimensional manifold



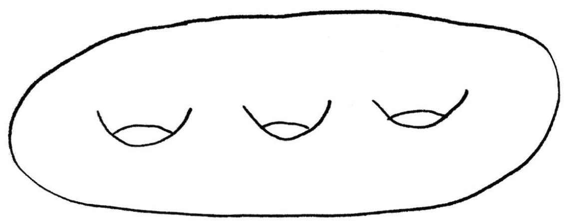
is intrinsically, 'as a manifold', just a closed curve, that is, a circle. The problem of distinguishing embeddings of a circle into $\mathbb{R}^3$ is one of the goals of *knot theory*, a deep and difficult area of mathematics.
b) Such complications disappear if one goes to higher dimensions. For example, the above knot (and indeed any knot in $\mathbb{R}^3$) can be disentangled inside $\mathbb{R}^4$ (with $\mathbb{R}^3$ viewed as a subspace). Thus, in $\mathbb{R}^4$ they become *unknots*.
c) The intrinsic description is sometimes much simpler to deal with than the extrinsic one. For instance, Equation (1.1) describing the torus $T^2 \subseteq \mathbb{R}^3$ is not especially simple or beautiful. But once we introduce the following parametrization of the torus

$$x = (R + r\cos\varphi)\cos\theta,\ \ y = (R + r\cos\varphi)\sin\theta,\ \ z = r\sin\varphi,$$

where θ, φ are determined up to multiples of 2π, we recognize that T^2 is simply a product:

$$T^2 = S^1 \times S^1. \tag{1.2}$$

That is, T^2 consists of ordered pairs of points on the circle, with the two factors corresponding to θ and φ. In contrast to (1.1), there is no distinction between 'small' circle (of radius r) and 'large' circle (of radius R). The new description suggests an embedding of T^2 into $\mathbb{R}^4$ which is 'nicer' than the embedding into $\mathbb{R}^3$. But then again, why not just work with the description (1.2), and avoid embeddings altogether!?
d) Often, there is no natural choice of an embedding of a given manifold inside $\mathbb{R}^N$, at least not in terms of concrete equations. For instance, while the triple torus



is easily pictured in 3-space $\mathbb{R}^3$, it is hard to describe it concretely as the level set of an equation.

e) While many examples of manifolds arise naturally as level sets of equations in some Euclidean space, there are also many examples for which the initial construction is different. For example, the set M whose elements are all affine lines in $\mathbb{R}^2$ (that is, straight lines that need not go through the origin) is naturally a 2-dimensional manifold. But some thought is required to realize it as a surface in $\mathbb{R}^3$. The next section deals with other such examples.

1.5 Soccer balls and linkages

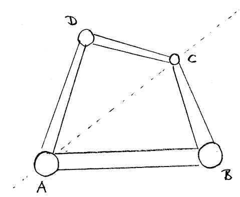
Mechanical systems typically have certain degrees of freedom, and hence may take on various *configurations*. The set of all possible configurations of such a system is called its *configurations space*, and is often (but not always) described by a manifold. As a simple example, consider the possible configurations of a soccer ball, positioned over the some fixed point of the lawn (the penalty mark, say).

From any fixed position of the ball, any other configuration is obtained by a rotation. It takes three parameters to describe a rotation, with two parameters specifying the axis of rotation and a third parameter specifying the angle of rotation. Hence we expect that the configuration space of the soccer ball is a 3-dimensional manifold, and this turns out to be true. Note that once an initial configuration is chosen, the configuration space of the soccer ball is *identified* with the group of rotations.

As a more elaborate example, consider a *spatial linkage* given by a collection of $N \geq 3$ rods, of prescribed lengths $l_1, \ldots, l_N > 0$, joined at their end points in such a way that they close up to a loop. (This is only possible if the length of the longest rod is less than or equal to the sum of the lengths of the remaining rods. We will assume that this is the case.) The rods may move freely around the joints. We shall consider two linkage configurations to be he same if they are obtained from each other by Euclidean motions (i.e., translations and rotations of the entire linkage). Denote the configuration space by

$$M(l_1, \ldots, l_N).$$

If $N = 3$, the linkage is a triangle, and there are no possibilities of changing the linkage: The configuration space $M(l_1, l_2, l_3)$ (if non-empty) is just a point. The following picture shows a typical linkage for $N = 4$. Note that this linkage has two degrees of freedom (other than rotations and translations), given by the 'bending' of the linkage along the dotted line through A, C, and a similar bending transformation along the straight line through B, D.



Hence, we expect that the configuration space $M(l_1, l_2, l_3, l_4)$ of a linkage with $N = 4$ rods, if non-empty, should typically be a 2-dimensional manifold (a surface).
To get an estimate for the number of degrees of freedom (i.e., the dimension of the configuration spaces, assuming the latter is a manifold) for general $N \geq 3$, note that the configuration of an N-linkage is realized by an ordered collection $\mathbf{u}_1, \ldots, \mathbf{u}_N$ of vectors of lengths[§] $||\mathbf{u}_1|| = l_1, \ldots,\ ||\mathbf{u}_N|| = l_N$, with the condition that the vectors add to zero:

$$\mathbf{u}_1 + \cdots + \mathbf{u}_N = \mathbf{0}. \tag{1.3}$$

Two such collections $\mathbf{u}_1, \ldots, \mathbf{u}_N$ and $\mathbf{u}'_1, \ldots, \mathbf{u}'_N$ describe the same linkage configuration if they are related by a rotation. Let us now count the number of independent parameters. Each vectors $\mathbf{u}_i$ is described by two parameters (its position on a sphere of radius l_i), giving $2N$ parameters. But these are not independent, due to the condition (1.3); the three components of this equation are cutting down the number of independent parameters by 3. Furthermore, using rotations we may arrange that $\mathbf{u}_1$ points in the x-direction, cutting down the number of parameters by another 2, and using a subsequent rotation about the x-axis, we may arrange that $\mathbf{u}_2$ lies in the xy-plane, cutting down the number of parameters by another 1. Hence we expect that configurations are described by $2N - 3 - 2 - 1 = 2N - 6$ parameters, consistent with

[§] Here we are using $||\cdot||$ for the usual Euclidean norm.

our observations for $N = 3$ and $N = 4$. Thus, letting M be the space of all configurations, and assuming this is a manifold, we expect its dimension to be

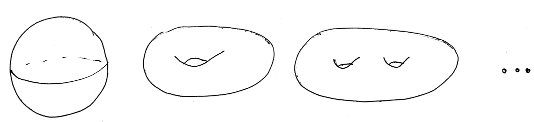
$$\dim M(l_1, \dots, l_N) = 2N - 6. \tag{1.4}$$

1 (answer on page ??). For any straight line through non-adjacent vertices of a linkage, one can define a 'bending transformation' similar to what we had for $N = 4$. How many straight lines with this property are there for $N = 5$? Does it match with the expected dimension of $M(l_1, \dots, l_5)$?

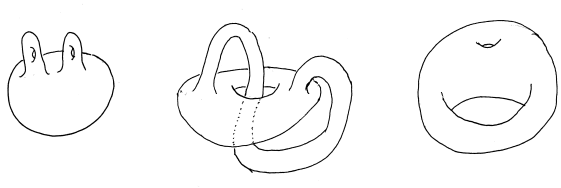
Of course, our discussion oversimplifies matters – for example, if $l_N = l_1 + \cdots + l_{N-1}$, there is only one configuration, hence our rough count is wrong in this case. More generally, whenever it is possible to make all rods 'parallel', which happens whenever there are sign choices such that $\pm l_1 \pm l_2 \pm \cdots \pm l_N = 0$, the space M will have singularities or be a manifold of a lower dimension. But for *typical* rod lengths, this cannot happen, and it turns out that the configuration space $M(l_1, \dots, l_N)$ (if non-empty) is indeed a manifold of dimension $2N - 6$. These manifolds have been much-studied, using techniques from symplectic geometry and algebraic geometry.

1.6 Surfaces

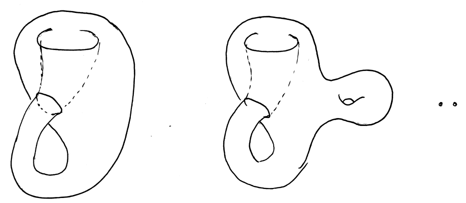
Let us briefly give a very informal discussion of *surfaces*. A surface is the same thing as a 2-dimensional manifold. We have already encountered some examples: The sphere, the torus, the double torus, triple torus, and so on:



All of these are 'orientable' surfaces, which essentially means that they have two sides which you might paint in two different colors. It turns out that these are *all* the orientable surfaces, if we consider the surfaces 'intrinsically' and only consider surfaces that are compact in the sense that they don't go off to infinity and do not have a boundary (thus excluding a cylinder, for example). For instance, each of the following drawings depicts a double torus:

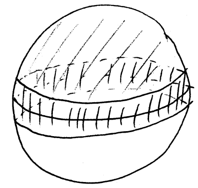


We also have one example of a non-orientable surface: The Klein bottle. More examples are obtained by attaching handles (just like we can think of the torus, double torus and so on as a sphere with handles attached).

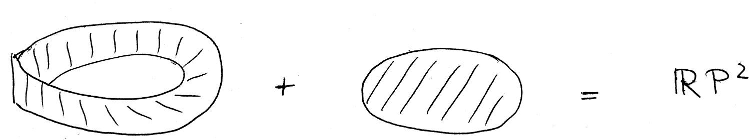


Are these *all* the non-orientable surfaces? In fact, the answer is *no*. We have missed what is in some sense the simplest non-orientable surface. Ironically, it is the surface which is hardest to visualize in 3-space. This surface is called the *projective plane* or *projective space*, and is denoted $\mathbb{R}P^2$. One can define $\mathbb{R}P^2$ as the set of all lines through the origin (i.e., 1-dimensional linear subspaces) in $\mathbb{R}^3$. It should be clear that this is a 2-dimensional manifold, since it takes 2 parameters to specify such a line. We can label such lines by their points of intersection with S^2, hence we can also think of $\mathbb{R}P^2$ as the set of antipodal (i.e., opposite) points on S^2. In other words, it is obtained from S^2 by identifying antipodal points. To get a better idea of how $\mathbb{R}P^2$ looks like, let us subdivide the sphere S^2 into two parts:

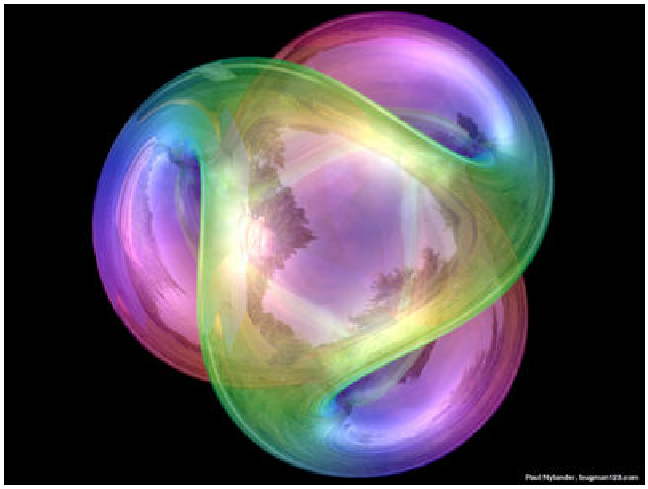
a) points having distance $\leq \varepsilon$ from the equator,
b) points having distance $\geq \varepsilon$ from the equator.



If we perform the antipodal identification for (i), we obtain a Möbius strip. If we perform antipodal identification for (ii), we obtain a 2-dimensional disk (think of it as the points of (ii) lying in the upper hemisphere). Hence, $\mathbb{R}P^2$ can also be regarded as gluing the boundary of a Möbius strip to the boundary of a disk:



Now, the question arises: *Is it possible to realize* $\mathbb{R}\mathrm{P}^2$ *smoothly as a surface inside* $\mathbb{R}^3$*, possibly with self-intersections* (similar to the Klein bottle)? Simple attempts of joining the boundary circle of the Möbius strip with the boundary of the disk will always create sharp edges or corners – try it. Around 1900, David Hilbert posed this problem to his student Werner Boy, who discovered that the answer is *yes*. The following picture of *Boy's surface* was created by Paul Nylander.



(There are some nice online videos illustrating the construction of the surface.) While these pictures are very beautiful, it certainly makes the projective space appear more complicated than it actually is. If one is only interested in $\mathbb{R}\mathrm{P}^2$ itself, rather than its realization as a surface in $\mathbb{R}^3$, it is much simpler to work with the definition (as a sphere with antipodal identification).

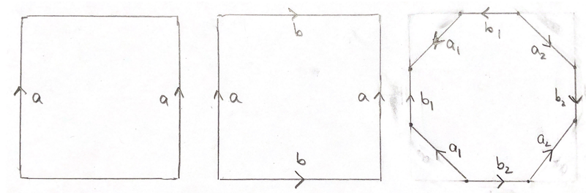
2 (answer on page ??). What surface results from "puncturing" the projective plane (i.e., removing a single point)?

Going back to the classification of surfaces, we have the following

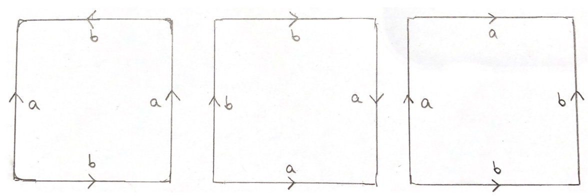
Fact: *All closed, connected surfaces are obtained from either the 2-sphere* S^2*, the Klein bottle, or the projective plane* $\mathbb{R}\mathrm{P}^2$*, by attaching handles.*

(We will not give a formal proof of this fact in this book.)

Another way of representing surfaces is with a so-called "gluing diagrams." In the diagram below boundaries are identified so that the arrows (and labels) match. For example, the diagrams below represent, from left to right, a cylinder, a 2-torus, and genus 2 surface.



3 (answer on page ??). What surfaces are obtained from the following gluing diagrams?



2

Manifolds

It is one of the goals of this book to develop the theory of manifolds in intrinsic terms, although we may occasionally use immersions or embeddings into Euclidean space in order to illustrate concepts. In physics terminology, we will formulate the theory of manifolds in terms that are 'manifestly coordinate-free'.

2.1 Atlases and charts

As we mentioned above, the basic feature of manifolds is the existence of 'local coordinates'. The transition from one set of coordinates to another should be *smooth*. We recall the following notions from multivariable calculus.

Definition 2.1. *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. A map $F: U \to V$ is called* smooth *if it is infinitely differentiable. The set of smooth functions from U to V is denoted $C^\infty(U,V)$. The map F is called a* diffeomorphism *from U to V if it is invertible and the inverse map $F^{-1}: V \to U$ is again smooth.*

Example 2.2. The exponential map $\exp: \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x) = e^x$ is smooth. It may be regarded as a map onto $\mathbb{R}_{>0} = \{y | y > 0\}$, and as such it is a diffeomorphism

$$\exp: \mathbb{R} \to \mathbb{R}_{>0}$$

with inverse $\exp^{-1} = \log$ (the natural logarithm). Similarly, the function $x \mapsto \tan(x)$ is a diffeomorphism from the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ onto $\mathbb{R}$, with inverse the function arctan.

Definition 2.3. *For a smooth map $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and any $\mathbf{x} \in U$, one defines the* Jacobian matrix *$DF(\mathbf{x})$ to be the $n \times m$-matrix of partial derivatives*

$$(DF(\mathbf{x}))^i_j = \frac{\partial F^i}{\partial x^j}$$

If $n = m$, it is a square matrix, and its determinant is called the Jacobian determinant *of F at $\mathbf{x}$.*

The *inverse function theorem* states that F is a diffeomorphism if and only if (i) F is invertible, and (ii) for all $\mathbf{x} \in U$, the Jacobian matrix $DF(\mathbf{x})$ is invertible. (That is, one does not actually have to check smoothness of the inverse map!)
The following definition formalizes the concept of introducing local coordinates.

Definition 2.4 (Charts). *Let M be a set.*

a) *An m-dimensional* (coordinate) chart (U, φ) *on M is a subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^m$, such that $\varphi(U) \subseteq \mathbb{R}^m$ is open and φ is a bijection from U to $\varphi(U)$. The set U is the* chart domain*, and φ is the* coordinate map*.*
b) *Two charts (U, φ) and (V, ψ) are called* compatible *if the subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open, and the* transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism.

As a special case, charts with $U \cap V = \emptyset$ are always compatible.

4 (answer on page ??). The bijection requirement on φ plays an important role; hence this may be a good opportunity to think through some set theory. (Also see Appendix A.) Prove the following (from now on, we shall use the properties below without further comment):
Let X, Y be sets, $f : X \to Y$ a map, and suppose $A, B \subseteq X$, and $C, D \subseteq Y$.

a) Show that

$$\begin{aligned} f(A \cup B) &= f(A) \cup f(B), \\ f(A \cap B) &= f(A) \cap f(B) \quad \text{if } f \text{ injective}, \\ f(A \setminus B) &= f(A) \setminus f(B) \quad \text{if } f \text{ injective}, \\ f(A^c) &= f(A)^c \qquad \text{if } f \text{ bijective}. \end{aligned}$$

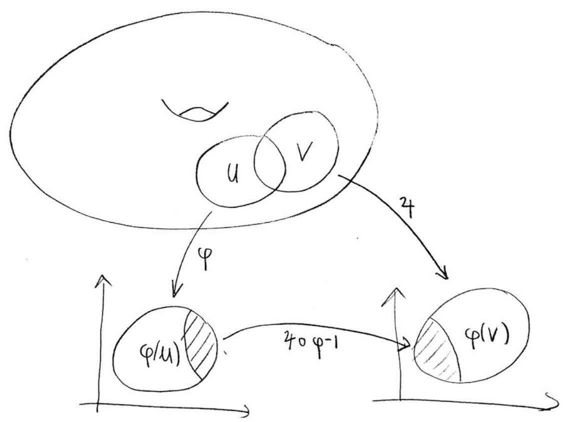
Here the superscript c denotes the complement. By giving counter-examples, show that the second and third equality may fail if f is not injective, and that the last equality may fail if f is only injective or only surjective.

b) Let us denote by $f^{-1}(C) = \{x : f(x) \in C\}$ the preimage of C. Show that:

$$\begin{aligned} f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D), \\ f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D), \\ f^{-1}(C^c) &= \left(f^{-1}(C)\right)^c, \\ f^{-1}(C \setminus D) &= f^{-1}(C) \setminus f^{-1}(D). \end{aligned}$$

5 (answer on page ??). Is compatibility of charts an equivalence relation? (See Appendix A for a reminder on equivalence relations.)

Let (U,φ) be a coordinate chart. Given a point $p \in U$, and writing $\varphi(p) = (u^1, \ldots, u^m)$, we say that the u^i are the *coordinates of p* in the given chart. (Note the convention of indexing by superscripts; be careful not to confuse indices with powers.) Letting p vary, these become real-valued functions $p \mapsto u^i(p)$; they are simply the component functions of φ.
Transition maps $\psi \circ \varphi^{-1}$ are also called *change of coordinates*. Here is a picture of a 'coordinate change':



Definition 2.5 (Atlas). *Let M be a set. An m-dimensional* atlas *on M is a collection of coordinate charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that*

a) The U_α cover all of M, i.e., $\bigcup_\alpha U_\alpha = M$.
b) For all indices α, β, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible.

In this definition, $\alpha, \beta, \ldots$ are indices used to distinguish the different charts; the indexing set may be finite or infinite, perhaps even uncountable.

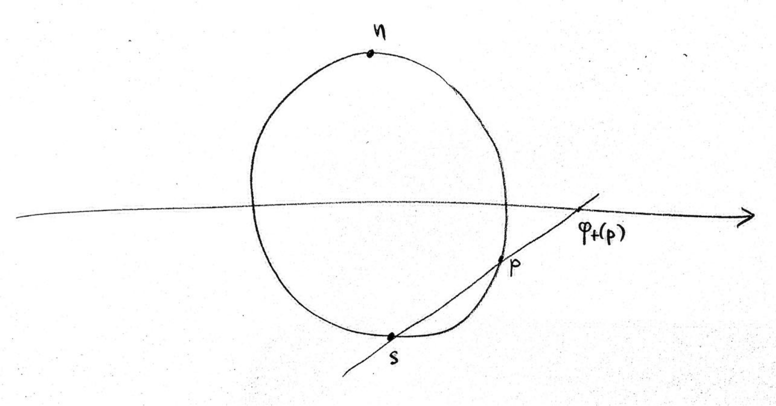
Example 2.6 (An atlas on the 2-sphere). Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere, consisting of all $(x,y,z) \in \mathbb{R}^3$ satisfying the equation $x^2+y^2+z^2 = 1$. We shall define an atlas with two charts (U_+, φ_+) and (U_-, φ_-). Let $n = (0,0,1)$ be the north pole, let $s = (0,0,-1)$ be the south pole, and put

$$U_+ = S^2 \setminus \{s\}, \quad U_- = S^2 \setminus \{n\}.$$

Regard $\mathbb{R}^2$ as the coordinate subspace of $\mathbb{R}^3$ on which $z = 0$. Let

$$\varphi_+ :\ U_+ \to \mathbb{R}^2, \quad p \mapsto \varphi_+(p)$$

be *stereographic projection from the south pole*. That is, $\varphi_+(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and s.



Similarly,

$$\varphi_- : U_- \to \mathbb{R}^2, \quad p \mapsto \varphi_-(p)$$

is *stereographic projection from the north pole*, where $\varphi_-(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and n.
A calculation gives the explicit formulas, for $(x,y,z) \in S^2 \subseteq \mathbb{R}^3$ in U_+, respectively U_-:

$$\varphi_+(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right), \quad \varphi_-(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \tag{2.1}$$

6 (answer on page ??). Verify (2.1).

Both $\varphi_\pm : U_\pm \to \mathbb{R}^2$ are bijections onto $\mathbb{R}^2$. Indeed, given $(u,v) \in \mathbb{R}^2$ we may solve the equation $(u,v) = \varphi_\pm(x,y,z)$, using the condition that $x^2+y^2+z^2 = 1$ and $z \pm 1 \neq 0$. The calculation gives

$$\varphi_\pm^{-1}(u,v) = \left(\frac{2u}{1+(u^2+v^2)}, \frac{2v}{1+(u^2+v^2)}, \pm\frac{1-(u^2+v^2)}{1+(u^2+v^2)}\right) \tag{2.2}$$

7 (answer on page ??). Verify (2.2).

Note that $\varphi_+(U_+ \cap U_-) = \mathbb{R}^2 \backslash \{(0,0)\}$. The transition map on the overlap of the two charts is

$$(\varphi_- \circ \varphi_+^{-1})(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

which is smooth on $\mathbb{R}^2 \backslash \{(0,0)\}$ as required. $\square$

Here is another simple, but less familiar example where one has an atlas with two charts.

Example 2.7 (Affine lines in $\mathbb{R}^2$). By an *affine line* in a vector space E, we mean a subset $\ell \subseteq E$ that is obtained by adding a fixed vector $\mathbf{v}_0$ to all elements of a 1-dimensional subspace. In plain terms, an affine line is simply a straight line that does not necessarily pass through the origin. (We reserve the term *line*, without prefix, for 1-dimensional subspaces, that is, for straight lines that *do* pass through the origin.) Let

$$M = \{\ell \mid \ell \text{ is an affine line in } \mathbb{R}^2\}.$$

Let $U \subseteq M$ be the subset of lines that are not vertical, and $V \subseteq M$ the lines that are not horizontal. Any $\ell \in U$ is given by an equation of the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept. The map $\varphi\colon\ U \to \mathbb{R}^2$ taking ℓ to (m,b) is a bijection. On the other hand, lines in V are given by equations of the form

$$x = ny + c,$$

and we also have the map $\psi\colon\ V \to \mathbb{R}^2$ taking such ℓ to (n,c). The intersection $U \cap V$ are lines ℓ that are neither vertical nor horizontal. Hence, $\varphi(U \cap V)$ is the set of all (m,b) such that $m \neq 0$, and similarly $\psi(U \cap V)$ is the set of all (n,c) such that $n \neq 0$.

8 (answer on page ??). Compute the transition maps $\psi \circ \varphi^{-1}$, $\varphi \circ \psi^{-1}$ and show they are smooth. Conclude that (U,φ) and (V,ψ) define a 2-dimensional atlas on M.

It turns out that M is a 2-dimensional manifold – a surface. Of course, we should be able to identify this mysterious surface:

9 (answer on page ??). What is this surface?

We return to our objective of giving a general definition of the concept of manifolds. As a first approximation, we may take an m-dimensional manifold to be a set with an m-dimensional atlas. This is almost the right definition, but we will make a few adjustments. A first criticism is that we may not want any *particular* atlas as part of the definition. For example, the 2-sphere with the atlas given by stereographic projections onto the xy-plane, and the 2-sphere with the atlas given by stereographic projections onto the yz-plane, should be one and the same manifold: S^2. To resolve this problem, we will use the following notion.

Definition 2.8. *Suppose $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ is an m-dimensional atlas on M, and let (U,φ) be another chart. Then (U,φ) is said to be* compatible *with $\mathscr{A}$ if it is compatible with all charts $(U_\alpha, \varphi_\alpha)$ of $\mathscr{A}$.*

Example 2.9. On the 2-sphere S^2, we have constructed the atlas

$$\mathscr{A} = \{(U_+, \varphi_+), (U_-, \varphi_-)\}$$

given by stereographic projection. Consider the chart (V, ψ), with domain V the set of all $(x,y,z) \in S^2$ such that $y < 0$, with $\psi(x,y,z) = (x,z)$. To check that it is compatible with (U_+, φ_+), note that $U_+ \cap V = V$, and

$$\varphi_+(U_+ \cap V) = \{(u,v) |\, v < 0\}, \quad \psi(U_+ \cap V) = \{(x,z) |\, x^2 + z^2 < 1\}.$$

10 (answer on page ??). Find explicit formulas for $\psi \circ \varphi_+^{-1}$ and $\varphi_+ \circ \psi^{-1}$. Conclude that (V, ψ) is compatible with (U_+, φ_+).

Note that (U, φ) is compatible with the atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ if and only if the union $\mathscr{A} \cup \{(U, \varphi)\}$ is again an atlas on M. This suggests defining a bigger atlas, by using *all* charts that are compatible with the given atlas. In order for this to work, we need the new charts to be compatible not only with the charts of $\mathscr{A}$, but also with each other. This is not entirely obvious, since compatibility of charts is not an equivalence relation (see 5).

Lemma 2.10. *Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a given atlas on the set M. If two charts (U, φ), (V, ψ) are compatible with $\mathscr{A}$, then they are also compatible with each other.*

Proof. For every chart $(U_\alpha, \varphi_\alpha)$, the sets $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi_\alpha(V \cap U_\alpha)$ are open, hence their intersection is open. This intersection is (see 11 below)

$$\varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(V \cap U_\alpha) = \varphi_\alpha(U \cap V \cap U_\alpha). \tag{2.3}$$

Since $\varphi \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U \cap U_\alpha) \to \varphi(U \cap U_\alpha)$ is a diffeomorphism, it follows that

$$\varphi(U \cap V \cap U_\alpha) = (\varphi \circ \varphi_\alpha^{-1})\big(\varphi_\alpha(U \cap V \cap U_\alpha)\big)$$

is open. Taking the union over all α, we see that

$$\varphi(U \cap V) = \bigcup_\alpha \varphi(U \cap V \cap U_\alpha)$$

is open. A similar argument applies to $\psi(U \cap V)$. The transition map $\psi \circ \varphi^{-1} : \ \varphi(U \cap V) \to \psi(U \cap V)$ is smooth since for all α, its restriction to $\varphi(U \cap V \cap U_\alpha)$ is a composition of two smooth maps $\varphi_\alpha \circ \varphi^{-1} : \ \varphi(U \cap V \cap U_\alpha) \longrightarrow \varphi_\alpha(U \cap V \cap U_\alpha)$ and $\psi \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U \cap V \cap U_\alpha) \longrightarrow \psi(U \cap V \cap U_\alpha)$. Likewise, the composition $\varphi \circ \psi^{-1} : \ \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. $\square$

11 (answer on page ??). Explain why (2.3) is true.

12 (answer on page ??). Suppose (U,φ) is a chart, with image $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Let $V \subseteq U$ be a subset such that $\widetilde{V} = \varphi(V) \subseteq \widetilde{U}$ is open, and let $\psi = \varphi|_V$ be the restriction of φ. Prove that (V,ψ) is again a chart, and is compatible with (U,φ). Furthermore, if (U,φ) is a chart from an atlas $\mathscr{A}$, then (V,ψ) is compatible with that atlas.

Theorem 2.11. *Given an atlas $\mathscr{A} = \{(U_\alpha,\varphi_\alpha)\}$ on M, let $\widetilde{\mathscr{A}}$ be the collection of all charts (U,φ) that are compatible with $\mathscr{A}$. Then $\widetilde{\mathscr{A}}$ is itself an atlas on M, containing $\mathscr{A}$. In fact, $\widetilde{\mathscr{A}}$ is the largest atlas containing $\mathscr{A}$.*

Proof. Note first that $\widetilde{\mathscr{A}}$ contains $\mathscr{A}$, since the set of charts compatible with $\mathscr{A}$ contains the charts from the atlas $\mathscr{A}$ itself. In particular, the charts in $\widetilde{\mathscr{A}}$ cover M. By the lemma above, any two charts in $\widetilde{\mathscr{A}}$ are compatible. Hence $\widetilde{\mathscr{A}}$ is an atlas. If (U,φ) is a chart compatible with all charts in $\widetilde{\mathscr{A}}$, then in particular it is compatible with all charts in $\mathscr{A}$; hence $(U,\varphi) \in \widetilde{\mathscr{A}}$ by the definition of $\widetilde{\mathscr{A}}$. This shows that $\widetilde{\mathscr{A}}$ cannot be extended to a larger atlas. □

Definition 2.12. *An atlas $\mathscr{A}$ is called* maximal *if it is not properly contained in any larger atlas. Given an arbitrary atlas $\mathscr{A}$, one calls $\widetilde{\mathscr{A}}$ (as in Theorem 2.11) the* maximal atlas determined by *$\mathscr{A}$.*

Remark 2.13. Although we will not need it, let us briefly discuss the notion of equivalence of atlases. (For background on equivalence relations, see the Appendix A.) Two atlases $\mathscr{A} = \{(U_\alpha,\varphi_\alpha)\}$ and $\mathscr{A}' = \{(U'_\alpha,\varphi'_\alpha)\}$ are called *equivalent* if every chart of $\mathscr{A}$ is compatible with every chart in $\mathscr{A}'$. For example, the atlas on the 2-sphere given by the two stereographic projections to the *xy*-plane is equivalent to the atlas $\mathscr{A}'$ given by the two stereographic projections to the *yz*-plane. Using Lemma 2.10, one sees that equivalence of atlases is indeed an equivalence relation. (In fact, two atlases are equivalent if and only if their union is an atlas.) Furthermore, two atlases are equivalent if and only if they are contained in the same maximal atlas. That is, *any maximal atlas determines an equivalence class of atlases, and vice versa.*

2.2 Definition of manifold

As our next approximation towards the right definition, we can take an *m*-dimensional manifold to be a set *M* together with an *m*-dimensional *maximal* atlas. This is already quite close to what we want, but for technical reasons we would like to impose two further conditions.

First of all, we insist that *M* can be covered by *countably many* coordinate charts. In most of our examples, *M* is in fact covered by finitely many coordinate charts. This countability condition is used for various arguments involving a proof by induction.

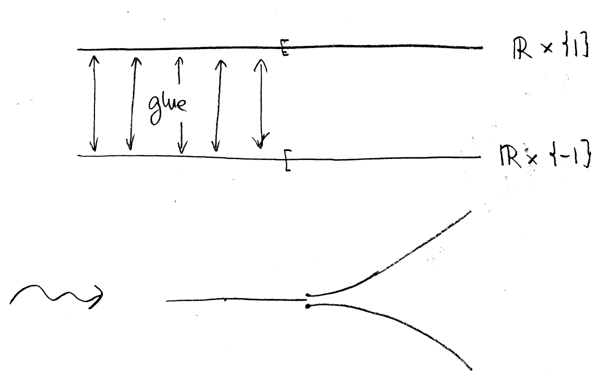
Example 2.14. A simple non-example that is not countable: Let $M = \mathbb{R}$, with $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ the 0-dimensional atlas, where each U_α consists of a single point, and $\varphi_\alpha : U_\alpha \to \{0\}$ is the unique map to $\mathbb{R}^0 = \{0\}$. Compatibility of charts is obvious. But M cannot be covered by countably many of these charts. Thus, we will not consider $\mathbb{R}$ to be a zero-dimensional manifold.

Secondly, we would like to avoid the following type of example.

Example 2.15. Let X be a disjoint union of two copies of the real line $\mathbb{R}$. We denote the two copies by $\mathbb{R}\times\{1\}$ and $\mathbb{R}\times\{-1\}$, just so that we can tell them apart. Consider the equivalence relation on X generated by

$$(x,1) \sim (x',-1) \;\Leftrightarrow x' = x < 0,$$

and let $M = X/\sim$ the set of equivalence classes. That is, we 'glue' the two real lines along their negative real axes (taking care that no glue gets on the origins of the axes). Here is a (not very successful) attempt to sketch the resulting space:



As a set, M is a disjoint union of $\mathbb{R}_{<0}$ with two copies of $\mathbb{R}_{\geq 0}$. Let $\pi : X \to M$ be the quotient map, and let

$$U = \pi(\mathbb{R}\times\{1\}), \quad V = \pi(\mathbb{R}\times\{-1\})$$

the images of the two real lines. The projection map $X \to \mathbb{R}$, $(x,\pm 1) \mapsto x$ is constant on equivalence classes, hence it descends to a map $f : M \to \mathbb{R}$; let $\varphi : U \to \mathbb{R}$ be the restriction of f to U and $\psi : V \to \mathbb{R}$ the restriction to V. Then

$$\varphi(U) = \psi(V) = \mathbb{R}, \quad \varphi(U\cap V) = \psi(U\cap V) = \mathbb{R}_{<0},$$

and the transition map is the identity map. Hence, $\mathscr{A} = \{(U,\varphi),\ (V,\psi)\}$ is an atlas for M. A strange feature of M with this atlas is that although the points

$$p = \varphi^{-1}(\{0\}) \in U, \quad q = \psi^{-1}(\{0\}) \in V,$$

are distinct ($p \neq q$), they are 'arbitrarily close': for any $\varepsilon, \delta > 0$ the preimages $\varphi^{-1}(-\varepsilon,\varepsilon) \subseteq U$ and $\psi^{-1}(-\delta,\delta) \subseteq V$ have non-empty intersection. There is no really satisfactory way of drawing M (our picture above is inadequate).

Since such a behavior is inconsistent with the idea of a manifold that 'locally looks like $\mathbb{R}^n$' (where, e.g. every converging sequence has a unique limit), we shall insist that for any two distinct points $p,q \in M$, there are always disjoint coordinate charts separating the two points. This is called the *Hausdorff condition*, after *Felix Hausdorff* (1868-1942).

Definition 2.16. *An m-dimensional manifold is a set M, together with a maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ with the following properties:*

a) **(Countability condition)** *M is covered by countably many coordinate charts in $\mathscr{A}$. That is, there are indices $\alpha_1, \alpha_2, \ldots$ (not necessarily distinct) with*

$$M = \bigcup_{i=1}^{\infty} U_{\alpha_i}$$

b) **(Hausdorff condition)** *For any two distinct points $p,q \in M$ there are coordinate charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) in $\mathscr{A}$ such that $p \in U_\alpha$, $q \in U_\beta$, and*

$$U_\alpha \cap U_\beta = \emptyset.$$

The charts $(U, \varphi) \in \mathscr{A}$ are called (coordinate) charts on the manifold M.

Before giving examples, let us note the following useful fact concerning the Hausdorff condition. We shall use the following result, concerning the shrinking of a chart domain:

Lemma 2.17. *Let M be a set with a maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$, and suppose $p,q \in M$ are distinct points contained in a single coordinate chart $(U, \varphi) \in \mathscr{A}$. Then we can find indices α, β such that $p \in U_\alpha$, $q \in U_\beta$, with $U_\alpha \cap U_\beta = \emptyset$.*

Proof. Let (U, φ) be as in the lemma, and $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Since

$$\widetilde{p} = \varphi(p), \quad \widetilde{q} = \varphi(q)$$

are distinct points in $\widetilde{U}$, we can choose disjoint open subsets $\widetilde{U}_\alpha,\ \widetilde{U}_\beta \subseteq \widetilde{U}$ containing $\widetilde{p} = \varphi(p)$ and $\widetilde{q} = \varphi(q)$, respectively. Let $U_\alpha,\ U_\beta \subseteq U$ be their preimages, and take $\varphi_\alpha = \varphi|_{U_\alpha},\ \varphi_\beta = \varphi|_{U_\beta}$. Then $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in $\mathscr{A}$, with disjoint chart domains, and by construction we have that $p \in U_\alpha$ and $q \in U_\beta$. $\square$

Example 2.18. Consider the 2-sphere S^2 with the atlas given by the two coordinate charts (U_+, φ_+) and (U_-, φ_-). This atlas extends uniquely to a maximal atlas. The countability condition is satisfied, since S^2 is already covered by two charts. The Hausdorff condition is satisfied as well: Given distinct points $p,q \in S^2$, if both are contained in U_+ or both in U_-, we can apply the lemma. The only remaining case is if one point (say p) is the north pole and the other (say q) the south pole. But here we can construct U_α, U_β by replacing U_+ and U_- with the open upper hemisphere and open lower hemisphere, respectively. Alternatively, we can use the chart given by stereographic projection to the xz plane, noting that this is also in the maximal atlas.

Remark 2.19. As we explained above, the Hausdorff condition rules out some strange examples that don't quite fit our idea of a space that is locally like $\mathbb{R}^n$. Nevertheless, so-called *non-Hausdorff manifolds* (with non-Hausdorff more properly called *not necessarily Hausdorff*) do arise in some important applications. Much of the theory can be developed without the Hausdorff property, but there are some complications. For instance, initial value problems for vector fields need not have unique solutions for non-Hausdorff manifolds. Let us also note that while the classification of 1-dimensional manifolds is very easy, there is no nice classification of 1-dimensional non-Hausdorff manifolds.

Remark 2.20 (Charts taking values in 'abstract' vector spaces). In the definition of an m-dimensional manifold M, rather than letting the charts $(U_\alpha, \varphi_\alpha)$ take values in $\mathbb{R}^m$ we could just as well let them take values in m-dimensional real vector spaces E_α:

$$\varphi_\alpha : \; U_\alpha \to E_\alpha.$$

Transition functions are defined as before, except they now take an open subset of E_β to an open subset of E_α. A choice of basis identifies $E_\alpha = \mathbb{R}^m$, and takes us back to the original definition.
As far as the definition of manifolds is concerned, nothing has been gained by adding this level of abstraction. However, it often happens that the E_α's are given to us 'naturally'. For example, if M is a surface inside $\mathbb{R}^3$, one would typically use xy-coordinates, or xz-coordinates, or yz-coordinates on appropriate chart domains. It can then be useful to regard the xy-plane, xz-plane, and yz-plane as the target spaces of the coordinate maps, and for notational reasons it may be convenient not to associate them with a single $\mathbb{R}^2$.

2.3 Examples of manifolds

We will now discuss some basic examples of manifolds. In each case, the manifold structure is given by a finite atlas; hence the countability property is immediate. We will not spend too much time on verifying the Hausdorff property; while it may be done 'by hand', we will later have better ways of doing this.

We begin the list of examples with the observation that any open subset U of $\mathbb{R}^n$ is a manifold, with atlas determined by the chart (U, id_U).

2.3.1 Spheres

The construction of an atlas for the 2-sphere S^2, by stereographic projection, also works for the n-sphere

$$S^n = \{(x^0, \ldots, x^n) \,|\, (x^0)^2 + \cdots + (x^n)^2 = 1\}.$$

Let $U_\pm$ be the subsets obtained by removing $(\mp 1, 0, \ldots, 0)$. Stereographic projection from these two points defines bijections $\varphi_\pm : U_\pm \to \mathbb{R}^n$, and by calculations similar to those for the 2-sphere, we see that

$$\varphi_\pm(x^0, x^1, \ldots, x^n) = \frac{1}{1 \pm x^0}(x^1, \ldots, x^n) \tag{2.4}$$

with inverse (writing $\mathbf{u} = (u^1, \ldots, u^n)$)

$$\varphi_\pm^{-1}(\mathbf{u}) = \frac{1}{1 + ||\mathbf{u}||^2}\big(\pm(1 - ||\mathbf{u}||^2), 2u^1, \ldots, 2u^n\big). \tag{2.5}$$

For the transition function one finds

$$(\varphi_- \circ \varphi_+^{-1})(\mathbf{u}) = \frac{\mathbf{u}}{||\mathbf{u}||^2}. \tag{2.6}$$

We leave it as an exercise to check the details. An equivalent atlas, with $2n+2$ charts, is given by the subsets $U_0^+, \ldots, U_n^+, U_0^-, \ldots, U_n^-$ where

$$U_j^+ = \{\mathbf{x} \in S^n |\, x^j > 0\}, \quad U_j^- = \{\mathbf{x} \in S^n |\, x^j < 0\}$$

for $j = 0, \ldots, n$, with $\varphi_j^\pm : U_j^\pm \to \mathbb{R}^n$ the projection to the j-th coordinate plane (in other words, omitting the j-th component x^j):

$$\varphi_j^\pm(x^0, \ldots, x^n) = (x^0, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n).$$

2.3.2 Real projective spaces

The n-dimensional projective space, denoted $\mathbb{R}\mathrm{P}^n$, is the set of all lines $\ell \subseteq \mathbb{R}^{n+1}$, where *line* is taken to mean '1-dimensional subspace'. It may also be regarded as a quotient space (see Appendix A)

$$\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \backslash \{\mathbf{0}\}) / \sim$$

for the equivalence relation

$$\mathbf{x} \sim \mathbf{x}' \Leftrightarrow \exists \lambda \in \mathbb{R} \backslash \{\mathbf{0}\} : \mathbf{x}' = \lambda \mathbf{x}.$$

Indeed, any nonzero vector $\mathbf{x} \in \mathbb{R}^{n+1} \backslash \{\mathbf{0}\}$ determines a line, while two vectors $\mathbf{x}, \mathbf{x}'$ determine the same line if and only if they agree up to a non-zero scalar multiple. The equivalence class of $\mathbf{x} = (x^0, \ldots, x^n)$ under this relation is commonly denoted

$$[\mathbf{x}] = (x^0 : \ldots : x^n).$$

The $(\cdot : \ldots : \cdot)$ are called *homogeneous coordinates*.

13 (answer on page ??). Show that the following are equivalent characterizations of $\mathbb{R}\mathrm{P}^n$ (in the sense that there are 'natural' set-theoretic bijections):

a) The sphere S^n with antipodal identification.
b) The closed ball $B^n := \{\mathbf{x} \in \mathbb{R}^n \,|\, ||\mathbf{x}||^2 \le 1\}$, with antipodal identification on its boundary sphere S^{n-1}. (That is, $\mathbf{x} \sim -\mathbf{x}$ for $\mathbf{x} \in S^{n-1}$.)

Specializing to $n = 1$, define a bijection $\mathbb{R}\mathrm{P}^1 \cong S^1$.

The projective space $\mathbb{R}\mathrm{P}^n$ has a *standard atlas*

$$\mathscr{A} = \{(U_0, \varphi_0), \ldots, (U_n, \varphi_n)\}$$

defined as follows. For $j = 0, \ldots, n$, let

$$U_j = \{(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n |\, x^j \neq 0\}$$

be the set for which the j-th coordinate is non-zero, and put

$$\varphi_j : \; U_j \to \mathbb{R}^n, \;\; (x^0 : \ldots : x^n) \mapsto \Big(\frac{x^0}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^n}{x^j}\Big).$$

This is well-defined, since the quotients do not change when all x^i are multiplied by a fixed scalar. Put differently, given an element $[\mathbf{x}] \in \mathbb{R}\mathrm{P}^n$ for which the j-th component x^j is non-zero, we first rescale the representative $\mathbf{x}$ to make the j-th component equal to 1, and then use the remaining components as our coordinates. As an example (with $n = 2$),

$$\varphi_1(7:3:2) = \varphi_1\Big(\frac{7}{3} : 1 : \frac{2}{3}\Big) = \Big(\frac{7}{3}, \frac{2}{3}\Big).$$

From this description, it is immediate that φ_j is a bijection from U_j onto $\mathbb{R}^n$, with inverse map

$$\varphi_j^{-1}(u^1, \ldots, u^n) = (u^1 : \ldots : u^j : 1 : u^{j+1} : \ldots : u^n).$$

Geometrically, viewing $\mathbb{R}\mathrm{P}^n$ as the set of lines in $\mathbb{R}^{n+1}$, the subset $U_j \subseteq \mathbb{R}\mathrm{P}^n$ consists of those lines ℓ which intersect the affine hyperplane

$$H_j = \{\mathbf{x} \in \mathbb{R}^{n+1} |\, x^j = 1\},$$

and the map φ_j takes such a line ℓ to its unique point of intersection $\ell \cap H_j$, followed by the identification $H_j \cong \mathbb{R}^n$ (dropping the coordinate $x^j = 1$).
Let us verify that $\mathscr{A}$ is indeed an atlas. Clearly, the domains U_j cover $\mathbb{R}\mathrm{P}^n$, since any element $[\mathbf{x}] \in \mathbb{R}\mathrm{P}^n$ has at least one of its components non-zero. For $i \neq j$, the intersection $U_i \cap U_j$ consists of elements $\mathbf{x}$ with the property that both components x^i, x^j are non-zero.

14 (answer on page ??). Compute the transition maps $\varphi_i \circ \varphi_j^{-1}$, and verify that they are smooth. (You will need to distinguish between the cases $i < j$ and $i > j$.)

To complete the proof that this atlas (or the unique maximal atlas containing it) defines a manifold structure, it remains to check the Hausdorff property. This can be done with the help of Lemma 2.17, but we postpone the proof since we will soon have a simple argument in terms of smooth functions. See Proposition 3.5 below.
In summary, the real projective space $\mathbb{R}\mathrm{P}^n$ is a manifold of dimension n. For $n = 1$ it is called the *(real) projective line*, for $n = 2$ the *(real)* projective plane.

Remark 2.21. Geometrically, U_i consists of all lines in $\mathbb{R}^{n+1}$ meeting the affine hyperplane H_i, hence its complement consists of all lines that are parallel to H_i, i.e., the lines in the coordinate subspace defined by $x^i = 0$. The set of such lines is $\mathbb{R}\mathrm{P}^{n-1}$. In other words, the complement of U_i in $\mathbb{R}\mathrm{P}^n$ is identified with $\mathbb{R}\mathrm{P}^{n-1}$. Thus, as sets, $\mathbb{R}\mathrm{P}^n$ is a disjoint union

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}\mathrm{P}^{n-1},$$

where $\mathbb{R}^n$ is identified (by the coordinate map φ_i) with the open subset U_n, and $\mathbb{R}\mathrm{P}^{n-1}$ with its complement. Inductively, we obtain a decomposition

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R} \sqcup \mathbb{R}^0, \tag{2.7}$$

where $\mathbb{R}^0 = \{0\}$. At this stage, it is simply a decomposition into subsets; later it will be recognized as a decomposition into submanifolds (see Example 4.7).

15 (answer on page ??). Find an identification of the space of rotations in $\mathbb{R}^3$ with the 3-dimensional projective space $\mathbb{R}\mathrm{P}^3$. (Suggestion: Associate a rotation to every $\mathbf{x} \in \mathbb{R}^3$ with $||\mathbf{x}|| \le \pi$.)

2.3.3 Complex projective spaces

In a similar fashion, one can define a *complex projective space* $\mathbb{C}\mathrm{P}^n$ as the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$. We have

$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$$

where the equivalence relation is given by the condition that $\mathbf{z} \sim \mathbf{z}'$ if and only if there exists a complex λ with $\mathbf{z}' = \lambda \mathbf{z}$. (Note that the scalar λ is then unique, and is non-zero.) Identify $\mathbb{C}$ with $\mathbb{R}^2$, thus $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. Letting $S^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ be the 'unit sphere' consisting of complex vectors of length $||\mathbf{z}|| = 1$, we have

$$\mathbb{C}\mathrm{P}^n = S^{2n+1} / \sim,$$

where $\mathbf{z}' \sim \mathbf{z}$ if and only if there exists a complex number λ with $\mathbf{z}' = \lambda \mathbf{z}$. (Note that the scalar λ is then unique, and has absolute value 1.) One defines charts (U_j, φ_j) similarly to those for the real projective space:

$$U_j = \left\{(z^0 : \ldots : z^n)\,|\, z^j \neq 0\right\}, \;\; \varphi_j :\; U_j \to \mathbb{C}^n = \mathbb{R}^{2n},$$

$$\varphi_j(z^0 : \ldots : z^n) = \Big(\frac{z^0}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j}\Big).$$

The transition maps between charts are given by similar formulas as for $\mathbb{R}\mathrm{P}^n$ (just replace $\mathbf{x}$ with $\mathbf{z}$); they are smooth maps between open subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$. Thus $\mathbb{C}\mathrm{P}^n$ is a smooth manifold of dimension $2n$. As with $\mathbb{R}\mathrm{P}^n$ (see Equation (2.7)) there is a decomposition

$$\mathbb{C}\mathrm{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \mathbb{C}^0.$$

We will show later (Section 3.7.2) that $\mathbb{C}\mathrm{P}^1 \cong S^2$ as manifolds; for larger n we obtain genuinely ‘new’ manifolds.

Remark 2.22. (For those who know a little bit of complex analysis.) We took m-dimensional manifolds to be modeled on open subsets of $\mathbb{R}^m$, with smooth transition maps. In a similar way, one can define *complex manifolds M* of complex dimension m to be modeled on open subsets of $\mathbb{C}^m$, with transition maps that are infinitely differentiable in the complex sense, i.e. *holomorphic*. In more detail, a complex manifold of dimension m may be defined as a real manifold M of dimension $2m$, with an atlas $\mathscr{A}$, such that all coordinate charts $(U_\alpha, \varphi_\alpha)$ take values in $\mathbb{C}^m \cong \mathbb{R}^{2m}$, and all transition maps $\varphi_\alpha \circ \varphi_\beta^{-1} :\; \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are holomorphic . The complex projective space $\mathbb{C}\mathrm{P}^n$ is an important example of a complex manifold, of complex dimension n. For $n = 1$ it is called the *complex projective line*, for $n = 2$ the *complex projective plane*.

2.3.4 Real Grassmannians

The set $\mathrm{Gr}(k,n)$ of all k-dimensional subspaces of $\mathbb{R}^n$ is called the *Grassmannian of k-planes in $\mathbb{R}^n$*. (Named after *Hermann Grassmann* (1809-1877).) As a special case, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$.

We will show that the Grassmannian is a manifold of dimension

$$\dim(\mathrm{Gr}(k,n)) = k(n-k).$$

An atlas for $\mathrm{Gr}(k,n)$ may be constructed as follows. The idea is to present linear subspaces $E \subseteq \mathbb{R}^n$ of dimension k as graphs of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$. Here $\mathbb{R}^k$ is viewed as the coordinate subspace corresponding to a choice of k components from $\mathbf{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$, and $\mathbb{R}^{n-k}$ the coordinate subspace for the remaining coordinates.

To make it precise, we introduce some notation. For any subset $I \subseteq \{1, \ldots, n\}$ of the set of indices, let

$$I' = \{1, \ldots, n\} \backslash I$$

be its complement. Let $\mathbb{R}^I \subseteq \mathbb{R}^n$ be the coordinate subspace

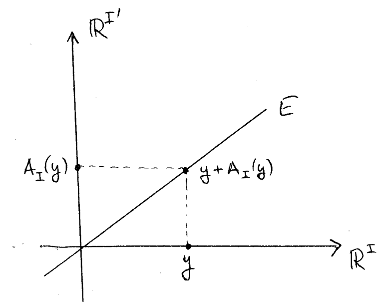
$$\mathbb{R}^I = \{\mathbf{x} \in \mathbb{R}^n |\, x^i = 0 \text{ for all } i \in I'\}.$$

If I has cardinality* $|I| = k$, then $\mathbb{R}^I \in \mathrm{Gr}(k,n)$. Note that $\mathbb{R}^{I'} = (\mathbb{R}^I)^\perp$. Let

$$U_I = \{E \in \mathrm{Gr}(k,n) |\, E \cap \mathbb{R}^{I'} = \{\mathbf{0}\}\},$$

the set of k-dimensional subspaces that are transverse to $\mathbb{R}^{I'}$. Each $E \in U_I$ is described as the graph of a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$, that is,

$$E = \{\mathbf{y} + A_I(\mathbf{y}) |\, \mathbf{y} \in \mathbb{R}^I\}.$$



16 (answer on page ??). Verify the claim that every $E \in U_I$ determines a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$ such that $E = \{\mathbf{y} + A_I(\mathbf{y}) |\, \mathbf{y} \in \mathbb{R}^I\}$.

This gives a bijection

$$\varphi_I : U_I \to \mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}),\ E \mapsto \varphi_I(E) = A_I,$$

where $\mathrm{Hom}(V,W)$ (also common is $L(V,W)$) denotes the space of linear maps from a vector space V to a vector space W. Note $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$, because the bases of $\mathbb{R}^I$ and $\mathbb{R}^{I'}$ identify the space of linear maps with $(n-k) \times k$-matrices, which in turn is just $\mathbb{R}^{k(n-k)}$ by listing the matrix entries. On the other hand, as explained in 2.20 it is not necessary to make this identification, and indeed it is better to work with the vector space $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R}^{I'})$ as the chart codomain. In terms of A_I, the subspace $E \in U_I$ is the range of the injective linear map

$$\begin{pmatrix} 1 \\ A_I \end{pmatrix} :\ \mathbb{R}^I \to \mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n \tag{2.8}$$

* It is common to use $|\cdot|$ for the cardinality ("size") of a set. Context will distinguish it from absolute value or complex modulus.

where we write elements of $\mathbb{R}^n$ as column vectors.

To check that the charts are compatible, suppose $E \in U_I \cap U_J$, and let A_I and A_J be the linear maps describing E in the two charts. We have to show that the map

$$\varphi_J \circ \varphi_I^{-1} : \operatorname{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \to \operatorname{Hom}(\mathbb{R}^J, \mathbb{R}^{J'}), \;\; A_I = \varphi_I(E) \mapsto A_J = \varphi_J(E)$$

is smooth. By assumption, E is described as the range of (2.8) and also as the range of a similar map for J. Here we are using the two decompositions $\mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n$ and $\mathbb{R}^J \oplus \mathbb{R}^{J'} \cong \mathbb{R}^n$. It is convenient to describe everything in terms of $\mathbb{R}^J \oplus \mathbb{R}^{J'}$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$$

be the matrix corresponding to the identification $\mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^n$ followed by the inverse of $\mathbb{R}^J \oplus \mathbb{R}^{J'} \to \mathbb{R}^n$. For example, the lower left block 'c' is the inclusion $\mathbb{R}^I \to \mathbb{R}^n$ as the corresponding coordinate subspace, followed by projection to the coordinate subspace $\mathbb{R}^{J'}$. By definition of E, the injective linear maps

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ A_I \end{pmatrix} : \mathbb{R}^I \to \mathbb{R}^J \oplus \mathbb{R}^{J'}, \quad \begin{pmatrix} 1 \\ A_J \end{pmatrix} : \mathbb{R}^J \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$$

have the same range (namely, E once we identify $\mathbb{R}^J \oplus \mathbb{R}^J \cong \mathbb{R}^n$). In other words, there is an isomorphism $S : \mathbb{R}^I \to \mathbb{R}^J$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ A_I \end{pmatrix} = \begin{pmatrix} 1 \\ A_J \end{pmatrix} S$$

as maps $\mathbb{R}^I \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$. We obtain

$$\begin{pmatrix} a + bA_I \\ c + dA_I \end{pmatrix} = \begin{pmatrix} S \\ A_J S \end{pmatrix}$$

Using the first row of this equation to eliminate the second row of this equation, we obtain the desired formula for the transition function $\varphi_J \circ \varphi_I^{-1}$, expressing A_J in terms of A_I:

$$A_J = (c + dA_I)\,(a + bA_I)^{-1}.$$

The dependence of the right hand side on the matrix entries of A_I is smooth, by Cramer's formula for the inverse matrix.

It follows that the collection of all $\varphi_I : U_I \to \operatorname{Hom}(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$ defines on $\operatorname{Gr}(k,n)$ the structure of a manifold of dimension $k(n-k)$. The number of charts of this atlas equals the number of subsets $I \subseteq \{1, \ldots, n\}$ of cardinality k, that is, it is equal to $\binom{n}{k}$.

The Hausdorff property may be checked in a similar fashion to $\mathbb{R}\mathrm{P}^n$. Here is a sketch of an alternative argument (later, we will have much simpler criteria for the Hausdorff property, avoiding these types of ad-hoc arguments). Given distinct $E_1, E_2 \in \operatorname{Gr}(k,n)$, choose a subspace $F \in \operatorname{Gr}(k,n)$ such that $F^\perp$ has zero intersection with both E_1, E_2.

(Such a subspace always exists.) One can then define a chart (U,φ), where U is the set of subspaces E transverse to $F^{\perp}$, and φ realizes any such map as the graph of a linear map $F \to F^{\perp}$. Thus $\varphi:\ U \to \mathrm{Hom}(F,F^{\perp})$. As above, we can check that this is compatible with all the charts (U_I,φ_I). Since both E_1,E_2 are in this chart U, we are done by Lemma 2.17.

17 (answer on page ??). Prove the parenthetical remark above: Given $E_1,E_2 \in \mathrm{Gr}(k,n)$ there exists $F \in \mathrm{Gr}(k,n)$ such that $F^{\perp} \cap E_1 = \{\mathbf{0}\}$ and $F^{\perp} \cap E_2 = \{\mathbf{0}\}$.

Remark 2.23. As already mentioned, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$. One can check that our system of charts in this case is the standard atlas for $\mathbb{R}\mathrm{P}^{n-1}$.

18 (answer on page ??). This is a preparation for the following remark. Recall that a linear map $\Pi:\ \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection onto some subspace $E \subseteq \mathbb{R}^n$ if $\Pi(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in E$ and $\Pi(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in E^{\perp}$. Show that a square matrix $P \in \mathrm{Mat}_{\mathbb{R}}(n)$ is the matrix of an orthogonal projection if and only if it has the properties

$$P^{\top} = P,\ PP = P,$$

where the superscript $\top$ indicates 'transpose'. What is the matrix of the orthogonal projection onto $E^{\perp}$?

For any k-dimensional subspace $E \subseteq \mathbb{R}^n$, let $P_E \in \mathrm{Mat}_{\mathbb{R}}(n)$ be the matrix of the linear map given by orthogonal projection onto E. By the 18,

$$P_E^{\top} = P_E,\ \ P_E P_E = P_E;$$

conversely, any square matrix P with the properties $P^{\top} = P,\ PP = P$ with $\mathrm{rank}(P) = k$ corresponds to a k-dimensional subspace $E = \{P\mathbf{x} |\ \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^n$. This identifies the Grassmannian $\mathrm{Gr}(k,n)$ with the set of orthogonal projections of rank k. In summary, we have an inclusion

$$\mathrm{Gr}(k,n) \hookrightarrow \mathrm{Mat}_{\mathbb{R}}(n) \cong \mathbb{R}^{n^2}, \quad E \mapsto P_E. \tag{2.9}$$

Note that this inclusion takes values in the subspace $\mathrm{Sym}_{\mathbb{R}}(n) \cong \mathbb{R}^{n(n+1)/2}$ of *symmetric* $n \times n$*-matrices*.

19 (answer on page ??). Describe a natural bijection $\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n)$, both in terms of subspaces and in terms of orthogonal projections.

20 (answer on page ??). Given a real n-dimensional vector space V, let $\mathrm{Gr}(k,V)$ be the set of k-dimensional subspaces of V. It is identified with $\mathrm{Gr}(k,n)$ once a basis of V is chosen; in particular it is a manifold of dimension $k(n-k)$. Letting V^* be the dual space, describe a bijection $\mathrm{Gr}(k,V) \cong \mathrm{Gr}(n-k,V^*)$, which is 'natural' in the sense that it does not depend on additional choices (such as a choice of basis).

Remark 2.24. Similar to $\mathbb{R}\mathrm{P}^2 = S^2/\sim$, the quotient modulo antipodal identification, one can also consider

$$M = (S^2 \times S^2)/\sim$$

the quotient space by the equivalence relation

$$(\mathbf{x},\mathbf{x}') \sim (-\mathbf{x},-\mathbf{x}').$$

It turns out (see, e.g., [4]) that this manifold M is the same as $\mathrm{Gr}(2,4)$, in the sense that there is a bijection of sets identifying the atlases.

2.3.5 Complex Grassmannians

Similar to the case of projective spaces, one can also consider the *complex Grassmannian* $\mathrm{Gr}_{\mathbb{C}}(k,n)$ of complex k-dimensional subspaces of $\mathbb{C}^n$. It is a manifold of dimension $2k(n-k)$, which can also be regarded as a complex manifold of complex dimension $k(n-k)$.

2.4 Open subsets

Let M be a set equipped with an m-dimensional maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$.

Definition 2.25. *A subset $U \subseteq M$ is* open *if and only if for all charts $(U_\alpha, \varphi_\alpha) \in \mathscr{A}$ the set $\varphi_\alpha(U \cap U_\alpha)$ is open.*

To check that a subset U is open, it is not actually necessary to verify this condition for all charts. As the following proposition shows, it is enough to check for any collection of charts whose union contains U. In particular, we may take $\mathscr{A}$ in Definition 2.25 to be any atlas, not necessarily a maximal atlas.

Proposition 2.26. *Given $U \subseteq M$, let $\mathscr{B} \subseteq \mathscr{A}$ be any collection of charts whose union contains U. Then U is open if and only if for all charts (U_β, φ_β) from $\mathscr{B}$, the sets $\varphi_\beta(U \cap U_\beta)$ are open.*

Proof. In what follows, we reserve the index β to indicate charts (U_β, φ_β) from $\mathscr{B}$. Suppose $\varphi_\beta(U \cap U_\beta)$ is open for all such β. Let $(U_\alpha, \varphi_\alpha)$ be a given chart in the maximal atlas $\mathscr{A}$. We have that

$$\begin{aligned}\varphi_\alpha(U \cap U_\alpha) &= \bigcup_\beta \varphi_\alpha(U \cap U_\alpha \cap U_\beta)\\ &= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U \cap U_\alpha \cap U_\beta)\big)\\ &= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U_\alpha \cap U_\beta) \cap \varphi_\beta(U \cap U_\beta)\big).\end{aligned}$$

Since $\mathscr{B} \subseteq \mathscr{A}$, all $\varphi_\beta(U_\alpha \cap U_\beta)$ are open. Hence the intersection with $\varphi_\beta(U \cap U_\beta)$ is open, and so is the preimage under the diffeomorphism $\varphi_\alpha \circ \varphi_\beta^{-1}$. Finally, we use that a union of open sets of $\mathbb{R}^m$ is again open. This proves the 'if' part; the 'only if' part is obvious. $\square$

If $\mathscr{A}$ is an atlas on M, and $U \subseteq M$ is open, then U inherits an atlas by restriction:

$$\mathscr{A}_U = \{(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha})\}.$$

21 (answer on page ??). Verify that if $\mathscr{A}$ is a maximal atlas, then so is $\mathscr{A}_U$, and if this maximal atlas $\mathscr{A}$ satisfies the countability and Hausdorff properties, then so does $\mathscr{A}_U$.

This then proves:

Proposition 2.27 (Open subsets are manifolds). *An open subset of a manifold is again a manifold.*

The collection of open sets of M with respect to an atlas has properties similar to those for $\mathbb{R}^n$:

Proposition 2.28. *Let M be a set with an m-dimensional maximal atlas. The collection of all open subsets of M has the following properties:*

- *$\emptyset, M$ are open.*
- *The intersection $U \cap U'$ of any two open sets U, U' is again open.*
- *The union $\bigcup_{i\in I} U_i$ of a collection $U_i,\ i \in I$ of open sets is again open. Here I is any indexing set (not necessarily countable).*

Proof. All of these properties follow from similar properties of open subsets in $\mathbb{R}^m$. For instance, if U, U' are open, then

$$\varphi_\alpha((U \cap U') \cap U_\alpha) = \varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(U' \cap U_\alpha)$$

is an intersection of open subsets of $\mathbb{R}^m$, hence it is open and therefore $U \cap U'$ is open. $\square$

These properties mean, by definition, that the collection of open subsets of M defines a *topology* on M. This allows us to adopt various notions from topology:

a) If U is an open subset and $p \in U$, then U is called an *open neighborhood of* p. More generally, if $A \subseteq U$ is a subset contained in M, then U is called an *open neighborhood of* A.
b) A subset $A \subseteq M$ is called *closed* if its complement $M \backslash A$ is open.
c) M is called *disconnected* if it can be written as the disjoint union $M = U \sqcup V$ of two open subsets $U, V \subseteq M$ (with $U \cap V = \emptyset$). M is called *connected* if it is not disconnected; equivalently, if the only subsets $A \subseteq M$ that are both closed and open are $A = \emptyset$ and $A = M$.

The Hausdorff condition in the definition of manifolds can now be restated as the condition that *any two distinct points* p, q *in* M *have disjoint open neighborhoods*. (It is not necessary to take them to be domains of coordinate charts.)
It is immediate from the definition that domains of coordinate charts are open. Indeed, this gives an alternative way of defining the open sets:

22 (answer on page ??). Let M be a set with a maximal atlas. Show that a subset $U \subseteq M$ is open if and only if it is either empty, or is a union $U = \bigcup_{i \in I} U_i$ where the U_i are domains of coordinate charts.

23 (answer on page ??). Let M be a set with an m-dimensional maximal atlas $\mathscr{A}$, and let (U, φ) be a chart in $\mathscr{A}$. Let $V \subseteq \mathbb{R}^m$ be open. Prove that $\varphi^{-1}(V)$ is open.

Regarding the notion of connectedness, we have:

24 (answer on page ??). Let M be a set with an m-dimensional maximal atlas. A function $f : M \to \mathbb{R}$ is called *locally constant* if every point $p \in M$ has an open neighborhood over which f is constant. Show that M is connected if and only if every locally constant function is in fact constant.

2.5 Compact subsets

Another important concept from topology that we will need is the notion of *compactness*. Recall (e.g. Munkres [13], Chapter 1§4) that a subset $A \subseteq \mathbb{R}^m$ is *compact* if it has the following property: For every collection $\{U_\alpha\}$ of open subsets of $\mathbb{R}^m$

whose union contains A, the set A is already covered by finitely many subsets from that collection. One then proves the important result (see Munkres [13], Theorems 4.2 and 4.9)

Theorem 2.29 (Heine-Borel). *A subset $A \subseteq \mathbb{R}^m$ is compact if and only if it is closed and bounded.*

While 'closed and bounded' is a simpler characterization of compactness to work with, it does not directly generalize to manifolds (or other topological spaces), while the original definition does:

Definition 2.30. *A subset $A \subseteq M$ of a manifold M is* compact *if it has the following property: For every collection $\{U_\alpha\}$ of open subsets of M whose union contains A, the set A is already covered by finitely many subsets from that collection.*

In short, $A \subseteq M$ is compact if every open cover admits a finite subcover. Definition 2.30 works more generally for any topological space M, in particular also for non-Hausdorff manifolds.

Proposition 2.31. *Let M be a manifold, and $A \subseteq M$ a subset which is contained in the domain of a coordinate chart (U, φ). Then A is compact in M if and only if $\varphi(A)$ is compact in $\mathbb{R}^n$.*

Proof. Suppose $\varphi(A)$ is compact. Let $\{U_\alpha\}$ be an open cover of A. Then the sets $U \cap U_\alpha$ are again an open cover of A, and their images $\varphi(U \cap U_\alpha)$ are an open cover of $\varphi(A)$. Since $\varphi(A)$ is compact, there are indices $\alpha_1, \ldots, \alpha_N$ such that

$$\varphi(A) \subseteq \varphi(U \cap U_{\alpha_1}) \cup \cdots \cup \varphi(U \cap U_{\alpha_N}).$$

Since $\varphi: \ U \to \varphi(U)$ is a bijection, this then implies that

$$A \subseteq (U \cap U_{\alpha_1}) \cup \cdots \cup (U \cap U_{\alpha_N}) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}.$$

Consequently, $U_{\alpha_1}, \ldots, U_{\alpha_N}$ are the desired finite subcover of A, which proves the direction '$\varphi(A)$ compact' $\Rightarrow$ 'A compact'. We invite the reader to prove the converse. □

25 (answer on page ??). Complete the proof, by proving that if A is compact, so is $\varphi(A)$.

The proposition is useful, since we can check compactness of $\varphi(A)$ by using the Heine-Borel criterion, Theorem 2.29. For more general subsets of M, we can often decide compactness by combining this result with the following:

Proposition 2.32. *If $A_1, \ldots, A_k \subseteq M$ is a finite collection of compact subsets, then their union $A = A_1 \cup \cdots \cup A_k$ is again compact.*

Proof. If $\{U_\alpha\}$ is an open cover of A, then in particular it is an open cover of each of the sets $A_1, \ldots, A_k$. For each A_i, we can choose a finite subcover. The collection of all U_α's that appear in at least one of these subcovers is then a finite subcover for A.

Example 2.33. Let $M = S^n$. The closed upper hemisphere $\{\mathbf{x} \in S^n |\ x^0 \geq 0\}$ is compact, because it is contained in the coordinate chart (U_+, φ_+) for stereographic projection, and its image under φ_+ is the closed and bounded subset $\{\mathbf{u} \in \mathbb{R}^n |\ ||\mathbf{u}|| \leq 1\}$. Likewise the closed lower hemisphere is compact, and hence S^n itself (as the union of the upper and lower hemispheres) is compact.

Example 2.34. Let $\{(U_i, \varphi_i) |\ i = 0, \ldots, n\}$ be the standard atlas for $\mathbb{R}\mathrm{P}^n$. Let

$$A_i = \{(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n |\ ||\mathbf{x}||^2 \leq (n+1)\,(x^i)^2\}.$$

Then $A_i \subseteq U_i$ (since necessarily $x^i \neq 0$ for elements of A_i). Furthermore, $\bigcup_{i=0}^n A_i = \mathbb{R}\mathrm{P}^n$: Indeed, given any $(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n$, let i be an index for which $|x^i|$ is maximal (here $|\cdot|$ denotes the absolute value). Then $||\mathbf{x}||^2 \leq (n+1)(x^i)^2$ (since the right hand side is obtained from the left hand side by replacing each $(x^j)^2$ with $(x^i)^2 \geq (x^j)^2$), hence $(x^0 : \ldots : x^n) \in A_i$. Finally, one checks that $\varphi_i(A_i) \subseteq \mathbb{R}^n$ is a closed ball of radius $\sqrt{n+1}$, and in particular is compact.

In a similar way, one can prove the compactness of $\mathbb{C}\mathrm{P}^n$, $\mathrm{Gr}(k,n)$, $\mathrm{Gr}_{\mathbb{C}}(k,n)$. However, soon we will have a simpler way of verifying compactness, by showing that they are closed and bounded subsets of $\mathbb{R}^N$ for a suitable N, and applying the Heine-Borel criterion 2.29.
For manifolds M, we may define a subset $A \subseteq M$ to be *bounded* if it is contained in some compact subset of M. By 26 below, it is then true that closed, bounded subsets of manifolds are compact. (But we cannot use this as a definition of compactness, since we used compactness to define boundedness.)

26 (answer on page ??). Show that if $A \subseteq M$ is compact, and $C \subseteq M$ is closed, then $A \cap C$ is compact. In particular, closed subsets of compact subsets are compact.

The following fact uses the Hausdorff property (and holds in fact for any Hausdorff topological space).

Proposition 2.35. *If M is a manifold, then every compact subset $A \subseteq M$ is closed.*

Proof. Suppose $A \subseteq M$ is compact. Let $p \in M \backslash A$ be given. For every $q \in A$ we may choose, by the Hausdorff property, disjoint open neighborhoods V_q of q and U_q of p. The collection $\{V_q,\ q \in A\}$ is an open cover of A, hence there exists a finite subcover $V_{q_1}, \ldots, V_{q_k}$. The intersection $U = U_{q_1} \cap \cdots \cap U_{q_k}$ is an open subset of M with $p \in M$ and not meeting $V_{q_1} \cup \cdots \cup V_{q_k}$, hence not meeting A. We have thus shown that every $p \in M \backslash A$ has an open neighborhood $U \subseteq M \backslash A$. The union over all such open neighborhoods for all $p \in M \backslash A$ is all of $M \backslash A$, which hence is open. It follows that A is closed. $\square$

For non-Hausdorff manifolds, compact subsets need not be closed. See Problem **??** at the end of this chapter.

2.6 Oriented manifolds

Our next aim is to give an intrinsic definition of *orientation* on a manifold. The spheres or complex projective spaces will be examples of orientable manifolds, whereas the Möbius strip, the projective plane $\mathbb{R}\mathrm{P}^2$, and the Klein bottle, are typical examples of non-orientable manifolds.
For the definition, observe that since the Jacobian matrix $D(\psi\circ\varphi^{-1})$ of the transition map between any two charts (U,φ) and (V,ψ) on a set M is invertible, its determinant (the Jacobian determinant) is non-zero everywhere on $\varphi(U\cap V)$.

Definition 2.36. *Two charts $(U,\varphi),(V,\psi)$ for a set M are* oriented-compatible *if the Jacobian determinant is positive everywhere:*

$$\det\big(D(\psi\circ\varphi^{-1}(\mathbf{x}))\big)>0$$

for all $\mathbf{x}\in\varphi(U\cap V)$. An oriented atlas *on M is an atlas such that any two of its charts are oriented-compatible; a* maximal oriented atlas *is one containing every chart which is oriented-compatible with all charts in this atlas. An* oriented manifold *is a set with a maximal oriented atlas, satisfying the Hausdorff and countability conditions as in definition 2.16. A manifold is called* orientable *if it admits an oriented atlas.*

The notion of an orientation on a manifold will become crucial later, since integration of differential forms over manifolds is only defined if the manifold is oriented.

Example 2.37. The spheres S^n are orientable. To see this, consider the atlas with the two charts (U_+,φ_+) and (U_-,φ_-), given by stereographic projections. (Section 2.3.1.) Here $\varphi_-(U_+\cap U_-)=\varphi_+(U_+\cap U_-)=\mathbb{R}^n\backslash\{\mathbf{0}\}$, with transition map $\varphi_-\circ\varphi_+^{-1}(\mathbf{u})=\mathbf{u}/||\mathbf{u}||^2$. The Jacobian matrix $D(\varphi_-\circ\varphi_+^{-1})(\mathbf{u})$ has entries[†]

$$\big(D(\varphi_-\circ\varphi_+^{-1})(\mathbf{u})\big)_{ij}=\frac{\partial}{\partial u^j}\Big(\frac{u^i}{||\mathbf{u}||^2}\Big)=\frac{1}{||\mathbf{u}||^2}\delta_{ij}-\frac{2u_iu_j}{||\mathbf{u}||^4}. \tag{2.10}$$

Its determinant is (see ✎ 27 below)

$$\det\big(D(\varphi_-\circ\varphi_+^{-1})(\mathbf{u})\big)=-||\mathbf{u}||^{-2n}<0. \tag{2.11}$$

Hence, the given atlas is *not* an oriented atlas. But this is easily remedied: Simply compose one of the charts, say U_-, with the map $(u_1,u_2,\dots,u_n)\mapsto(-u_1,u_2,\dots,u_n)$; then with the resulting new coordinate map $\widetilde{\varphi_-}$ the atlas $(U_+,\varphi_+),(U_-,\widetilde{\varphi_-})$ will be an oriented atlas.

[†] Here $\delta_{ij}=\begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$ is Kronecker delta function.

27 (answer on page ??). Check that the given vector $\mathbf{u}$ (regarded as a vector in $\mathbb{R}^n$) is an eigenvector of the matrix A with entries A_{ij} given by (2.10), as is any vector orthogonal to $\mathbf{u}$. Find the corresponding eigenvalues, and use this to compute $\det(A)$.

Example 2.38. The complex projective spaces $\mathbb{C}\mathrm{P}^n$ and the complex Grassmannians $\mathrm{Gr}_{\mathbb{C}}(k,n)$ are all orientable. This follows because they are *complex manifolds* (see Remark 2.22), i.e., the transition maps for their standard charts, as maps between open subsets of $\mathbb{C}^m$, are actually complex-holomorphic. This implies that as real maps, their Jacobian determinant is positive everywhere (28 below).

28 (answer on page ??). Let $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ be a complex square matrix, and $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ the same matrix regarded as a real-linear transformation of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Show that

$$\det_{\mathbb{R}}(A_{\mathbb{R}}) = |\det_{\mathbb{C}}(A)|^2.$$

(Here $|\cdot|$ signifies the complex modulus. You may want to start with the case $n = 1$.)

We shall see shortly that if a connected manifold is orientable, then there are exactly two orientations. Given one orientation, the other one is obtained by the following procedure.

Definition 2.39. *Let M be an oriented manifold, with oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$. Then the* opposite orientation *on M is obtained by replacing each φ_α by its composition with the map $(u^1,\dots,u^m) \mapsto (-u^1,u^2,\dots,u^m)$.*

Proposition 2.40. *Let M be an oriented manifold, and (U,φ) a connected chart compatible with the atlas of M. Then this chart is compatible either with the given orientation of M, or with the opposite orientation.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas for M. Given $p \in U$, with image $\mathbf{x} \in \widetilde{U} = \varphi(U)$, let

$$\varepsilon(p) = \pm 1$$

be the sign of the determinant of $D(\varphi_\alpha \circ \varphi^{-1})(\mathbf{x})$ for any chart $(U_\alpha, \varphi_\alpha)$ containing p. This is well-defined, for if (U_β, φ_β) is another such chart, then the Jacobian matrix $D(\varphi_\beta \circ \varphi^{-1})(\mathbf{x})$ is the product of $D(\varphi_\alpha \circ \varphi^{-1})(\mathbf{x})$ and $D(\varphi_\beta \circ \varphi_\alpha^{-1})(\mathbf{x})$, and the latter has positive determinant since the atlas is oriented by assumption. It is also clear that the function $\varepsilon : U \to \{1,-1\}$ is constant near any given p, hence it is locally constant and therefore constant (since U is connected). It follows that (U,φ) is either compatible with the orientation of M (if $\varepsilon = +1$) or with the opposite orientation (if $\varepsilon = -1$). $\square$

29 (answer on page ??). Show that if M is a connected, orientable manifold, then there are exactly two orientations on M. In fact, any connected chart determines a unique orientation for which it is an oriented chart.

Example 2.41. $\mathbb{R}\mathrm{P}^2$ is non-orientable. To see this, consider its standard atlas with charts (U_i, φ_i) for $i = 0, 1, 2$. Suppose $\mathbb{R}\mathrm{P}^2$ has an orientation. By the proposition, each of the charts is compatible either with the given orientation, or with the opposite orientation. But the transition map between (U_0, φ_0), (U_1, φ_1) is

$$(\varphi_1 \circ \varphi_0^{-1})(u^1, u^2) = \Big(\frac{1}{u^1}, \frac{u^2}{u^1}\Big)$$

defined on $\varphi_0(U_0 \cap U_1) = \{(u^1, u^2) :\ u^1 \neq 0\}$. This has Jacobian determinant $-u_1^{-3}$, which changes sign. Thus, the two charts cannot be oriented-compatible, even after composing the coordinate map of one of them with $(u^1, u^2) \mapsto (-u^1, u^2)$. This contradiction shows that $\mathbb{R}\mathrm{P}^2$ is not orientable.

Using similar arguments, one can show that $\mathbb{R}\mathrm{P}^n$ for $n \geq 2$ is orientable if and only if n is odd. See Problem **??** at the end of this chapter. More generally, the real Grassmannian $\mathrm{Gr}(k, n)$ for $n \geq 2$ is orientable if and only if n is even.

2.7 Building new manifolds

2.7.1 Disjoint Union

Given manifolds M, M' of the same dimensions m, with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the disjoint union $N = M \sqcup M'$ is again an m-dimensional manifold with atlas $\{(U_\alpha, \varphi_\alpha)\} \cup \{(U'_\beta, \varphi'_\beta)\}$. This manifold N is not much more interesting than considering M and M' separately, but is the first step towards "gluing" M and M' in an interesting way, which often results in genuinely new manifold (more below). More generally, given a countable collection of manifolds of the same dimension, their disjoint union is a manifold.

2.7.2 Products

Given manifolds M, M' of dimensions m, m' (not necessarily the same), with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the cartesian product $M \times M'$ is a manifold of dimension $m + m'$. An atlas is given by the product charts $U_\alpha \times U'_\beta$ with the product maps $\varphi_\alpha \times \varphi'_\beta :\ (x, x') \mapsto (\varphi_\alpha(x), \varphi'_\beta(x'))$. For example, the 2-torus $T^2 = S^1 \times S^1$ becomes a manifold in this way, and more generally the n-torus

$$T^n = S^1 \times \cdots \times S^1.$$

2.7.3 Connected sums

Let M_1, M_2 be connected, oriented manifolds of the same dimension m. The connected sum $M_1 \# M_2$ is obtained by first removing chosen points $p_1 \in M_1,\ p_2 \in M_2$, and gluing in an open cylinder to connect the two 'punctured' manifolds. In more detail let (U_i, φ_i) be coordinate charts around chosen points $p_i \in M_i$, with $\varphi_i(p_i) = \mathbf{0}$, and let $\varepsilon > 0$ be such that $B_\varepsilon(\mathbf{0}) \subseteq \varphi_i(U_i)$. We assume that (U_2, φ_2) is oriented-compatible, and (U_1, φ_1) is oriented-compatible with M_1^{op}, the manifold with *opposite* orientation (!) to that of M_1. Denote

$$Z = S^{m-1} \times (-\varepsilon, \varepsilon)$$

the 'open cylinder'; elements of this cylinder will be denoted as pairs $(\mathbf{v}, t)$. We define

$$M_1 \# M_2 = \Big((M_1 \backslash \{p_1\}) \sqcup Z \sqcup (M_2 \backslash \{p_2\}) \Big) / \sim,$$

where the equivalence relation identifies

$$(\mathbf{v}, t) \sim \varphi_2^{-1}(t\mathbf{v}) \in M_2 \quad \text{for} \quad 0 < t < \varepsilon$$

$$(\mathbf{v}, t) \sim \varphi_1^{-1}(-t\mathbf{v}) \in M_1 \quad \text{for} \quad -\varepsilon < t < 0$$

Note that both maps

$$S^{m-1} \times (0, \varepsilon) \to M_2,\ (\mathbf{v}, t) \to \varphi_1^{-1}(t\mathbf{v})$$

and

$$S^{m-1} \times (-\varepsilon, 0) \to M_1,\ (\mathbf{v}, t) \to \varphi_1^{-1}(-t\mathbf{v})$$

are orientation preserving; for the latter this follows since it is a composition of two orientation reversing maps. The result is an oriented manifold. It is a non-trivial fact that up to diffeomorphism of oriented manifolds, the connected sum does not depend on the choices made.

Generalizing the connected sum of two different manifolds, one can apply a similar construction to a pair of points $p_1 \neq p_2$ of a connected, oriented manifold M. This is the higher-dimensional version of 'attaching a handle'.

2.7.4 Quotients

We have seen various examples of manifolds defined as quotient spaces for equivalence relations on given manifolds.

a) The real projective space $\mathbb{R}\mathrm{P}^n$ can be defined as the quotient $(\mathbb{R}^{n+1} \backslash \{0\}) / \sim$ under the equivalence relation $\mathbf{x} \sim \mathbf{x}' \Leftrightarrow \mathbb{R}\mathbf{x} = \mathbb{R}\mathbf{x}'$, or also as a quotient $S^n / \sim$ for the equivalence relation $\mathbf{x} \sim -\mathbf{x}$ (antipodal identification).
b) The non-Hausdorff manifold 2.15 was also defined as the quotient under an equivalence relation, $(\mathbb{R} \sqcup \mathbb{R}) / \sim$. The non-example illustrates a typical problem for such constructions: even if the quotient inherits an atlas, it may fail to be Hausdorff.

c) Our construction of the connected sum of oriented manifolds, from the previous section, also involved a quotient by an equivalence relation.
d) Let M be a manifold with a given countable atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$. Let

$$Q = \bigsqcup_\alpha U_\alpha$$

be the manifold given as the disjoint union of the chart domains, and let $\pi : Q \to M$ be the map whose restriction to U_α is the obvious inclusion into M. Since π is surjective, this realizes M as a quotient $Q/\sim$, formalizing the idea that every manifold is obtained by gluing charts.

There is a general criterion for deciding when the quotient space of a manifold M under an equivalence relation determines a manifold structure on the quotient space $M/\sim$. However, we shall need more tools to formulate this result. See Theorem 4.35 below.

3

Smooth Maps

3.1 Smooth functions on manifolds

A real-valued function on an open subset $U \subseteq \mathbb{R}^m$ is called *smooth at* $\mathbf{x} \in U$ if it is infinitely differentiable on an open neighborhood of $\mathbf{x}$. It is called *smooth on* U if it is smooth at all points of U. The notion of smooth functions on open subsets of Euclidean spaces carries over to manifolds: A function is smooth if its expression in local coordinates is smooth.

Definition 3.1. *Let M be a manifold. A function $f: M \to \mathbb{R}$ is called* smooth at $p \in M$ *if there exists a chart (U, φ) around p such that the function*

$$f \circ \varphi^{-1}: \ \varphi(U) \to \mathbb{R}$$

is smooth at $\varphi(p)$; it is called smooth *if it is smooth at all points of M. The set of smooth functions on M is denoted $C^\infty(M)$.*

The condition for smoothness at p does not depend on the choice of chart: If (U', φ') is another chart containing p, then the two maps

$$(f \circ \varphi^{-1})|_{\varphi(U \cap U')}, \quad (f \circ (\varphi')^{-1})|_{\varphi'(U \cap U')}$$

are related by the transition map $\varphi \circ (\varphi')^{-1}$, which is a diffeomorphism. It follows that the first map is smooth at $\varphi(p)$ if and only if the second map is smooth at $\varphi'(p)$. Consequently, to check if f is smooth on M, it suffices to take *any* atlas $\{(U_\alpha, \varphi_\alpha)\}$ for M (not necessarily the maximal atlas), and verify that for all charts from this atlas, the maps $f \circ \varphi_\alpha^{-1}: \ \varphi(U_\alpha) \to \mathbb{R}$ are smooth.

Example 3.2. The 'height function'

$$f: \ S^2 \to \mathbb{R}, \quad (x, y, z) \mapsto z$$

is smooth. This may be checked, for example, by using the 6-chart atlas given by projection onto the coordinate planes: E.g., in the chart $U = \{(x, y, z) | \ z > 0\}$ with $\varphi(x, y, z) = (x, y)$, we have that

$$(f \circ \varphi^{-1})(x,y) = \sqrt{1-(x^2+y^2)}$$

which is smooth on $\varphi(U) = \{(x,y) | \, x^2+y^2 < 1\}$. (The argument for the other charts in this atlas is similar.) Alternatively, we could also use the atlas with two charts, given by stereographic projection.
A similar argument shows, more generally, that for any smooth function $h \in C^\infty(\mathbb{R}^3)$ (for example the coordinate functions), the restriction $f = h|_{S^2}$ is again smooth.

30 (answer on page ??). Check the whether or not the map $f: \ S^2 \to \mathbb{R}, \ (x,y,z) \mapsto \sqrt{1-z^2}$ is smooth.

31 (answer on page ??). Check that the map

$$f: \mathbb{R}\mathrm{P}^2 \to \mathbb{R}, \ (x:y:z) \mapsto \frac{yz+xz+xy}{x^2+y^2+z^2}$$

is well-defined, and use charts to show that it is smooth.

From the properties of smooth functions on $\mathbb{R}^m$, one gets the following properties of smooth $\mathbb{R}$-valued functions on manifolds M:

- If $f,g \in C^\infty(M)$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda f + \mu g \in C^\infty(M)$.
- If $f,g \in C^\infty(M)$, then $fg \in C^\infty(M)$.
- $1 \in C^\infty(M)$ (where '1' denotes the constant function taking on the value 1).

32 (answer on page ??). Prove the assertion that $f,g \in C^\infty(M) \implies fg \in C^\infty(M)$.

These properties say that $C^\infty(M)$ is an *algebra*, with unit the constant function 1. (See Appendix A.3 for some background information on algebras.) Below, we will develop many of the concepts of manifolds in terms of the algebra of smooth functions. In particular, M itself may be recovered from this algebra. (See Problem **??**.)

Definition 3.3. *The* support *of a function* $f: \ M \to \mathbb{R}$ *is the smallest closed subset*

$$\mathrm{supp}(f) \subseteq M$$

with the property that f *is zero outside of* $\mathrm{supp}(f)$.

In other words, $\mathrm{supp}(f)$ is the *closure* (see Appendix B) of the subset where f is non-zero. The following result will be needed later on.

Lemma 3.4 (Extension by zero). *Suppose U is an open subset of a manifold M, and let $g \in C^\infty(U)$ be such that* $\mathrm{supp}(g) \subseteq U$ *is closed as a subset of M. Then the function $f : M \to \mathbb{R}$, given by $f|_U = g$ and $f|_{M\setminus U} = 0$, is smooth.*

Proof. By assumption, $V = M\setminus\mathrm{supp}(g)$ is open, and contains $M\setminus U$. Thus, U and V are an open cover of M. Since both $f|_U = g$ and $f|_V = 0$ are smooth, it follows that f is smooth.

33 (answer on page ??). Give examples of an open subset $U \subseteq M$ and a smooth function $g \in C^\infty(U)$ such that

a) g does not extend to a smooth function $f \in C^\infty(M)$.

b) g extends by zero to a smooth function $f \in C^\infty(M)$, even though $\mathrm{supp}(g) \subseteq U$ is not closed in M.

3.2 The Hausdorff property via smooth functions

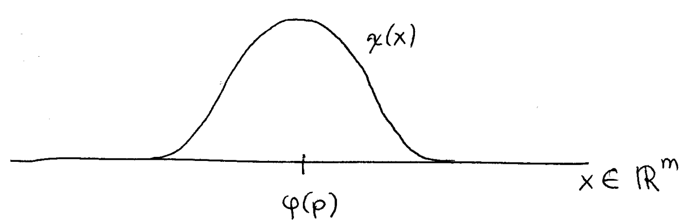
Suppose M is any set with a maximal atlas. The definition of the algebra of smooth functions $C^\infty(M)$ does not use the Hausdorff or countability conditions; hence it makes sense in this more general context. In fact, we may use smooth functions to check the Hausdorff property:

Proposition 3.5. *Suppose M is any set with an m-dimensional maximal atlas, and p,q are two distinct points in M. Then the following are equivalent:*

a) There are disjoint open neighborhoods U of p and V of q.

b) There exists a smooth function $f : M \to \mathbb{R}$ with $f(p) \neq f(q)$.

Proof. "$(i) \Rightarrow (ii)$". Suppose (i) holds. As explained in Section 2.4, we may take U and V to be the domains of coordinate charts (U,φ) around p and (V,ψ) around q. Choose $\chi \in C^\infty(\mathbb{R}^m)$ with $\mathrm{supp}(\chi) \subseteq \varphi(U)$, and such that $\chi(p) = 1$. (For example, we may take χ to be a 'bump function' on a small ball centered at $\varphi(p)$, see Lemma B.8 in the Appendix.)



Then
$$\operatorname{supp}(\chi \circ \varphi) = \varphi^{-1}(\operatorname{supp}(\chi))$$
is closed *as a subset of* M, hence Lemma 3.4 shows that $\chi \circ \varphi$ extends by 0 to a smooth function $f \in C^\infty(M)$. This function satisfies $f(p) = \chi(\varphi(p)) = 1$ while $f(q) = 0$ since $q \in V \subseteq M \backslash \operatorname{supp}(\chi \circ \varphi)$.

"$(ii) \Rightarrow (i)$". Given $f \in C^\infty(M)$ as in (ii), let $a = f(p),\ b = f(q)$. Choose coordinate charts (U, φ) around p and (V, ψ) around q. Since the function $f \circ \varphi^{-1}$ on $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$ is smooth, it is in particular continuous. Hence, the set
$$\widetilde{U}' = \{\mathbf{x} \in \widetilde{U} |\ \big| f \circ \varphi^{-1}(\mathbf{x}) - a \big| < |b - a|/2\}$$
is open, and so is $U' = \varphi^{-1}(\widetilde{U}')$. Replacing U with U', we may thus arrange that $|f|_U - a| < |b - a|/2$. Similarly, we may arrange that $|f|_V - b| < |a - b|/2$. Then U, V are the desired disjoint open neighborhoods of p, q.

In other words, the separability of points by disjoint open neighborhoods (Hausdorff condition) is equivalent to separability by smooth functions. A consequence of this result is:

Corollary 3.6 (Criterion for Hausdorff condition). *A set M with a maximal atlas satisfies the Hausdorff condition if and only if for any two distinct points $p, q \in M$, there exists a smooth function $f:\ M \to \mathbb{R}$ with $f(p) \neq f(q)$. In particular, if there exists a smooth injective map $F:\ M \to \mathbb{R}^N$, then M is Hausdorff.*

Here, a map $F:\ M \to \mathbb{R}^N$ is called smooth if its component functions are smooth.

34 (answer on page ??). Justify the last assertion: if there exists a smooth injective map $F : M \to \mathbb{R}^N$, then M is Hausdorff.

Example 3.7 (Projective spaces). Write vectors $\mathbf{x} \in \mathbb{R}^{n+1}$ as column vectors, hence $\mathbf{x}^\top$ is the corresponding row vector. The matrix product $\mathbf{x}\mathbf{x}^\top$ is a square matrix with entries $x^j x^k$. The map
$$F:\ \mathbb{R}\mathrm{P}^n \to \mathrm{Mat}_{\mathbb{R}}(n+1) \cong \mathbb{R}^{(n+1)^2},\ \ (x^0 : \ldots : x^n) \mapsto \frac{\mathbf{x}\,\mathbf{x}^\top}{||\mathbf{x}||^2} \tag{3.1}$$
is well-defined, since the right hand side does not change when $\mathbf{x}$ is replaced with $\lambda\mathbf{x}$ for a nonzero scalar λ. It is also smooth, as one may check by considering the map in local charts, similarly to 31. Finally, it is injective: Given $F(\mathbf{x})$, one recovers the 1-dimensional subspace $\mathbb{R}\mathbf{x} \subseteq \mathbb{R}^{n+1}$ as the range of the rank 1 orthogonal projection $F(\mathbf{x})$. Hence, the criterion applies, and the Hausdorff condition follows.

35 (answer on page ??). Use a similar argument to verify the Hausdorff condition for $\mathbb{C}\mathrm{P}^n$.

The criterion may also be used for the real and complex Grassmannians (see Problem **??**), and many other examples. In the opposite direction, the criterion tells us that for a set M with a maximal atlas, if the Hausdorff condition does *not* hold then no smooth injective map into $\mathbb{R}^N$ exists. This is one reason why it is often difficult to 'visualize' non-Hausdorff manifolds.

Example 3.8. Consider the non-Hausdorff manifold M from Example 2.15. Here, there are two points p, q that do not admit disjoint open neighborhoods, and we see directly that every $f \in C^\infty(M)$ must take on the same values at p and q: With the coordinate charts $(U, \varphi),\ (V, \psi)$ in that example,

$$f(p) = f(\varphi^{-1}(0)) = \lim_{t \to 0^-} f(\varphi^{-1}(t)) = \lim_{t \to 0^-} f(\psi^{-1}(t)) = f(\psi^{-1}(0)) = f(q),$$

since $\varphi^{-1}(t) = \psi^{-1}(t)$ for $t < 0$.

3.3 Continuous functions

Smooth functions on open subsets of $\mathbb{R}^m$ are, in particular, continuous. The same is true for smooth functions on manifolds M:

Proposition 3.9. *Let M be a manifold (or a set with a maximal atlas), and $f \in C^\infty(M)$. Then f is* continuous: *For every open subset $J \subseteq \mathbb{R}$, the preimage $f^{-1}(J) \subseteq M$ is open.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas of M. For all α, the function $f \circ \varphi_\alpha^{-1} :\ \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth, hence continuous. Hence, for all open $J \subseteq \mathbb{R}$ the preimage

$$(f \circ \varphi_\alpha^{-1})^{-1}(J) = \varphi_\alpha\big(f^{-1}(J) \cap U_\alpha\big)$$

is open. By Definition 2.25, this means that $f^{-1}(J)$ is open. $\square$

We have characterized smooth functions as functions that are smooth "in charts." There is a similar characterization for continuous functions:

36 (answer on page ??). Show that $f :\ M \to \mathbb{R}$ is continuous (as in Proposition 3.9) if and only if for all charts (U, φ) the function $f \circ \varphi^{-1}$ is continuous.

Continuous functions form an algebra $C(M)$, containing $C^\infty(M)$ as a subalgebra. In our criterion for the Hausdorff condition, Corollary 3.6, we may replace smooth functions with continuous functions, with essentially the same proof.

Remark 3.10. The analogous result does not hold for arbitrary topological spaces: separation of points by continuous functions implies separation by disjoint open neighborhoods, but not conversely.

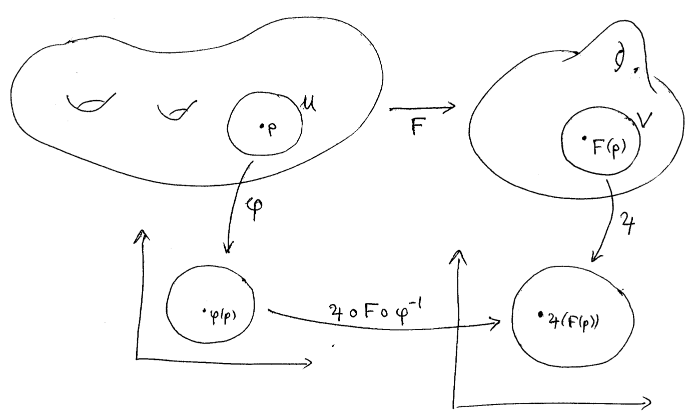
3.4 Smooth maps between manifolds

The notion of smooth maps from M to $\mathbb{R}$ generalizes to smooth maps between manifolds.

Definition 3.11. *A map $F: M \to N$ between manifolds is* smooth at $p \in M$ *if there are coordinate charts (U, φ) around p and (V, ψ) around $F(p)$ such that $F(U) \subseteq V$ and such that the composition*

$$\psi \circ F \circ \varphi^{-1}: \varphi(U) \to \psi(V)$$

is smooth. The function F is called a smooth map from M to N *if it is smooth at all $p \in M$. The collection of smooth maps $f: M \to N$ is denoted $C^\infty(M, N)$.*



As before, to check smoothness of F, it suffices to take *any* atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M with the property that $F(U_\alpha) \subseteq V_\alpha$ for *some* chart (V_α, ψ_α) of N, and then check smoothness of the maps

$$\psi_\alpha \circ F \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha) \to \psi_\alpha(V_\alpha).$$

This is because the condition for smoothness at p does not depend on the choice of charts (compare to the remark following Definition 3.11): Given a different choice of charts (U', φ') and (V', ψ') with $F(U') \subseteq V'$, we have

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

on $\varphi'(U \cap U')$. Since $(\psi' \circ \psi^{-1})$ and $(\varphi \circ (\varphi')^{-1})$ are smooth, we see that $\psi' \circ F \circ (\varphi')^{-1}$ is smooth at $\varphi'(p)$ if and only if $(\psi \circ F \circ \varphi^{-1})$ is smooth at $\varphi(p)$.

The previous proof illustrates the motivation behind the requirement that transition charts be diffeomorphisms (i.e. that an atlas is comprised of compatible charts): if some smoothness property holds 'locally' in a chart around a point, the compatibility condition is used to verify that the choice of chart is irrelevant.

37 (answer on page ??). Show that $C^\infty(M,\mathbb{R}) = C^\infty(M)$. (Part of the task is understanding the question!)

38 (answer on page ??). Show that the quotient maps

$$\pi: \ \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}\mathrm{P}^n$$
$$\pi: \ \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{C}\mathrm{P}^n$$

are smooth.

39 (answer on page ??).

a) Show that the map $F: \ \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ given by

$$\begin{cases} (1:0) \mapsto (1:0), \\ (t:1) \mapsto (\exp(t^2):1) \end{cases}$$

is smooth. (Why is it well-defined?)

b) Show that the map $F: \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ given by the same formula is *not* smooth.

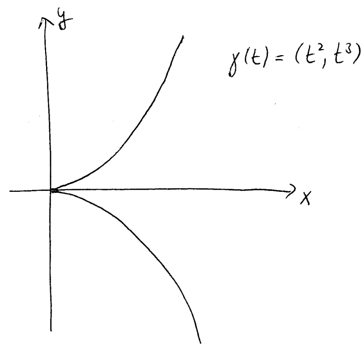
The discussion of Section 3.3 generalizes to $C^\infty(M,N)$:

Proposition 3.12. *Smooth functions $F \in C^\infty(M,N)$ are continuous: For any open subset $V \subseteq N$ the preimage $F^{-1}(V)$ is open.*

We leave the proof as an exercise; see Problem **??** at the end of this chapter. Furthermore, a map $F: \ M \to N$ is continuous at $p \in M$ if and only if its local coordinate expressions $\psi \circ F \circ \varphi^{-1}$ is continuous at $\varphi(p)$.

Smooth functions $\gamma: \ J \to M$ from an open interval $J \subseteq \mathbb{R}$ to M are called *(smooth) curves in M*. Note that the image of a smooth curve need not look smooth.

Example 3.13. The image of $\gamma: \ \mathbb{R} \to \mathbb{R}^2, \ t \mapsto (t^2, t^3)$ has a 'cusp singularity' at $(0,0)$.



3.5 Composition of smooth maps

Just as for smooth maps between open subsets of Euclidean spaces, the composition of smooth maps between manifolds is again smooth:

Proposition 3.14. *Suppose $F_1 : M_1 \to M_2$ is smooth at $p \in M_1$ and $F_2 : M_2 \to M_3$ is smooth at $F_1(p)$. Then the composition*

$$F_2 \circ F_1 : M_1 \to M_3$$

is smooth at p. Hence, if $F_1 \in C^\infty(M_1, M_2)$ and $F_2 \in C^\infty(M_2, M_3)$ then $F_2 \circ F_1 \in C^\infty(M_1, M_3)$.

Proof. Let (U_3, φ_3) be a chart around $F_2(F_1(p))$. Choose a chart (U_2, φ_2) around $F_1(p)$ with $F_2(U_2) \subseteq U_3$, as well as a chart (U_1, φ_1) around p with $F_1(U_1) \subseteq U_2$. Such charts always exist; see 40 below. Then $F_2(F_1(U_1)) \subseteq U_3$, and we have:

$$\varphi_3 \circ (F_2 \circ F_1) \circ \varphi_1^{-1} = (\varphi_3 \circ F_2 \circ \varphi_2^{-1}) \circ (\varphi_2 \circ F_1 \circ \varphi_1^{-1}).$$

Since $\varphi_2 \circ F_1 \circ \varphi_1^{-1} : \varphi_1(U_1) \to \varphi_2(U_2)$ is smooth at $\varphi_1(p)$ and $\varphi_3 \circ F_2 \circ \varphi_2^{-1} : \varphi_2(U_2) \to \varphi_3(U_3)$ is smooth at $\varphi_2(F_1(p))$, the composition is smooth at $\varphi_1(p)$.

40 (answer on page ??). Let $F \in C^\infty(M, N)$. Given $p \in M$ and *any* open neighborhood $V \subseteq N$ of $F(p)$, show that there exists a chart (U, φ) around p such that $F(U) \subseteq V$.

Example 3.15. As a simple application, once we know that the inclusion $i : S^2 \to \mathbb{R}^3$ of the 2-sphere is smooth, we see that for any open neighborhood $U \subseteq \mathbb{R}^3$ and any $h \in C^\infty(U)$, the restriction $h|_{S^2} : S^2 \to \mathbb{R}$ is smooth. (Cf. Example 3.2.) Indeed, this is immediate from Proposition 3.14 since

$$h|_{S^2} = h \circ i$$

(where we think of i as a map $S^2 \to U$). This simple observation applies to many similar examples.

41 (answer on page ??). Let $f \in C^\infty(M)$ be a function with $f > 0$ everywhere on M. Show (without using charts) that the function

$$\frac{1}{f}: M \to \mathbb{R}, \ \ p \mapsto \frac{1}{f(p)}$$

is smooth.

3.6 Diffeomorphisms of manifolds

Definition 3.16. *A smooth map $F : M \to N$ is called a* diffeomorphism *if it is invertible, with a smooth inverse $F^{-1} : N \to M$. Manifolds M, N are called* diffeomorphic *if there exists a diffeomorphism from M to N.*

You should verify that being diffeomorphic is an equivalence relation (transitivity is implied by Proposition 3.14). A diffeomorphism of manifolds is a bijection of the underlying sets that identifies the maximal atlases of the manifolds. Manifolds that are diffeomorphic are therefore considered 'the same manifold'.

Example 3.17. By definition, every coordinate chart (U, φ) on a manifold M gives a diffeomorphism $\varphi : U \to \varphi(U)$.

Example 3.18. In Section 3.7.2 we will describe explicit diffeomorphisms between $\mathbb{R}\mathrm{P}^1$ and S^1, and between $\mathbb{C}\mathrm{P}^1$ and S^2.

Example 3.19. The claim from Example 2.24 may now be rephrased as the assertion that the quotient of $S^2 \times S^2$ under the equivalence relation $(\mathbf{x}, \mathbf{x}') \sim (-\mathbf{x}, -\mathbf{x}')$ is diffeomorphic to $\mathrm{Gr}(2,4)$.

A continuous map $F : M \to N$ between manifolds (or, more generally, topological spaces) is called a *homeomorphism* if it is invertible, with a continuous inverse. Manifolds that are homeomorphic are considered 'the same topologically'. Since every smooth map is continuous, every diffeomorphism is a homeomorphism.

Example 3.20. The standard example of a homeomorphism of smooth manifolds that is not a diffeomorphism is the map

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3.$$

Indeed, this map is smooth and invertible, but the inverse map $y \mapsto y^{\frac{1}{3}}$ is not smooth.

The following 42 gives another way of looking at this example: we get two *distinct* manifold structures on $\mathbb{R}$, with the same collection of open sets.

42 (answer on page ??). Consider $M = \mathbb{R}$ with the trivial atlas $\mathscr{A} = \{(\mathbb{R}, \mathrm{id})\}$, and let $M' = \mathbb{R}$ with the atlas $\mathscr{A}' = \{(\mathbb{R}, \varphi)\}$ where $\varphi(x) = x^3$.

a) Show that $\mathbb{R}$ equipped with the atlas $\mathscr{A}'$ is a 1-dimensional manifold, whose open sets are just the usual open subsets of $\mathbb{R}$.
b) Show that the maximal atlases generated by $\mathscr{A}$ and $\mathscr{A}'$ are different.
c) Show that the map $f: M \to M'$ given by $f(x) = x^{1/3}$ is a diffeomorphism.

While the two manifold structures on $\mathbb{R}$ are not equal 'on the nose' (the identity map $\mathbb{R} \to \mathbb{R}$ is a homeomorphism $M \to M'$ but not a diffeomorphism), they are still diffeomorphic.

Remark 3.21. In the introduction (Section 1.6), we presented the classification of 2-dimensional compact, connected manifolds (i.e., surfaces) up to diffeomorphism. This classification coincides with their classification up to homeomorphism. That is, homeomorphic 2-manifolds Σ, Σ' are also diffeomorphic. In higher dimensions, it becomes much more complicated: It is possible for two manifolds to be homeomorphic but not diffeomorphic. The first example of 'exotic' manifold structures was discovered by John Milnor [12] in 1956, who found that the 7-sphere S^7 admits manifold structures that are not diffeomorphic to the standard manifold structure, even though they induce the standard topology. Kervaire and Milnor [9] proved in 1963 that up to diffeomorphism, there are exactly 28 distinct manifold structures on S^7, and in fact classified all manifold structures on all spheres S^n with the exception of the case $n = 4$. For example, they showed that S^3, S^5, S^6 do not admit exotic (i.e., non-standard) manifold structures, while S^{15} has 16256 different manifold structures. For S^4 the existence of exotic manifold structures is an open problem; this is known as the *smooth Poincaré conjecture*. Around 1982, Michael Freedman [7] (using results of Simon Donaldson [6]) discovered the existence of exotic manifold structures on $\mathbb{R}^4$; in 1987 Clifford Taubes [19] showed that there are uncountably many such. For $\mathbb{R}^n$ with $n \neq 4$, it is known that there are no exotic manifold structures on $\mathbb{R}^n$.

3.7 Examples of smooth maps

3.7.1 Products, diagonal maps

a) If M, N are manifolds, then the projection maps

$$\mathrm{pr}_M : M \times N \to M, \quad \mathrm{pr}_N : M \times N \to N$$

are smooth. (This follows immediately by taking product charts $U_\alpha \times V_\beta$.)

b) The diagonal inclusion
$$\mathrm{diag}_M : M \to M \times M$$
is smooth. (In a coordinate chart (U, φ) around p and the chart $(U \times U, \varphi \times \varphi)$ around (p, p), the map is the restriction to $\varphi(U) \subseteq \mathbb{R}^n$ of the diagonal inclusion $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$.)

c) Suppose $F : M \to N$ and $F' : M' \to N'$ are smooth maps. Then the direct product
$$F \times F' : M \times M' \to N \times N'$$
is smooth. This follows from the analogous statement for smooth maps on open subsets of Euclidean spaces.

3.7.2 The diffeomorphisms $\mathbb{R}\mathrm{P}^1 \cong S^1$ and $\mathbb{C}\mathrm{P}^1 \cong S^2$

We have stated before that $\mathbb{R}\mathrm{P}^1 \cong S^1$. We will now describe an explicit diffeomorphism $F : \mathbb{R}\mathrm{P}^1 \to S^1$, using the homogeneous coordinates $(w^0 : w^1)$ for $\mathbb{R}\mathrm{P}^1$ and regarding S^1 as the unit circle in $\mathbb{R}^2$. Consider the standard atlas $\{(U_0, \varphi_0), (U_1, \varphi_1)\}$ for $\mathbb{R}\mathrm{P}^1$, as described in Section 2.3.2. Thus U_i consists of all $(w^0 : w^1)$ such that $w^i \neq 0$; the coordinate maps are
$$\varphi_0(w^0 : w^1) = \frac{w^1}{w^0}, \quad \varphi_1(w^0 : w^1) = \frac{w^0}{w^1}$$
and have range the entire real line. The image of $U_0 \cap U_1$ under each of the coordinate maps is $\mathbb{R}\backslash\{0\}$, and the transition map is
$$\varphi_1 \circ \varphi_0^{-1} : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\},\ u \mapsto \frac{1}{u}.$$
Under the desired identification $\mathbb{R}\mathrm{P}^1 \cong S^1$, this atlas should correspond to an atlas with two charts for the circle, S^1, with the same chart images (namely, $\mathbb{R}$) and the same transition functions. A natural candidate is the atlas $\{(U_+, \varphi_+), (U_-, \varphi_-)\}$ given by stereographic projection, see Section 2.3.1: Thus $U_+ = S^1\backslash\{(0,-1)\}$, $U_- = S^1\backslash\{(0,1)\}$, with the coordinate maps
$$\varphi_+(x,y) = \frac{x}{1+y}, \quad \varphi_-(x,y) = \frac{x}{1-y}.$$
Again, the range of each coordinate map is the real line $\mathbb{R}$, the image of $U_+ \cap U_-$ is $\mathbb{R}\backslash\{0\}$, and by (2.6) the transition map is $\varphi_- \circ \varphi_+^{-1}(u) = 1/u$. Hence, there is a unique diffeomorphism $F : \mathbb{R}\mathrm{P}^1 \to S^1$ identifying the coordinate charts, in the sense that $F(U_0) = U_+,\ F(U_1) = U_-$, and such that
$$\varphi_+ \circ F|_{U_0} = \varphi_0, \quad \varphi_- \circ F|_{U_1} = \varphi_1.$$
For $(w^0 : w^1) \in U_0$, we obtain

$$F(w^0 : w^1) = \varphi_+^{-1}(\varphi_0(w^0 : w^1)) = \varphi_+^{-1}\big(\frac{w^1}{w^0}\big).$$

Using the formula for the inverse map of stereographic projection,

$$\varphi_+^{-1}(u) = \frac{1}{1+u^2}\big(2u,\ (1-u^2)\big),$$

we arrive at

$$F(w^0 : w^1) = \frac{1}{||\mathbf{w}||^2}\big(2w^1w^0,\ \ (w^0)^2 - (w^1)^2\big). \tag{3.2}$$

See Problem **??** at the end of this chapter for an explanation of (3.2) in terms of the squaring map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$.

43 (answer on page ??). Our derivation of (3.2) used the assumption $(w^0 : w^1) \in U_0$, but one obtains the same result for $(w^0 : w^1) \in U_1$. In fact, this is clear without repeating the calculation – why?

44 (answer on page ??). Use a similar strategy to compute the inverse map $G :\ S^1 \to \mathbb{R}\mathrm{P}^1$. (You may want to consider $(w^0 : w^1) = G(x,y)$ for the two cases $(x,y) \in U_+$ and $(x,y) \in U_-$.)

A similar strategy works for the complex projective line. Again, we compare the standard atlas for $\mathbb{C}\mathrm{P}^1$ with the atlas for the 2-sphere S^2, given by stereographic projection. This results in the following

Proposition 3.22. *There is a unique diffeomorphism $F :\ \mathbb{C}\mathrm{P}^1 \to S^2$ with the property $F|_{U_0} = \varphi_+^{-1} \circ \varphi_0$. In homogeneous coordinates, it is given by the formula*

$$F(w^0 : w^1) = \frac{1}{|w^0|^2 + |w^1|^2}\Big(2\,\mathrm{Re}(w^1\overline{w^0}),\ 2\,\mathrm{Im}(w^1\overline{w^0}),\ |w^0|^2 - |w^1|^2\Big) \tag{3.3}$$

where $\mathrm{Re}(z)$ *and* $\mathrm{Im}(z)$ *are the real and imaginary part of a complex number z.*

45 (answer on page ??). Prove Proposition 3.22. (You will discover that the transition map for the standard atlas of $\mathbb{C}\mathrm{P}^1$ is not quite the same as for the stereographic atlas of S^2, and a small adjustment is needed.) Also find expressions for the restriction of the inverse map $G = F^{-1} :\ S^2 \to \mathbb{C}\mathrm{P}^1$ to $U_\pm$.

3.7.3 Maps to and from projective space

In 38 you have verified that the quotient map

$$\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}\mathrm{P}^n,\ x = (x^0, \ldots, x^n) \mapsto (x^0 : \ldots : x^n)$$

is smooth. Given a map $F : \mathbb{R}\mathrm{P}^n \to N$, we take its *lift* to be the composition $\widetilde{F} = F \circ \pi : \mathbb{R}^{n+1} \setminus \{0\} \to N$. That is,

$$\widetilde{F}(x^0, \ldots, x^n) = F(x^0 : \ldots : x^n).$$

Note that $\widetilde{F}(\lambda x^0, \ldots, \lambda x^n) = \widetilde{F}(x^0, \ldots, x^n)$ for all non-zero $\lambda \in \mathbb{R} \setminus \{0\}$; conversely, every map $\widetilde{F}$ with this property descends to a map F on projective space. Similarly, maps $F : \mathbb{C}\mathrm{P}^n \to N$ are in 1-1 correspondence with maps $\widetilde{F} : \mathbb{C}^{n+1} \setminus \{\mathbf{0}\} \to N$ that are invariant under scalar multiplication.

Lemma 3.23. *A map $F : \mathbb{R}\mathrm{P}^n \to N$ is smooth if and only the lifted map $\widetilde{F} : \mathbb{R}^{n+1}\setminus\{\mathbf{0}\} \to N$ is smooth. Similarly, a map $F : \mathbb{C}\mathrm{P}^n \to N$ is smooth if and only the lifted map $\widetilde{F} : \mathbb{C}^{n+1}\setminus\{\mathbf{0}\} \to N$ is smooth.*

Proof. Let $\{(U_i, \varphi_i)\}$ be the standard atlas. If $\widetilde{F}$ is smooth,then

$$(F \circ \varphi_i^{-1})(u^1, \ldots, u^n) = \widetilde{F}(u^1, \ldots, u^i, 1, u^{i+1}, \ldots, u^n)$$

are smooth for all i, hence F is smooth. Conversely, if F is smooth then $\widetilde{F} = F \circ \pi$ is smooth. $\square$

46 (answer on page ??). Show that the map

$$\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2,\ (z^0 : z^1) \mapsto \big((z^0)^2 : (z^1)^2 : z^0 z^1\big)$$

is smooth.

As remarked earlier, the projective space $\mathbb{R}\mathrm{P}^n$ may also be regarded as a quotient of the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$, since every point $[\mathbf{x}] = (x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n$ has a representative $\mathbf{x} \in \mathbb{R}^{n+1}$ with $||\mathbf{x}|| = 1$. The quotient map

$$\pi : S^n \to \mathbb{R}\mathrm{P}^n$$

is again smooth, since it is a composition of the inclusion $\iota : S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ with the quotient map $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}\mathrm{P}^n$.
Similarly, the complex projective space $\mathbb{C}\mathrm{P}^n$ may also be regarded as a quotient of the unit sphere S^{2n+1} inside $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$, since any equivalence class $[\mathbf{z}] = (z^0 : \ldots : z^n)$ has a representative $\mathbf{z} \in \mathbb{C}^{n+1}$ with $||\mathbf{z}|| = 1$, with a smooth quotient map

$$\pi : S^{2n+1} \to \mathbb{C}\mathrm{P}^n \tag{3.4}$$

is smooth. Note that for any point $p \in \mathbb{C}\mathrm{P}^n$, the fiber $\pi^{-1}(p) \subseteq S^{2n+1}$ is diffeomorphic to a circle S^1, regarded as complex numbers of absolute value 1. Indeed, given any point $(z^0, \ldots, z^n) \in \pi^{-1}(p)$ in the fiber, the other points are obtained as $(\lambda z^0, \ldots, \lambda z^n)$ where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This defines a decomposition of the odd dimensional sphere

$$S^{2n+1} = \bigsqcup_{p \in \mathbb{C}\mathrm{P}^n} \pi^{-1}(p)$$

as a disjoint union of circles, parametrized by the points of $\mathbb{C}\mathrm{P}^n$. This is an example of what differential geometers call a *fiber bundle* or *fibration*. We won't give a formal definition here, but remark that this fibration is 'non-trivial' since S^{2n+1} is *not* diffeomorphic to a product $\mathbb{C}\mathrm{P}^n \times S^1$, as we will see later.

3.8 The Hopf fibration

The case $n = 1$ of the fibration (3.4) is of particular importance. Let us describe some of the properties of this fibration; our discussion will be somewhat informal, with details deferred to homework problems.

Identifying $\mathbb{C}\mathrm{P}^1 \cong S^2$ as in Proposition 3.22, the map (3.4) becomes a smooth map

$$\pi: \; S^3 \to S^2$$

with fibers diffeomorphic to S^1. Explicitly, by Equation (3.3),

$$\pi(z, w) = \Big(2\mathrm{Re}(w\bar{z}),\; 2\mathrm{Im}(w\bar{z}),\; |z|^2 - |w|^2\Big) \tag{3.5}$$

for $(z, w) \in S^3 \subseteq \mathbb{R}^4 \cong \mathbb{C}^2$, i.e., $|z|^2 + |w|^2 = 1$. This map appears in many contexts; it is called the *Hopf fibration* (after *Heinz Hopf* (1894-1971)). To get a picture of the Hopf fibration, recall that stereographic projection identifies the complement of a given point p in S^3 with $\mathbb{R}^3$; hence we obtain a decomposition of $\mathbb{R}^3$ into a collection of circles together with one line (corresponding to the circle in S^3 containing p); the line may be thought of as the circle through the 'point at infinity'.

To be specific, take p to be the 'south pole' $(0, i) \in S^3$. Stereographic projection from this point is the map

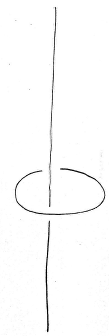
$$F: \; S^3 \backslash \{(0, i)\} \xrightarrow{\cong} \mathbb{R}^3, \quad (z, w) \mapsto \frac{1}{1 + \mathrm{Im}(w)}(z, \mathrm{Re}(w)). \tag{3.6}$$

Denote by $n = (0, 0, 1)$ the north pole and by $s = (0, 0, -1)$ the south pole of S^2. The fiber $\pi^{-1}(s)$ consists of elements of the form $(0, w) \in C^2$ with $|w| = 1$, and after removing $(0, i)$ from this circle, the image under F is the 3-axis in $\mathbb{R}^3$:

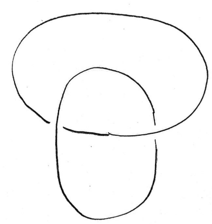
$$F\big(\pi^{-1}(s) \backslash (0, i)\big) = \{\mathbf{x} \in \mathbb{R}^3 |\; x^1 = x^2 = 0\}.$$

Similarly, $\pi^{-1}(n)$ consists of elements $(z, 0) \in \mathbb{C}^2$ with $|z|^2 = 1$; its image under stereographic projection is the unit circle in the x^1-x^2-coordinate plane,

$$F\big(\pi^{-1}(n)\big) = \{\mathbf{x} \in \mathbb{R}^3 |\; ||\mathbf{x}|| = 1,\; x^3 = 0\}.$$



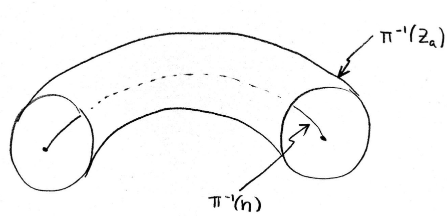
Note that this circle winds around the vertical line exactly once, and cannot be continuously shrunk to a point without intersecting the vertical line. This illustrates that the two circles $\pi^{-1}(s)$ and $\pi^{-1}(n)$ are *linked*. In particular, these two circles cannot be separated in S^3 through continuous movements of the circles, without intersecting the circles. When n,s are replaced with a different pair of distinct points $p,q \in S^2$, the corresponding circle $\pi^{-1}(p), \pi^{-1}(q)$ will also be linked. (Indeed, we can continuously move n,s to the new pair p,q, and the corresponding circles will move continuously also.) That is, *any two distinct circles of the Hopf fibration are linked.*



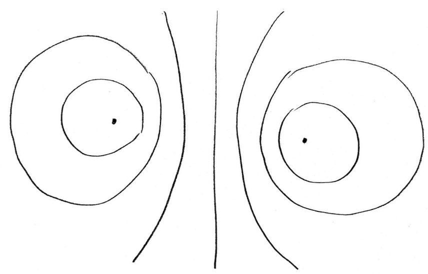
To get a more complete picture, consider the pre-image $\pi^{-1}(Z_a)$ of a circle of latitude $a \in (-1,1)$, i.e.,

$$Z_a = \{\mathbf{x} \in S^2 \,|\, x^3 = a\}.$$

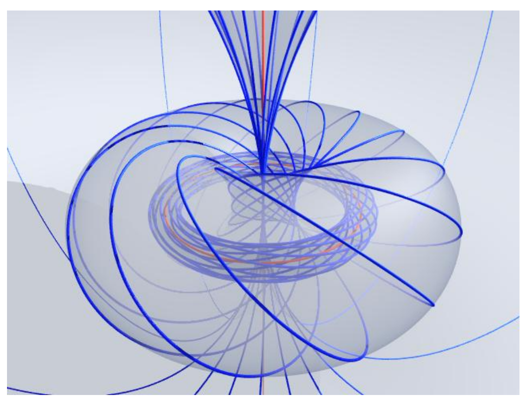
The fiber $\pi^{-1}(p)$ of any $p \in Z_a$ is a circle; since Z_a itself is a circle we expect that $\pi^{-1}(Z_a)$ is a 2-torus, and so is its image $F(\pi^{-1}(Z_a))$. This is confirmed by explicit calculation (see Problem **??** at the end of this chapter). For $a \to 1$, the circle of latitude Z_a approaches the north pole n, hence this torus is rather 'thin', and surrounds the circle $\pi^{-1}(n)$. The circle $F(\pi^{-1}(p))$ for $p \in Z_a$ lies inside this torus winding around it slowly, in such a way that it is linked with the circle $F(\pi^{-1}(n))$. Note that for p close to n, this circle should be just a small perturbation of the circle $F(\pi^{-1}(n))$.



As a moves towards -1, the tori get 'fatter' and larger. The intersection of this collection of 2-tori with the coordinate plane $\{\mathbf{x} \in \mathbb{R}^3 | \, x^2 = 0\}$ looks like this:



The full picture looks as follows:



Here is another interesting feature of the Hopf fibration. Let $U_+ = S^2 \backslash \{s\}$ and $U_- = S^2 \backslash \{n\}$. A calculation (see Problem **??** at the end of this chapter) gives diffeomorphisms

$$\pi^{-1}(U_\pm) \xrightarrow{\cong} U_\pm \times S^1,$$

in such a way that π becomes simply projection onto the first factor, $U_\pm$. In particular, the preimage of the closed upper hemisphere of S^2 is a *solid 2-torus*

$$D^2 \times S^1$$

(with $D^2 = \{z \in \mathbb{C} | \; |z| \leq 1\}$ the unit disk), geometrically depicted as a 2-torus in $\mathbb{R}^3$ together with its interior.* Likewise, the preimage of the closed lower hemisphere is a solid 2-torus $D^2 \times S^1$. The preimage of the equator is a 2-torus $S^1 \times S^1$. We hence see that the S^3 may be obtained by gluing two solid 2-tori along their boundaries $\partial(D^2 \times S^1) = S^1 \times S^1$. More precisely, the gluing identifies $(z,w) \in S^1 \times S^1$ in the boundary of the first solid torus with (w,z) in the boundary of the second solid torus. Note that one can also glue two copies of $D^2 \times S^1$ to produce $S^2 \times S^1$. However, here one uses a different gluing map, and indeed S^3 is *not* diffeomorphic to $S^2 \times S^1$. (We will prove this fact later.)

Remark 3.24. (For those who are familiar with quaternions – see Example A.18 in the appendix for a brief discussion.) Let $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$ be the quaternion numbers. The unit quaternions are a 3-sphere S^3. Generalizing the definition of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$, there are also quaternionic projective spaces, $\mathbb{H}\mathrm{P}^n$. These are quotients of the unit sphere inside $\mathbb{H}^{n+1} = \mathbb{R}^{4n+4}$, hence one obtains a quotient map

$$S^{4n+3} \to \mathbb{H}\mathrm{P}^n;$$

its fibers are diffeomorphic to S^3. For $n = 1$, one can show that $\mathbb{H}\mathrm{P}^1 = S^4$, hence one obtains a smooth map

$$\pi : \; S^7 \to S^4$$

with fibers diffeomorphic to S^3.

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* A solid torus is an example of a "manifold with boundary", a concept we haven't properly discussed yet.

4

Submanifolds

4.1 Submanifolds

Let M be a manifold of dimension m. We will define a k-dimensional submanifold $S \subseteq M$ to be a subset that looks locally like $\mathbb{R}^k \subseteq \mathbb{R}^m$, regarded as the coordinate subspace defined by $x^{k+1} = \cdots = x^m = 0$.

Definition 4.1. *A subset $S \subseteq M$ is called a* submanifold *of dimension $k \leq m$, if it has the following property: for every $p \in S$ there is a coordinate chart (U, φ) around p such that*

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k. \tag{4.1}$$

Charts (U, φ) of M with this property are called submanifold charts *for S.*

Remark 4.2.

a) A chart (U, φ) such that $U \cap S = \emptyset$ and $\varphi(U) \cap \mathbb{R}^k = \emptyset$ is considered a submanifold chart.
b) We stress that the existence of submanifold charts is only required for points p that lie in S. For example, the half-open line $S = (0, \infty)$ is a submanifold of $\mathbb{R}$ (of dimension 1). There does not exist a submanifold chart around $p = 0$, but this is not a problem since $0 \notin S$.

Strictly speaking, a submanifold chart for S is not a chart for S, but rather a chart for M which is adapted to S. On the other hand, submanifold charts *restrict* to charts for S, and this may be used to construct an atlas for S:

Proposition 4.3. *Suppose S is a submanifold of M. Then S is a k-dimensional manifold in its own right, with atlas consisting of all charts $(U \cap S, \varphi')$ such that (U, φ) is a submanifold chart, and $\varphi' = \pi \circ \varphi|_{U \cap S}$ where $\pi : \mathbb{R}^m \to \mathbb{R}^k$ is projection onto the first k-coordinates. The inclusion map*

$$i : S \to M$$

(taking $p \in S$ to the same point p viewed as an element of M) is smooth.

Proof. Let (U,φ) and (V,ψ) be two submanifold charts for S. We have to show that the charts $(U\cap S,\varphi')$ and $(V\cap S,\psi')$ are compatible. The map

$$\psi'\circ(\varphi')^{-1}:\ \varphi'(U\cap V\cap S)\to\psi'(U\cap V\cap S)$$

is smooth, because it is the restriction of $\psi\circ\varphi^{-1}:\ \varphi(U\cap V\cap S)\to\psi(U\cap V\cap S)$ to the coordinate subspace $\mathbb{R}^k$. Likewise its inverse map is smooth. The Hausdorff condition follows because any two distinct points $p,q\in S$, one can take disjoint submanifold charts around p,q. (Just take any submanifold charts around p,q, and restrict the chart domains to the intersection with disjoint open neighborhoods.)
The argument that S admits a *countable* atlas is unfortunately a bit technical. Our plan is to construct a countable collection of submanifold charts covering S. (The atlas for S itself is then obtained by restriction.) We will use the following

> **47 (answer on page ??).** Prove that every open subset of $\mathbb{R}^m$ is a union of *rational* ε-balls $B_\varepsilon(\mathbf{x}),\ \varepsilon>0$. Here, 'rational' means that both the center of the ball and its radius are rational: $\mathbf{x}\in\mathbb{Q}^n,\ \varepsilon\in\mathbb{Q}$.

Start with *any* countable atlas $\{(U_\alpha,\varphi_\alpha)\}$ for M. Given $p\in S\cap U_\alpha$, we can choose a submanifold chart (V,ψ) for S containing p, and by 47, we can choose a rational ε-ball $B_\varepsilon(\mathbf{x})\subseteq\mathbb{R}^m$ with

$$\varphi(p)\in B_\varepsilon(\mathbf{x})\subseteq\varphi_\alpha(U_\alpha\cap V).$$

This shows that $S\cap U_\alpha$ is covered by preimages of rational ε-balls $B_\varepsilon(\mathbf{x})\subseteq\varphi_\alpha(U_\alpha)$ with the additional property that $\varphi_\alpha^{-1}(B_\varepsilon(\mathbf{x}))$ is contained in *some* submanifold chart. In particular, we have a countable cover of $S\cap U_\alpha$ by submanifold charts (take $\varphi_\alpha^{-1}(B_\varepsilon(\mathbf{x}))$ as the domains, and the restriction of ψ as coordinate map.) Taking the union over all α, we obtain the desired countable collection of submanifold charts covering S. (Recall (cf. Appendix A.1.1) that a countable union of countable sets is again countable.)

> **48 (answer on page ??).** Complete the proof by verifying that the inclusion map $i:S\to M$ is smooth. $\square$

Example 4.4 (Open subsets). The m-dimensional submanifolds of an m-dimensional manifold are exactly the open subsets.

Example 4.5 (Euclidean space). For $k\le n$, we may regard $\mathbb{R}^k\subseteq\mathbb{R}^n$ as the subset where the last $n-k$ coordinates are zero. These are submanifolds, with any chart as submanifold chart.

Example 4.6 (Spheres). Let $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid ||\mathbf{x}||^2 = 1\}$. Write $\mathbf{x} = (x^0, \dots, x^n)$, and regard

$$S^k \subseteq S^n$$

for $k < n$ as the subset where the last $n-k$ coordinates are zero. These are submanifolds: The charts $(U_\pm, \varphi_\pm)$ for S^n given by stereographic projection

$$\varphi_\pm(x^0, \dots, x^n) = \frac{1}{1 \pm x^0}(x^1, \dots, x^n)$$

are submanifold charts. Alternatively, the charts $(U_i^\pm, \varphi_i^\pm)$, where $U_i^\pm \subseteq S^n$ is the subset where $\pm x^i > 0$, with $\varphi_i^\pm$ the projection to the remaining coordinates, are submanifold charts as well.

Example 4.7 (Projective spaces). For $k < n$, regard

$$\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$$

as the subset of all $(x^0 : \ldots : x^n)$ for which $x^{k+1} = \cdots = x^n = 0$. These are submanifolds, with the standard charts (U_i, φ_i) for $\mathbb{R}\mathrm{P}^n$ as submanifold charts. (Note that the charts $U_{k+1}, \dots, U_n$ do not intersect $\mathbb{R}\mathrm{P}^k$, but this does not cause a problem.) In fact, the resulting charts for $\mathbb{R}\mathrm{P}^k$ obtained by restricting these submanifold charts, are just the standard charts of $\mathbb{R}\mathrm{P}^k$. Similarly,

$$\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$$

are submanifolds, and for $n < n'$ we have $\mathrm{Gr}(k,n) \subseteq \mathrm{Gr}(k,n')$ as a submanifold. This proves the claim that the decomposition (2.7) is a decomposition into submanifolds.

Proposition 4.8 (Graphs are submanifolds). *Let $F: M \to N$ be a smooth map between manifolds of dimensions m and n. Then*

$$\mathrm{graph}(F) = \{(F(p), p) \mid p \in M\} \subseteq N \times M \tag{4.2}$$

is a submanifold of $N \times M$, of dimension equal to the dimension of M.

Remark 4.9. Many authors define the graph as a subset of $M \times N$ rather than $N \times M$. An advantage of the convention above is that it gives

$$\mathrm{graph}(F' \circ F) = \mathrm{graph}(F') \circ \mathrm{graph}(F)$$

under composition of relations.

Proof. Given $p \in M$, choose charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$. We claim that (W, κ) with $W = V \times U$ and

$$\kappa(q, p) = (\varphi(p),\ \psi(q) - \psi(F(p))) \tag{4.3}$$

is a submanifold chart for (4.2). Note that this is indeed a chart of $N \times M$, because it is obtained from the product chart $(V \times U, \psi \times \varphi)$ by composition with the diffeomorphism (see ✎ 49 below)

$$\psi(V) \times \varphi(U) \to \kappa(W), \quad (v, u) \mapsto (u, v - \widetilde{F}(u)). \tag{4.4}$$

where $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$. Furthermore, the second component in (4.3) vanishes if and only if $F(p) = q$. That is,

$$\kappa(W \cap \mathrm{graph}(F)) = \kappa(W) \cap \mathbb{R}^m$$

as required. $\square$

49 (answer on page ??). Prove that the map (4.4) is a diffeomorphism.

This result has the following consequence: If $S \subseteq M$ is a subset of a manifold, such that S can be *locally* described as the graph of a smooth map, then S is a submanifold. In more detail, suppose that S can be covered by open sets $U \subseteq M$, such that for each U there is a diffeomorphism $U \to P \times Q$ taking $S \cap U$ to the graph of a smooth map $Q \to P$, then S is a submanifold.

Example 4.10. The 2-torus $S = f^{-1}(0) \subseteq \mathbb{R}^3$, where

$$f(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2$$

is a submanifold of $\mathbb{R}^3$, since it can locally be expressed as the graph of a function of two of the coordinates (see 50 below).

50 (answer on page ??). Show that on the subset where $z > 0$, S is the graph of a smooth function on the annulus

$$\{(x, y) |\ (R - r)^2 < x^2 + y^2 < (R + r)^2\}.$$

How many open subsets of this kind (where S is given as the graph of a function of two of the coordinates) are necessary to cover S?

Example 4.11. More generally, suppose $S \subseteq \mathbb{R}^3$ is given as a level set $S = f^{-1}(0)$ of a smooth map $f \in C^\infty(\mathbb{R}^3)$. (Actually, we only need f to be defined on an open neighborhood of S.) Let $p \in S$, and suppose

$$\left.\frac{\partial f}{\partial x}\right|_p \neq 0.$$

By the *implicit function theorem* from multivariable calculus, there is an open neighborhood $U \subseteq \mathbb{R}^3$ of p on which the equation $f(x, y, z) = 0$ can be uniquely solved for x. That is,

$$S \cap U = \{(x,y,z) \in U \,|\, x = F(y,z)\}$$

for a smooth function F, defined on a suitable open subset of $\mathbb{R}^2$. By Proposition 4.8, this shows that S is a submanifold near p, and that we may use y,z as coordinates on S near p. Similar arguments apply for $\frac{\partial f}{\partial y}|_p \neq 0$ or $\frac{\partial f}{\partial z}|_p \neq 0$. Hence, if the gradient

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is non-vanishing at all points $p \in S = f^{-1}(0)$, then S is a 2-dimensional submanifold. Of course, there is nothing special about 2-dimensional submanifolds of $\mathbb{R}^3$, and below we will put this discussion in a more general framework.

Suppose $S \to M$ is a submanifold, and $F \in C^\infty(M,N)$. Then the restriction

$$F|_S : \ S \to N$$

is again smooth. Indeed, since the inclusion $i : \ S \to M$ is smooth (Proposition 4.3), the restriction may be seen as a composition of smooth maps $F|_S = F \circ i$. This is useful in practice, because in such cases there is no need to verify smoothness in the local coordinates of S. For example, the map $S^2 \to \mathbb{R}, \ (x,y,z) \mapsto z$ is smooth since it is the restriction of a smooth map $\mathbb{R}^3 \to \mathbb{R}$ to the submanifold S^2. We now invite you to prove a related result:

51 (answer on page ??). Let $S \subseteq M$ be a submanifold, with inclusion map i, and let $F : \ Q \to S$ be a map from another manifold Q. Then F is smooth if and only if $i \circ F$ is smooth. (In other words, F is smooth as a map to S if and only if it is smooth as a map to M.)

The following proposition shows that the topology of S as a manifold (i.e., its collection of open subsets) coincides with the 'subspace topology' as a subset of the manifold M.

Proposition 4.12. *Suppose S is a submanifold of M. Then the open subsets of S for its manifold structure are exactly those of the form $U \cap S$, where U is an open subset of M.*

Proof. We have to show:

$$U' \subseteq S \quad \text{is open} \quad \Leftrightarrow \quad U' = U \cap S \quad \text{where} \quad U \subseteq M \quad \text{is open.}$$

"$\Leftarrow$". Suppose $U \subseteq M$ is open, and let $U' = U \cap S$. For any submanifold chart (V, ψ), with corresponding chart $(V \cap S, \psi')$ for S (where, as before, $\psi' = \pi \circ \psi|_{V \cap S}$), we have that

$$\psi'((V \cap S) \cap U') = \pi \circ \psi(V \cap S \cap U) = \pi(\psi(U) \cap \psi(V) \cap \mathbb{R}^k).$$

Now, $\psi(U) \cap \psi(V) \cap \mathbb{R}^k$ is the intersection of the open set $\psi(U) \cap \psi(V) \subseteq \mathbb{R}^n$ with the subspace $\mathbb{R}^k$, hence is open in $\mathbb{R}^k$. Since charts of the form $(V \cap S, \psi')$ cover all of S, this shows that U' is open.

"⇒". Suppose $U' \subseteq S$ is open in S. Define

$$U = \bigcup_V \psi^{-1}(\psi'(U' \cap V) \times \mathbb{R}^{m-k}) \subseteq M,$$

where the union is over any collection of submanifold charts (V, ψ) that cover all of S. This satisfies

$$U \cap S = U' \tag{4.5}$$

52 (answer on page ??). Verify (4.5).

To show that U is open, it suffices to show that for all submanifold charts (V, ψ), the set $\psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k})$ is open. Indeed:

$$\begin{aligned} U' \text{ is open in } S &\Rightarrow U' \cap V \text{ is open in } S \\ &\Rightarrow \psi'(U' \cap V) \text{ is open in } \mathbb{R}^k \\ &\Rightarrow \psi'(U' \cap V) \times \mathbb{R}^{m-k} \text{ is open in } \mathbb{R}^m \\ &\Rightarrow \psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k}) \text{ is open in } M. \square \end{aligned}$$

Corollary 4.13. *A submanifold $S \subseteq M$ is compact with respect to its manifold topology if and only if it is compact as a subset of M.*

In particular, if a manifold M can be realized as a submanifold $M \subseteq \mathbb{R}^n$, then M is compact with respect to its manifold topology if and only if it is a closed and bounded subset of $\mathbb{R}^n$. This can be used to give quick proofs of the facts that the real or complex projective spaces, as well as the real or complex Grassmannians, are all compact.

4.2 The rank of a smooth map

Let $F \in C^\infty(M,N)$ be a smooth map. Then the fibers (level sets)

$$F^{-1}(q) \subseteq M$$

for $q \in N$ need not be submanifolds, in general. Similarly, the image

$$F(M) \subseteq N$$

need not be a submanifold – even if we allow self-intersections. (More precisely, there may be points p such that the image $F(U) \subseteq N$ of any open neighborhood U of p is never a submanifold.) Here are some counter-examples:

a) The fibers $f^{-1}(c)$ of the map $f(x,y) = xy$ are hyperbolas for $c \neq 0$, but $f^{-1}(0)$ is the union of coordinate axes. What makes this possible is that the gradient of f is zero at the origin.

b) As we mentioned earlier (cf. Example 3.13), the image of the smooth map

$$\gamma:\ \mathbb{R} \to \mathbb{R}^2,\ \ \gamma(t) = (t^2, t^3)$$

does not look smooth near $(0,0)$ (and replacing $\mathbb{R}$ by an open interval around 0 does not help). What makes this is possible is that the velocity $\dot{\gamma}(t)$ vanishes for $t = 0$: the curve described by γ 'comes to a halt' at $t = 0$, and then turns around.

In both cases, the problems arise at points where the map does not have maximal rank. After reviewing the notion of rank of a map from multivariable calculus we will generalize to manifolds.

4.2.1 The rank of the Jacobian matrix

We shall need some notions from multivariable calculus. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets, and $F \in C^\infty(U,V)$ a smooth map. Recall from Definition 2.3 that the *Jacobian matrix* of F at p is the matrix of partial derivatives

$$D_pF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}\big|_p & \frac{\partial F^1}{\partial x^2}\big|_p & \cdots & \frac{\partial F^1}{\partial x^m}\big|_p \\ \frac{\partial F^2}{\partial x^1}\big|_p & \frac{\partial F^2}{\partial x^2}\big|_p & \cdots & \frac{\partial F^2}{\partial x^m}\big|_p \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1}\big|_p & \frac{\partial F^n}{\partial x^2}\big|_p & \cdots & \frac{\partial F^n}{\partial x^m}\big|_p \end{pmatrix}.$$

Definition 4.14. *The* rank *of F at $p \in U$ is the rank of the Jacobian matrix D_pF at p.*

Thus, the rank may be computed as the number of linearly independent rows, or equivalently the number of linearly independent columns. Note that

$$\mathrm{rank}_p(F) \le \min(m,n). \tag{4.6}$$

We will prefer to think of the Jacobian matrix not as an array of numbers, but as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$, more conceptually defined as follows:

Definition 4.15. *The* derivative of F at $p \in U$ *is the linear map*

$$D_pF:\ \mathbb{R}^m \to \mathbb{R}^n,\ \ \mathbf{v} \mapsto \frac{d}{dt}\bigg|_{t=0} F(p+t\mathbf{v}).$$

So, the rank of F at p is the rank of this linear map, i.e., the dimension of its range. Note that we will use the same notation for this linear map and its matrix. By the chain rule for differentiation, the derivative of a composition of two smooth maps satisfies

$$D_p(F' \circ F) = D_{F(p)}(F') \circ D_p(F). \tag{4.7}$$

In particular, if F' is a diffeomorphism then $\mathrm{rank}_p(F' \circ F) = \mathrm{rank}_p(F)$, and if F is a diffeomorphism then $\mathrm{rank}_p(F' \circ F) = \mathrm{rank}_{F(p)}(F')$.

4.2.2 The rank of smooth maps between manifolds

Using charts, we may generalize the notion of rank to smooth maps between manifolds:

Definition 4.16. *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds, and $p \in M$. The* rank *of F at $p \in M$ is defined as*

$$\operatorname{rank}_p(F) = \operatorname{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1}),$$

for any two coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$.

By (4.7), this is well-defined: if we use different charts (U',φ') and (V',ψ'), then the rank of

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

at $\varphi'(p)$ equals that of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$, since the two maps are related by diffeomorphisms.
By (4.6),

$$\operatorname{rank}_p(F) \leq \min(\dim M,\ \dim N)$$

for all $p \in M$.

Definition 4.17. *A smooth map $F \in C^\infty(M,N)$ has* maximal rank *at $p \in M$ if*

$$\operatorname{rank}_p(F) = \min(\dim M,\ \dim N).$$

A point $p \in M$ is called a critical point *for F if does not have maximal rank at p.*

53 (answer on page ??). Consider the lemniscate of Gerono:

$$F:\ \mathbb{R} \to \mathbb{R}^2,\ \ \theta \mapsto (\cos\theta, \sin\theta\cos\theta).$$

Find $\operatorname{rank}_p(F)$ for all $p \in \mathbb{R}$, and determine the critical points (if any). How does the graph look like?

54 (answer on page ??). Consider the map

$$F:\ \mathbb{R}^3 \to \mathbb{R}^4,\ \ (x,y,z) \mapsto (yz, xy, xz, x^2 + 2y^2 + 3z^2).$$

Find $\operatorname{rank}_p(F)$ for all $p \in \mathbb{R}^3$, and determine the critical points (if any).

4.3 Smooth maps of maximal rank

The following discussion will focus on maps $F \in C^\infty(M,N)$ of maximal rank (Definition 4.17). We will separate the three cases where $\dim M$ is equal to, greater than, or less than $\dim N$.

4.3.1 Local diffeomorphisms

In this section we will consider the case $\dim M = \dim N$. Our 'workhorse theorem' from multivariable calculus is going to be the following fact.

Theorem 4.18 (Inverse Function Theorem for $\mathbb{R}^m$). *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets of $\mathbb{R}^m$, and suppose that the derivative D_pF at $p \in U$ is invertible. Then there exists an open neighborhood $U_1 \subseteq U$ of p such that F restricts to a diffeomorphism $U_1 \to F(U_1)$.*

Among other things, the theorem tells us that if $F \in C^\infty(U,V)$ is a bijection from $U \to V$, then the inverse $F^{-1}: V \to U$ is smooth provided that the differential (i.e., the *first* derivative) of F is invertible everywhere. It is not necessary to check anything involving higher derivatives.
It is good to see, in just one dimension, how this is possible. Given an invertible smooth function $y = f(x)$, with inverse $x = g(y)$, and using $\frac{\mathrm{d}}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x}$, we have

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)}, \\ g''(y) &= \frac{-f''(x)}{f'(x)^3}, \\ g'''(y) &= \frac{-f'''(x)}{f'(x)^4} + 3\frac{f''(x)^2}{f'(x)^5}, \end{aligned}$$

and so on; only powers of $f'(x)$ appear in the denominator.
Using charts, we can pass from open subsets of $\mathbb{R}^m$ to manifolds.

Theorem 4.19 (Inverse function theorem for manifolds). *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of the same dimension $m = n$. If $p \in M$ is such that* $\operatorname{rank}_p(F) = m$, *then there exists an open neighborhood $U \subseteq M$ of p such that F restricts to a diffeomorphism $U \to F(U)$.*

Proof. Choose charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$. The map

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1}: \widetilde{U} := \varphi(U) \to \widetilde{V} := \psi(V)$$

has rank m at $\varphi(p)$. Hence, by the inverse function theorem for $\mathbb{R}^m$ (Theorem 4.18), after replacing $\widetilde{U}$ with a smaller open neighborhood of $\varphi(p)$ in $\mathbb{R}^m$ (equivalently, replacing U with a smaller open neighborhood of p in M) the map $\widetilde{F}$ becomes a diffeomorphism from $\widetilde{U}$ onto $\widetilde{F}(\widetilde{U}) = \psi(F(U))$. It then follows that

$$F = \psi^{-1} \circ \widetilde{F} \circ \varphi : \; U \to V$$

is a diffeomorphism $U \to F(U)$. $\square$

Definition 4.20. *A smooth map $F \in C^\infty(M,N)$ is called a* local diffeomorphism *if for every point $p \in M$ there exists an open neighborhood U of p such that $F(U)$ is open, and F restricts to a diffeomorphism $U \to F(U)$.*

By the theorem, this is equivalent to the condition that

$$\mathrm{rank}_p(F) = \dim M = \dim N$$

for all $p \in M$. It depends on the map in question which of these two conditions is easier to verify.

55 (answer on page ??). Show that the map $\mathbb{R} \to S^1$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is a local diffeomorphism. How does it fail to be a diffeomorphism?

Example 4.21. The quotient map $\pi : \; S^n \to \mathbb{R}\mathrm{P}^n$ is a local diffeomorphism. For example, one can see that π restricts to diffeomorphisms from the charts $U_j^\pm = \{x \in S^n | \pm x^j > 0\}$ (with coordinate map given by the remaining coordinates) to the standard chart U_j of the projective space. Note that π is not bijective and so cannot be a diffeomorphism.

Example 4.22. Let M be a manifold with a countable open cover $\{U_\alpha\}$. Then the *disjoint* union

$$Q = \bigsqcup_\alpha U_\alpha$$

is a manifold. The map $\pi : \; Q \to M$, given on $U_\alpha \subseteq Q$ by the inclusion into M, is a local diffeomorphism. Since π is surjective, it determines an equivalence relation on Q, with π as the quotient map and $M = Q/\sim$.

56 (answer on page ??). Why did we make the assumption that the cover is *countable*?

57 (answer on page ??). Show that if the U_α's are the domains of coordinate charts, then Q is diffeomorphic to an open subset of $\mathbb{R}^m$. We hence conclude that any manifold is realized as a quotient of an open subset of $\mathbb{R}^m$, in such a way that the quotient map is a local diffeomorphism.

Remark 4.23 (Mapping degree). Suppose $F \in C^\infty(M,N)$ is a smooth map between manifolds of the same dimension $m = n$, where M,N are *oriented*. If F has maximal rank at $p \in M$, and taking the open subset U in Theorem 4.19 to be connected, the diffeomorphism $F|_U : U \to F(U)$ is either orientation preserving or orientation reversing. Put $\varepsilon_p = +1$ in the first case, $\varepsilon_p = -1$ in the second case. If $q \in N$ is a regular value, so that F has maximal rank at all $p \in F^{-1}(q)$, and assuming that $F^{-1}(q)$ is finite, one calls

$$\deg_q(F) = \sum_{p \in F^{-1}(q)} \varepsilon_p \in \mathbb{Z}$$

the degree of F at q. If q is not in the range of F, we put $\deg_q(F) = 0$. We will see later that when N is connected and M is compact, then the degree does not depend on the choice of regular value $q \in N$. In particular, it is zero unless F is surjective.

4.3.2 Submersions

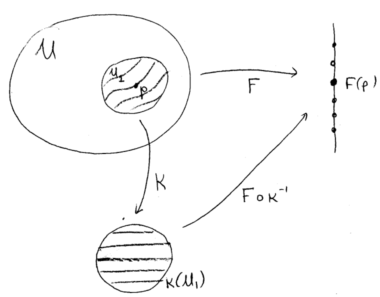
We next consider maps $F : M \to N$ of maximal rank between manifolds of dimensions $m \geq n$. This discussion will rely on the *implicit function theorem* from multivariable calculus.

Theorem 4.24 (Implicit Function Theorem for $\mathbb{R}^m$). *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is surjective. Then there exists an open neighborhood $U_1 \subseteq U$ of p and a diffeomorphism $\kappa : U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$ such that*

$$(F \circ \kappa^{-1})(u^1, \ldots, u^m) = (u^{m-n+1}, \ldots, u^m)$$

for all $u = (u^1, \ldots, u^m) \in \kappa(U_1)$.

Thus, in suitable coordinates F is given by a projection onto the last n coordinates.



Although it belongs to multivariable calculus, let us recall how to get this result from the inverse function theorem.

Proof. The idea is to extend F to a map between open subsets of $\mathbb{R}^m$, and then apply the inverse function theorem.
By assumption, the derivative D_pF has rank equal to n. Hence it has n linearly independent columns. By re-indexing the coordinates of $\mathbb{R}^m$ (this permutation is itself a change of coordinates, i.e. , a diffeomorphism) we may assume that these are the last n columns. That is, writing

$$D_pF = \begin{pmatrix} C\ D \end{pmatrix}$$

where C is the $n \times (m-n)$-matrix formed by the first $m-n$ columns and D the $n \times n$-matrix formed by the last n columns, the square matrix D is invertible. Write elements $\mathbf{x} \in \mathbb{R}^m$ in the form $\mathbf{x} = (x', x'')$ where x' are the first $m-n$ coordinates and x'' the last n coordinates. Let

$$G\colon\ U \to \mathbb{R}^m,\ \mathbf{x} = (x', x'') \mapsto (x', F(\mathbf{x})).$$

Then the derivative D_pG has block form

$$D_pG = \begin{pmatrix} I_{m-n} & 0 \\ C & D \end{pmatrix},$$

(where I_{m-n} is the square $(m-n) \times (m-n)$ identity matrix), and is therefore invertible. Hence, by the inverse function theorem there exists a smaller open neighborhood U_1 of p such that G restricts to a diffeomorphism $\kappa\colon\ U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$. We have,

$$G \circ \kappa^{-1}(u', u'') = (u', u'')$$

for all $(u', u'') \in \kappa(U_1)$. Since F is just G followed by projection to the x'' component, we conclude

$$F \circ \kappa^{-1}(u', u'') = u''.$$

Again, this result has a version for manifolds:

Theorem 4.25 (Normal form for Submersions). *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m \geq n$, and suppose $p \in M$ is such that* $\mathrm{rank}_p(F) = n$. *Then there exist coordinate charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, such that*

$$(\psi \circ F \circ \varphi^{-1})(u', u'') = u''$$

for all $u = (u', u'') \in \varphi(U)$. In particular, for all $q \in V$ the intersection

$$F^{-1}(q) \cap U$$

is a submanifold of dimension $m-n$.

Proof. Start with coordinate charts (U, φ) around p and (V, ψ) around $F(p)$ such that $F(U) \subseteq V$. Apply Theorem 4.24 to the map $\widetilde{F} = \psi \circ F \circ \varphi^{-1}\colon\ \varphi(U) \to \psi(V)$, to define a smaller neighborhood $\varphi(U_1) \subseteq \varphi(U)$ and change of coordinates κ so that

$$\widetilde{F} \circ \kappa^{-1}(u', u'') = u''.$$

After renaming $(U_1, \kappa \circ \varphi|_{U_1})$ as (U, φ) we have the desired charts for F. The last part of the theorem follows since the chart (U, φ) gives a submanifold chart for $F^{-1}(q) \cap U$. $\square$

Definition 4.26. *Let $F \in C^\infty(M,N)$. A point $p \in M$ is called a* regular point *of F, if* $\mathrm{rank}_p(F) = \dim N$*, otherwise it is called a* critical point *(or* singular point*).*
A point $q \in N$ is called a regular value *of $F \in C^\infty(M,N)$ if for all $p \in F^{-1}(q)$, one has*

$$\mathrm{rank}_p(F) = \dim N.$$

It is called a critical value *(or* singular value*) if it is not a regular value.*

Points of N that are not in the image of the map F are considered regular values. We may restate Theorem 4.25 as follows:

Theorem 4.27 (Regular Value Theorem). *For any regular value $q \in N$ of a smooth map $F \in C^\infty(M,N)$, the level set $S = F^{-1}(q)$ is a submanifold of dimension*

$$\dim S = \dim M - \dim N.$$

Example 4.28. The n-sphere S^n may be defined as the level set $F^{-1}(1)$ of the function $F \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ given by

$$F(x^0, \ldots, x^n) = ||\mathbf{x}||^2 = (x^0)^2 + \cdots + (x^n)^2.$$

The derivative of F at $p = \mathbf{x}$ is the $1 \times (n+1)$-matrix of partial derivatives, that is, the gradient ∇F:

$$D_p F = (2x^0, \ldots, 2x^n).$$

For $\mathbf{x} \neq \mathbf{0}$ this has maximal rank. A real number $q \in \mathbb{R}$ is a regular value of F if and only if $q \neq 0$ (since $\mathbf{0} \notin F^{-1}(q)$ in this case); hence all the level sets $F^{-1}(q)$ for $q \neq 0$ are submanifolds of dimension $(n+1) - 1 = n$. The number $q = 0$ is a critical value; the level set $F^{-1}(0) = \{\mathbf{0}\}$ is a submanifold but of the 'wrong' dimension.

58 (answer on page ??). Let $0 < r < R$. Show that

$$F(x,y,z) = (\sqrt{x^2 + y^2} - R)^2 + z^2$$

has r^2 as a regular value. What is the resulting submanifold?

Example 4.29. The *orthogonal group* $\mathrm{O}(n)$ is the group of matrices $A \in \mathrm{Mat}_{\mathbb{R}}(n)$ satisfying $\langle A\mathbf{x},\, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$; here $\langle \cdot, \cdot \rangle$ is the standard inner product (dot product) on $\mathbb{R}^n$. This is equivalent to the property

$$A^\top = A^{-1}$$

of the matrix A, or $A^\top A = I$. We claim that $\mathrm{O}(n)$ is a submanifold of $\mathrm{Mat}_\mathbb{R}(n)$. To see this, let us regard $\mathrm{O}(n)$ as the level set $F^{-1}(I)$ of the function

$$F:\ \mathrm{Mat}_\mathbb{R}(n) \to \mathrm{Sym}_\mathbb{R}(n),\ \ A \mapsto A^\top A,$$

where $\mathrm{Sym}_\mathbb{R}(n) \subseteq \mathrm{Mat}_\mathbb{R}(n)$ denotes the subspace of symmetric matrices. We want to show that the identity matrix I is a regular value of F. We compute the differential $D_A F:\ \mathrm{Mat}_\mathbb{R}(n) \to \mathrm{Sym}_\mathbb{R}(n)$ using the definition. (It would be confusing to work with the description of $D_A F$ as a matrix of partial derivatives.)

$$\begin{aligned}(D_A F)(X) &= \frac{d}{dt}\Big|_{t=0} F(A+tX)\\ &= \frac{d}{dt}\Big|_{t=0} ((A^\top + tX^\top)(A+tX))\\ &= A^\top X + X^\top A.\end{aligned}$$

To see that this is surjective for $A \in F^{-1}(I)$, we need to show that for any $Y \in \mathrm{Sym}_\mathbb{R}(n)$ there exists a solution $X \in \mathrm{Mat}_\mathbb{R}(n)$ for

$$A^\top X + X^\top A = Y.$$

Using $A^\top A = F(A) = I$ we see that $X = \frac{1}{2}AY$ is a solution. We conclude that I is a regular value, and hence that $\mathrm{O}(n) = F^{-1}(I)$ is a submanifold. Its dimension is

$$\dim \mathrm{O}(n) = \dim \mathrm{Mat}_\mathbb{R}(n) - \dim \mathrm{Sym}_\mathbb{R}(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

Note that it was important here to regard F as a map to $\mathrm{Sym}_\mathbb{R}(n)$; for F viewed as a map to $\mathrm{Mat}_\mathbb{R}(n)$ the identity would *not* be a regular value.

Definition 4.30. *A smooth map $F \in C^\infty(M,N)$ is a* submersion *if* $\mathrm{rank}_p(F) = \dim N$ *for all $p \in M$.*

Thus, for a submersion *all* level sets $F^{-1}(q)$ are submanifolds.

Example 4.31. Local diffeomorphisms are submersions; here the level sets $F^{-1}(q)$ are discrete points, i.e. 0-dimensional manifolds.

Example 4.32. For a product manifold $N \times Q$, the projection to the first factor

$$\mathrm{pr}_N:\ N \times Q \to N$$

is a submersion. The normal form theorem for submersions, Theorem 4.25, shows that locally, *any* submersion $F:\ M \to N$ is of this form. That is, given $p \in M$ there is an open neighborhood $U \subseteq M$ of p and a map $\psi \in C^\infty(U, N \times Q)$ (where Q is a manifold of dimension $m-n$, for example $Q = \mathbb{R}^{m-n}$) such that ψ is a diffeomorphism onto its image and such that the diagram

$$\begin{tikzcd} M \supseteq U \arrow[r, "\psi"] \arrow[dr, "F|_U"'] & N \times Q \arrow[d, "\mathrm{pr}_N"] \\ & N \end{tikzcd}$$

commutes. (That is, $F|_U = \mathrm{pr}_N \circ \psi$.)

4.3.3 Example: The Steiner surface

In this section, we give a more detailed example, investigating the smoothness of level sets.

Example 4.33 (Steiner's surface). Let $S \subseteq \mathbb{R}^3$ be the solution set of

$$y^2z^2 + x^2z^2 + x^2y^2 = xyz.$$

Is this a surface in $\mathbb{R}^3$? (We use *surface* as another term for *2-dimensional manifold*; by a *surface in M* we mean a 2-dimensional submanifold.) Actually, it is *not*. If we take one of x, y, z equal to 0, then the equation holds if and only if one of the other two coordinates is 0. Hence, the intersection of S with the set where $xyz = 0$ (the union of the coordinate hyperplanes) is the union of the three coordinate axes. Let $U \subseteq \mathbb{R}^3$ be the subset where $xyz \neq 0$, then $S \cap U$ is entirely contained in the set

$$\left\{(x,y,z) \in \mathbb{R}^3 \,|\, |x| \le 1, |y| \le 1, |z| \le 1\right\}.$$

Let $V \subseteq \mathbb{R}^3$ be an open set around, say, $(2,0,0)$; by replacing it with a possibly smaller open set we may assume that $V \cap S \subseteq \mathbb{R}^1$. Thus, $V \cap S$ is an open subset of $\mathbb{R}^1$, and thus a 1-dimensional manifold. On the other hand, Proposition 4.12 shows that $V \cap S$ is an open subset of S. Hence, if S were a surface, $V \cap S$ would be a 2-dimensional manifold, contradicting invariance of dimension (cf. Chapter 3, Problem **??**).

59 (answer on page ??). Show that $U \cap S$ is entirely contained in the set

$$\left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, |x| \le 1, |y| \le 1, |z| \le 1\right\}$$

as claimed.

Let us therefore rephrase the question: is $S \cap U$ a surface? To investigate the problem, consider the function

$$f(x,y,z) = y^2z^2 + x^2z^2 + x^2y^2 - xyz.$$

60 (answer on page ??). Find the critical points of

$$f(x,y,z) = y^2z^2 + x^2z^2 + x^2y^2 - xyz.$$

Conclude that $S \cap U$ is indeed a submanifold.

How does $S \cap U$ look like? It turns out that there is a nice answer. First, let's divide the defining equation by xyz. The equation takes on the form

$$xyz\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 1. \tag{4.8}$$

Since $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} > 0$, the solution set of (4.8) is contained in the set of all (x,y,z) such that $xyz > 0$. On this subset, we introduce new variables

$$\alpha = \frac{\sqrt{xyz}}{x}, \quad \beta = \frac{\sqrt{xyz}}{y}, \quad \gamma = \frac{\sqrt{xyz}}{z};$$

the old variables x, y, z are recovered as

$$x = \beta\gamma, \quad y = \alpha\gamma, \quad z = \alpha\beta.$$

In terms of α, β, γ, Equation (4.8) becomes the equation $\alpha^2 + \beta^2 + \gamma^2 = 1$. Actually, it is even better to consider the corresponding points

$$(\alpha : \beta : \gamma) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right) \in \mathbb{R}\mathrm{P}^2,$$

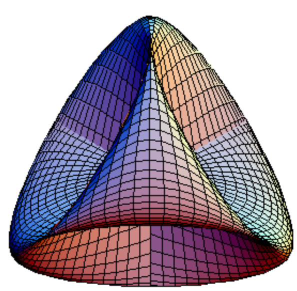
because we could take either square root of xyz (changing the sign of all α, β, γ doesn't affect x, y, z). We conclude that the map $U \to \mathbb{R}\mathrm{P}^2,\ (x,y,z) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$ restricts to a diffeomorphism from $S \cap U$ onto

$$\mathbb{R}\mathrm{P}^2 \backslash \{(\alpha : \beta : \gamma) |\ \alpha\beta\gamma = 0\}.$$

The image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3, \quad (\alpha : \beta : \gamma) \mapsto \frac{1}{|\alpha|^2 + |\beta|^2 + |\gamma|^2}(\beta\gamma, \alpha\beta, \alpha\gamma).$$

is called *Steiner's surface*, even though it is not a submanifold (not even an *immersed* submanifold). Here is a picture:



Note that the subset of $\mathbb{RP}^2$ defined by $\alpha\beta\gamma = 0$ is a union of three $\mathbb{RP}^1 \cong S^1$, each of which maps into a coordinate axis (but not the entire coordinate axis). For example, the circle defined by $\alpha = 0$ maps to the set of all $(x,0,0)$ with $-\frac{1}{2} \le x \le \frac{1}{2}$. In any case, S is the union of the Steiner surface with the three coordinate axes.

Example 4.34. Let $S \subseteq \mathbb{R}^4$ be the solution set of

$$y^2x^2 + x^2z^2 + x^2y^2 = xyz, \quad y^2x^2 + 2x^2z^2 + 3x^2y^2 = xyzw.$$

Again, this cannot quite be a surface because it contains the coordinate axes for x, y, z. Closer investigation shows that S is the union of the three coordinate axes, together with the image of an injective map

$$\mathbb{RP}^2 \to \mathbb{R}^4, \ (\alpha : \beta : \gamma) \mapsto \frac{1}{\alpha^2 + \beta^2 + \gamma^2}(\beta\gamma, \alpha\beta, \alpha\gamma, \alpha^2 + 2\beta^2 + 3\gamma^2).$$

It turns out (see Section 5.2.4 below) that the latter is a submanifold, which realizes $\mathbb{RP}^2$ as a surface in $\mathbb{R}^4$.

4.3.4 Quotient maps

A surjective submersion $F: M \to N$ may be regarded as the quotient map for an equivalence relation on M, where $p \sim p'$ if and only if p, p' are in the same fiber of F. It is natural to ask the converse (see Section 2.7.4): Under what conditions does an equivalence relation $\sim$ on a manifold M determine a manifold structure on the quotient space $N = M/\sim$, in such a way that the quotient map

$$\pi : M \to M/\sim$$

is a submersion. (In Problem **??**, you are asked to show that there can be at most one such manifold structure on $M/\sim$.) The answer involves the graph of the equivalence relation,

$$R = \{(p, p') \in M \times M | \ p \sim p'\}.$$

Theorem 4.35. *There is a manifold structure on $M/\sim$ with the property that the quotient map $\pi : M \to M/\sim$ is a submersion, if and only if the following conditions are satisfied:*

a) R is a closed submanifold of $M \times M$,
b) the map $\mathrm{pr}_1 : M \times M \to M,\ (p,q) \mapsto p$ restricts to a submersion $\mathrm{pr}_1|_R : R \to M$.

We will not present the proof of this result, which may be found, for example, in Bourbaki [1, Section 5.9]. One direction is ✎ 61 below. Also, the special case that $\mathrm{pr}_1|_R$ is a local diffeomorphism (in particular, $\dim R = \dim M$) is left as Problem **??** at the end of this chapter.

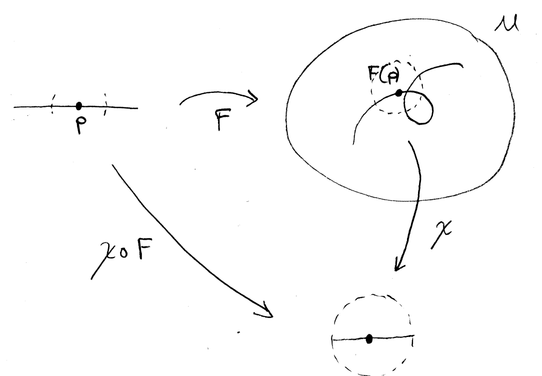
61 (answer on page ??). Suppose $M/\sim$ has the structure of a (possibly non-Hausdorff) manifold, in such a way that π is a submersion. Show that R is a submanifold of $M \times M$, which is closed if and only if $M/\sim$ satisfies the Hausdorff property.

4.3.5 Immersions

We next consider maps $F:\ M \to N$ of maximal rank between manifolds of dimensions $m \le n$. Once again, such a map can be put into a 'normal form': By choosing suitable coordinates it becomes linear.

Proposition 4.36. *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is injective. Then there exist smaller neighborhoods $U_1 \subseteq U$ of p and $V_1 \subseteq V$ of $F(p)$, with $F(U_1) \subseteq V_1$, and a diffeomorphism $\chi:\ V_1 \to \chi(V_1)$, such that*

$$(\chi \circ F)(u) = (u,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$$



Proof. Since D_pF is injective, it has m linearly independent rows. By re-indexing the rows (which amounts to a change of coordinates on V), we may assume that these are the first m rows. That is, writing

$$D_pF = \begin{pmatrix} A \\ C \end{pmatrix}$$

where A is the $m \times m$-matrix formed by the first m rows and C is the $(n-m) \times m$-matrix formed by the last $n-m$ rows, the square matrix A is invertible. Consider the map

$$H:\ U\times\mathbb{R}^{n-m}\to\mathbb{R}^n,\ (x,y)\mapsto F(x)+(0,y).$$

Its Jacobian at $(p,0)$ is

$$D_{(p,0)}H=\begin{pmatrix} A & 0\\ C & I_{n-m}\end{pmatrix}$$

which is invertible. Hence, by the inverse function theorem for $\mathbb{R}^n$ (Theorem 4.18), H is a diffeomorphism from some neighborhood of $(p,0)$ in $U\times\mathbb{R}^{n-m}$ onto some neighborhood V_1 of $H(p,0)=F(p)$, which we may take to be contained in V. Let

$$\chi:\ V_1\to\chi(V_1)\subseteq U\times\mathbb{R}^{n-m}$$

be the inverse; thus

$$(\chi\circ H)(x,y)=(x,y)$$

for all $(x,y)\in\chi(V_1)$. Replace U with the smaller open neighborhood

$$U_1=F^{-1}(V_1)\cap U$$

of p. Then $F(U_1)\subseteq V_1$, and

$$(\chi\circ F)(u)=(\chi\circ H)(u,0)=(u,0)$$

for all $u\in U_1$. □

The manifolds version reads as follows:

Theorem 4.37 (Normal Form for Immersions). *Let $F\in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m\le n$, and $p\in M$ a point with*

$$\mathrm{rank}_p(F)=m.$$

Then there are coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U)\subseteq V$ and

$$(\psi\circ F\circ\varphi^{-1})(u)=(u,0).$$

In particular, $F(U)\subseteq N$ is a submanifold of dimension m.

Proof. Once again, this is proved by introducing charts around p and $F(p)$, to reduce to a map between open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$, and then use the multivariable version of the result (Proposition 4.36) to obtain a change of coordinates, putting the map into normal form. We leave the details as an exercise to the reader (see Problem **??** at the end of this chapter). □

Definition 4.38. *A smooth map $F:\ M\to N$ is an* immersion *if* $\mathrm{rank}_p(F)=\dim M$ *for all $p\in M$.*

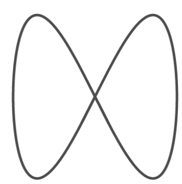
Theorem 4.37 gives a local normal form for immersions.

Example 4.39. Let $J \subseteq \mathbb{R}$ be an open interval, and $\gamma\colon\ J \to M$ a smooth map, i.e., a *smooth curve*. We see that the image of γ is an immersed submanifold, provided that $\operatorname{rank}_p(\gamma) = 1$ for all $p \in M$. In local coordinates (U, φ), this means that $\frac{d}{dt}(\varphi \circ \gamma)(t) \neq 0$ for all t with $\gamma(t) \in U$. For example, the curve $\gamma(t) = (t^2, t^3)$ from Example 3.13 fails to have this property at $t = 0$.

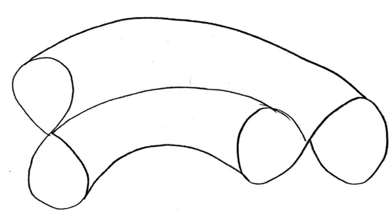
Example 4.40 (Figure eight). The map

$$\gamma\colon\ \mathbb{R} \to \mathbb{R}^2,\ \ t \mapsto \big(\sin(t), \sin(2t)\big)$$

is an immersion; indeed, for all $t \in \mathbb{R}$ we have $D_t\gamma \equiv \dot{\gamma}(t) \neq 0$. The image is a figure eight:



Example 4.41 (A mystery immersion). Consider the surface in $\mathbb{R}^3$ obtained by the following procedure. Consider the figure eight in the xz-plane, as in previous example. Shift in the x-direction by an amount $R > 1$, so that the resulting figure lies in the region where $x > 0$. Then rotate the plane containing the figure eight about the z-axis, while at the same time rotating the figure eight about its center, with exactly half the speed of rotation. That is, after a full turn $\varphi \mapsto \varphi + 2\pi$ the figure eight has performed a half turn. (Think of a single propeller plane flying in a circle, with the figure eight as its propeller.)



The picture suggests that the resulting subset $S \subseteq \mathbb{R}^3$ is the image of an immersion

$$\iota\colon\ \Sigma \to \mathbb{R}^3$$

of a *compact, connected surface* Σ. To make Σ uniquely defined, we should assume that the map ι is 1-1 on Σ. Since we know the classification of such surfaces, this raises the question: Which surface is it? Let us first try to come up with a good guess, without writing formulas:

62 (answer on page ??). What is the surface Σ, in terms of the classification of compact, connected surfaces? Hint: It may be instructive to investigate the subset $\Sigma_+ \subseteq \Sigma$ generated by half of the figure eight, corresponding to $-\pi/2 < t < \pi/2$ in terms of the parametrization from the previous example.

To get an explicit formula for the immersion, note that the procedure described above is a composition $F = F_3 \circ F_2 \circ F_1$ of the three maps. The map

$$F_1: \ (t,\varphi) \mapsto (\sin(t), \sin(2t), \varphi) = (u,v,\varphi)$$

describes the figure eight in the uv-plane (with φ just a bystander). Next,

$$F_2: \ (u,v,\varphi) \mapsto \Big(u\cos(\frac{\varphi}{2}) + v\sin(\frac{\varphi}{2}),\ v\cos(\frac{\varphi}{2}) - u\sin(\frac{\varphi}{2}), \varphi\Big) = (a,b,\varphi)$$

rotates the uv-plane as it moves in the direction of φ, by an angle of $\varphi/2$; thus $\varphi = 2\pi$ corresponds to a half-turn. Finally,

$$F_3: \ \big(a,b,\varphi) \mapsto ((a+R)\cos\varphi,\ (a+R)\sin\varphi,\ b\big) = (x,y,z)$$

takes this family of rotating uv-planes, and wraps it around the circle in the xy-plane of radius R, with φ now playing the role of the angular coordinate. The resulting map $F = F_3 \circ F_2 \circ F_1: \ \mathbb{R}^2 \to \mathbb{R}^3$ is given by $F(t,\varphi) = (x,y,z)$, where

$$\begin{aligned}
x &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\cos\varphi,\\
y &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\sin\varphi,\\
z &= \cos(\frac{\varphi}{2})\sin(2t) - \sin(\frac{\varphi}{2})\sin(t).
\end{aligned}$$

To verify that this is an immersion, it would be cumbersome to work out the Jacobian matrix of F directly. It is much easier to use that F_1 is an immersion, F_2 is a diffeomorphism, and F_3 is a local diffeomorphism from the open subset where $|a| < R$ onto its image.

63 (answer on page ??). With these formulas in place, confirm the result from 62.

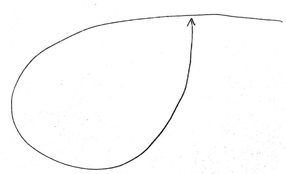
Example 4.42. Let M be a manifold, and $S \subseteq M$ a k-dimensional submanifold. Then the inclusion map $\iota: \ S \to M, \ \ x \mapsto x$ is an immersion. Indeed, if (V,ψ) is a submanifold chart for S, with $p \in U = V \cap S$, and letting $\varphi = \psi|_{V\cap S}$, we have that

$$(\psi \circ F \circ \varphi^{-1})(u) = (u, 0),$$

which shows

$$\operatorname{rank}_p(F) = \operatorname{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1}) = k.$$

By an *embedding*, we will mean an immersion given as the inclusion map for a submanifold. Not every injective immersion is an embedding; the following picture indicates an injective immersion $\mathbb{R} \to \mathbb{R}^2$ whose image is not a submanifold.



In practice, showing that an injective smooth map is an immersion (which amounts to showing that the rank is maximal everywhere) tends to be easier than proving that its image is a submanifold (which amounts to constructing submanifold charts). Fortunately, for compact manifolds we have the following fact:

Theorem 4.43. *Let $F: M \to N$ be an injective immersion, where the manifold M is compact. Then the image $F(M) \subseteq N$ is an embedded submanifold.*

Proof. We have to show that there exists a submanifold chart for $S = F(M)$ around any given point $F(p) \in N$, for $p \in M$. By Theorem 4.37, we can find charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, such that the local coordinate expression $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ is in normal form: i.e.,

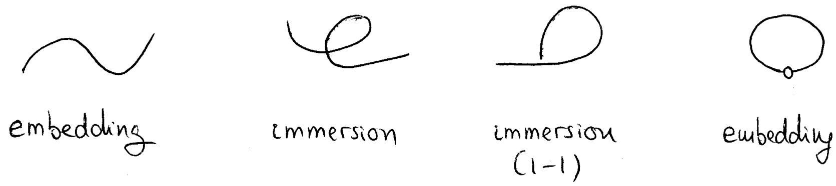
$$\widetilde{F}(u) = (u, 0).$$

We would like to take (V, ψ) as a submanifold chart for $S = F(M)$, but this may not work yet since the normal form above is only given for $F(U) \cap V$, and the set $F(M) \cap V = S \cap V$ may be strictly larger than that. Note however that $A := M \backslash U$ is compact, hence its image $F(A) \subseteq N$ is compact, and therefore closed (see Proposition 2.35; note we are using that N is Hausdorff). Since F is injective, we have that $p \notin F(A)$. Replace V with the smaller open neighborhood

$$V_1 = V \backslash (V \cap F(A)).$$

Then $(V_1, \psi|_{V_1})$ is the desired submanifold chart.

Remark 4.44. Unfortunately, the terminology for submanifolds used in the literature is not quite uniform. For example, some authors refer to injective immersions $\iota: S \to M$ as submanifolds (thus, a submanifold is taken to be a map rather than a subset). To clarify, 'our' submanifolds are sometimes called 'embedded submanifolds' or 'regular submanifolds'.



Example 4.45. Let A,B,C be distinct real numbers. We will leave it as a homework problem (Problem **??**) below) to verify that the map

$$F:\ \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4,\ \ (\alpha:\beta:\gamma) \mapsto (\beta\gamma, \alpha\gamma, \alpha\beta,\ A\alpha^2+B\beta^2+C\gamma^2),$$

where we use representatives (α,β,γ) such that $\alpha^2+\beta^2+\gamma^2=1$, is an injective immersion. Hence, by Theorem 4.43, it is an embedding of $\mathbb{R}\mathrm{P}^2$ as a submanifold of $\mathbb{R}^4$.

To summarize the outcome from the last few sections:

> If a smooth map $F \in C^\infty(M,N)$ has maximal rank near a given point $p \in M$, then one can choose local coordinates around p and around $F(p)$ such that the coordinate expression of F becomes a linear map.

In particular, near any given point of m, submersions look like surjective linear maps, while immersions look like injective linear maps.

Remark 4.46. This generalizes further to maps of *constant rank*. That is, if $\mathrm{rank}_p(F)$ is independent of p on some open subset $U \subseteq M$, then for all $p \in U$ one can choose coordinates near p and near $F(p)$ in which F becomes linear. In particular, the image of a sufficiently small open neighborhood of p is a submanifold of N.

4.3.6 Further remarks on embeddings and immersions

Remark 4.47. Let M be a manifold of dimension m. Given $k \in \mathbb{N}$, one may ask if there exists an embedding of M into $\mathbb{R}^k$, or at least an immersion into $\mathbb{R}^k$? For example, one knows that compact 2-manifolds (surfaces) Σ can be embedded into $\mathbb{R}^4$ (even $\mathbb{R}^3$ if Σ is orientable), and immersed into $\mathbb{R}^3$.
These and similar question belong to the realm of *differential topology*, with many deep and difficult results. The *Whitney embedding theorem* states that every m-dimensional manifold M can be realized as an embedded submanifold of $\mathbb{R}^{2m}$. (For a much weaker version of this result, see Theorem B.10 Appendix B.) This was improved later by various authors to $\mathbb{R}^{2m-1}$, provided that m is not a power of 2, but it is not known what the optimal bound is, in general. The *Whitney immersion theorem* states that every m-dimensional manifold M can be immersed into $\mathbb{R}^{2m-1}$. There had been conjectured optimal bounds $k = 2m-\alpha(m)$ (due to Massey), for a specific function $\alpha(m)$, so that any m-dimensional manifold can be immersed into $\mathbb{R}^{2m-\alpha(m)}$. This conjecture was proved in a 1985 paper of Cohen [5].

Remark 4.48. Another area of differential topology concerns the classification of immersions $M \to N$ up to isotopy, in particular for the case that the target is $N = \mathbb{R}^k$. We say that two immersions $F_0,\ F_1 :\ M \to N$ are *isotopic* if there exists a smooth map

$$F :\ \mathbb{R} \times M \to N$$

such that

$$F_0 = F(0,\cdot),\ F_1 = F(1,\cdot),$$

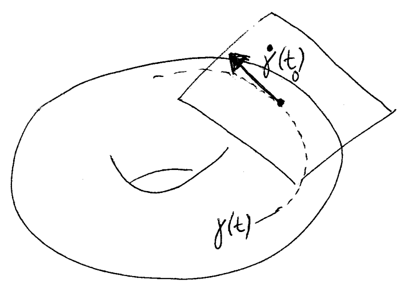
and such that all $F_t = F(t,\cdot)$ are immersions. In the late 1950s, Stephen Smale developed criteria for the existence of isotopies, and for example gave a striking application to the problem of 'sphere eversion' [18]. To explain this result, consider a standard 2-sphere in $\mathbb{R}^3$, with the 'outer side' of the sphere painted red, and the 'inner side' painted blue. Consider the following question: "Is it possible to turn the sphere inside out, without creating kinks or edges, ending up with the red paint on the inner side and the blue paint on the outer side?" In mathematical terms, letting $F_0 : S^2 \to \mathbb{R}^3$ be the standard inclusion of the sphere, and $F_1 :\ S^2 \to \mathbb{R}^3$ its composition with the map $\mathbf{x} \mapsto -\mathbf{x}$, this is the question whether the immersions F_0 and F_1 are isotopic. Our experience with immersions of the circle in $\mathbb{R}^2$ tends to suggest that this is probably not possible (a circle in $\mathbb{R}^2$ cannot be turned inside out). It hence came as quite a surprise when Stephen Smale proved, in 1957, that such a 'sphere eversion' does in fact exist. In subsequent years, various mathematicians developed concrete visualizations for sphere eversions, one of which is the subject of the 1994 movie 'Outside In'.

5

Tangent Spaces

5.1 Intrinsic definition of tangent spaces

For embedded submanifolds $M \subseteq \mathbb{R}^n$, the tangent space T_pM at $p \in M$ can be defined as the set of all *velocity vectors* $v = \dot{\gamma}(0)$, for smooth curves $\gamma\colon J \to M$ with $\gamma(0) = p$; here $J \subseteq \mathbb{R}$ is an open interval around 0.



It turns out (not entirely obvious!) that T_pM becomes a vector subspace of $\mathbb{R}^n$. (Warning: In pictures we tend to draw the tangent space as an *affine* subspace, where the origin has been moved to p.)

Example 5.1. Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$, given as the set of $\mathbf{x}$ such that $||\mathbf{x}|| = 1$. A curve $\gamma(t)$ lies in S^n if and only if $||\gamma(t)|| = 1$. Taking the derivative of the equation $\gamma(t) \cdot \gamma(t) = 1$ at $t = 0$, we obtain (after dividing by 2, and using $\gamma(0) = p$)

$$p \cdot \dot{\gamma}(0) = 0.$$

That is, T_pM consists of vectors $\mathbf{v} \in \mathbb{R}^{n+1}$ that are orthogonal to $p \in S^n$. Conversely, every such vector $\mathbf{v}$ is of the form $\dot{\gamma}(0)$: Given $\mathbf{v}$, we may take $\gamma(t) = (p + t\mathbf{v})/||p + t\mathbf{v}||$, for example. Hence

$$T_pS^n = (\mathbb{R}p)^\perp,$$

the hyperplane orthogonal to the line through p.

For general manifolds M, without a given embedding into a Euclidean space, we would like to make sense of ‘velocity vectors’ of curves, and hence of the tangent space, *intrinsically*. The basic observation is that the curve $t \mapsto \gamma(t)$ defines a ‘directional derivative’ on functions $f \in C^\infty(M)$:

$$f \mapsto \left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)\big)$$

64 (answer on page ??). Show that if M is a submanifold of $\mathbb{R}^n$, then the map $C^\infty(M) \to \mathbb{R},\ f \mapsto \frac{d}{dt}\big|_{t=0} f\big(\gamma(t)\big)$ depends only on $p = \gamma(0)$ and the velocity vector $v = \dot{\gamma}(0)$.

For a general manifold, we think of tangent vectors not as vectors in some ambient Euclidean space, but as the set of directional derivatives:

Definition 5.2 (Tangent spaces – first definition). *Let M be a manifold, $p \in M$. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form*

$$v(f) = \left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)\big),$$

for smooth curves $\gamma \in C^\infty(J,M)$ with $\gamma(0) = p$, for some open interval $J \subseteq \mathbb{R}$ around 0*. The elements $v \in T_pM$ are called the* tangent vectors *to M at p.*

As it stands, T_pM is defined as a certain subset of the infinite-dimensional vector space

$$L(C^\infty(M), \mathbb{R})$$

of all linear maps $C^\infty(M) \to \mathbb{R}$. The following local coordinate description makes it clear that T_pM is a linear subspace of this vector space, of dimension equal to the dimension of M.

Theorem 5.3. *Let (U, φ) be a coordinate chart around p. A linear map $v : C^\infty(M) \to \mathbb{R}$ is in T_pM if and only if it has the form,*

$$v(f) = \sum_{i=1}^m a^i \left.\frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\right|_{u=\varphi(p)} \tag{5.1}$$

for some $\mathbf{a} = (a^1, \ldots, a^m) \in \mathbb{R}^m$.

Proof. Given a linear map v of this form, let $\tilde{\gamma}\colon \mathbb{R} \to \varphi(U)$ be a curve with $\tilde{\gamma}(t) = \varphi(p) + t\mathbf{a}$ for $|t|$ sufficiently small. Let $\gamma = \varphi^{-1} \circ \tilde{\gamma}$. Then

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\bigg|_{t=0} (f \circ \varphi^{-1})(\varphi(p) + t\mathbf{a}) \\
&= \sum_{i=1}^{m} a^i \, \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\bigg|_{u=\varphi(p)}
\end{aligned}$$

by the chain rule. Conversely, given any curve γ with $\gamma(0) = p$, let $\tilde{\gamma} = \varphi \circ \gamma$ be the corresponding curve in $\varphi(U)$ (defined for small $|t|$). Then $\widetilde{\gamma}(0) = \varphi(p)$, and

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\bigg|_{t=0} (f \circ \varphi^{-1})(\tilde{\gamma}(t)) \\
&= \sum_{i=1}^{m} a^i \, \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\bigg|_{u=\gamma(p)}
\end{aligned}$$

where $\mathbf{a} = \dfrac{d\tilde{\gamma}}{dt}\bigg|_{t=0}$. $\square$

We can use this result as an alternative definition of the tangent space, namely:

Definition 5.4 (Tangent spaces – second definition). *Let (U, φ) be a chart around p. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form*

$$v(f) = \sum_{i=1}^{m} a^i \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\bigg|_{u=\varphi(p)} \tag{5.2}$$

for some $\mathbf{a} = (a^1, \ldots, a^m) \in \mathbb{R}^m$.

Remark 5.5. From this version of the definition, it is immediate that T_pM is an m-dimensional vector space. It is not immediately obvious from this second definition that T_pM is independent of the choice of coordinate chart, but this follows from the equivalence with the first definition. Alternatively, one may check directly that the subspace of $L(C^\infty(M), \mathbb{R})$ characterized by (5.2) does not depend on the chart, by studying the effect of a change of coordinates (see Problem **??** at the end of the chapter).

According to (5.2), any choice of coordinate chart (U, φ) around p defines a vector space isomorphism $T_pM \cong \mathbb{R}^m$, taking v to $\mathbf{a} = (a^1, \ldots, a^m)$. In particular, we see that if $U \subseteq \mathbb{R}^m$ is an open subset, and $p \in U$, then T_pU is the subspace of the space of linear maps $C^\infty(U) \to \mathbb{R}$ spanned by the partial derivatives at p. That is, T_pU has a basis

$$\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^m}\bigg|_p$$

identifying $T_pU \cong \mathbb{R}^m$. Given

$$v = \sum a^i \left.\frac{\partial}{\partial x^i}\right|_p$$

the coefficients a^i are obtained by applying v to the coordinate functions $x^1, \ldots, x^m : U \to \mathbb{R}$, that is, $a^i = v(x^i)$.

We now describe yet another approach to tangent spaces which again characterizes "directional derivatives" in a coordinate-free way, but without reference to curves γ. Note first that every tangent vector satisfies the *product rule*, also called the *Leibniz rule*:

Lemma 5.6. *Let $v \in T_pM$ be a tangent vector at $p \in M$. Then*

$$v(fg) = f(p)\, v(g) + v(f)\, g(p) \tag{5.3}$$

for all $f, g \in C^\infty(M)$.

Proof. Letting v be represented by a curve γ, this follows from

$$\begin{aligned} v(fg) &= \left.\frac{d}{dt}\right|_{t=0} \Big(f\big(\gamma(t)\big)g\big(\gamma(t)\big)\Big) \\ &= f(\gamma(0))\Big(\left.\frac{d}{dt}\right|_{t=0} g\big(\gamma(t)\big)\Big) + \Big(\left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)\big)\Big) g(\gamma(0)) \\ &= f(p)\, v(g) + v(f)\, g(p) \end{aligned}$$

where we used the product rule for functions of $t \in \mathbb{R}$. □

Alternatively, using local coordinates, Equation (5.3) amounts to the product rule for partial derivatives.

Note that there is an abundance of linear functionals $v \in L(C^\infty(M), \mathbb{R})$ that do *not* satisfy the product rule. For example, the evaluation map $\mathrm{ev}_p : f \mapsto f(p)$ is linear, but does not satisfy (5.3) with respect to p (or any other point).

65 (answer on page ??). Let $M = \mathbb{R}$. Give several examples of linear maps $v \in L(C^\infty(\mathbb{R}), \mathbb{R})$ that do not satisfy the product rule (5.3) with respect to $p = 0$.

66 (answer on page ??). Suppose that $v : C^\infty(M) \to \mathbb{R}$ is a linear map satisfying the product rule (5.3). Prove the following two facts:

a) v vanishes on constants. That is, if $f \in C^\infty(M)$ is the constant map, then $v(f) = 0$.
b) Suppose $f, g \in C^\infty(M)$ with $f(p) = g(p) = 0$. Then $v(fg) = 0$.

It turns out that the product rule completely characterizes tangent vectors:

Theorem 5.7. *A linear map $v : C^\infty(M) \to \mathbb{R}$ defines an element of T_pM if and only if it satisfies the product rule (5.3).*

The proof of Theorem 5.7 will require the following fact from multivariable calculus:

Lemma 5.8 (Hadamard's Lemma). *Let $U = B_R(\mathbf{0}) \subseteq \mathbb{R}^m$ be an open ball of radius $R > 0$ centered at $\mathbf{0}$, and $h \in C^\infty(U)$ a smooth function. Then there exist smooth functions $h_i \in C^\infty(U)$ with*

$$h(\mathbf{u}) = h(\mathbf{0}) + \sum_{i=1}^{m} u^i h_i(\mathbf{u}) \tag{5.4}$$

for all $\mathbf{u} \in U$. For any choice of such functions,

$$h_i(\mathbf{0}) = \frac{\partial h}{\partial u^i}(\mathbf{0}). \tag{5.5}$$

Proof. For fixed $\mathbf{u} \in U$, consider the function $t \mapsto h(t\mathbf{u}) = h(tu^1, \ldots, tu^n)$. We have that

$$h(\mathbf{u}) - h(\mathbf{0}) = \int_0^1 \frac{d}{dt} h(t\mathbf{u})\, dt = \int_0^1 \sum_{i=1}^{n} u^i \frac{\partial h}{\partial u^i}(t\mathbf{u})\, dt = \sum_{i=1}^{n} u^i h_i(\mathbf{u})$$

where

$$h_i(\mathbf{u}) = \int_0^1 \frac{\partial h}{\partial u^i}(t\mathbf{u})\, dt$$

are smooth functions of $\mathbf{u}$. Taking the derivative of (5.4)

$$\frac{\partial h}{\partial u^i} = \frac{\partial}{\partial u^i}\left(h(\mathbf{0}) + \sum_{i=1}^{m} u^i h_i(\mathbf{u}) \right) = h_i(\mathbf{u}) + \sum_k u^k \frac{\partial h_k}{\partial u^i}$$

and putting $\mathbf{u} = \mathbf{0}$, we see that $\frac{\partial h}{\partial u^i}\Big|_{\mathbf{u}=\mathbf{0}} = h_i(\mathbf{0})$. $\square$

Proof (of Theorem 5.7). Let $v : C^\infty(M) \to \mathbb{R}$ be a linear map satisfying the product rule (5.3). The proof consists of the following three steps.

Step 1: *If $f_1 = f_2$ on some open neighborhood U of p, then $v(f_1) = v(f_2)$.*

Equivalently, letting $f = f_1 - f_2$, we show that $v(f) = 0$ if $f|_U = 0$. Choose a 'bump function' $\chi \in C^\infty(M)$ with $\chi(p) = 1$, and $\chi|_{M \setminus U} = 0$. Then $f\chi = 0$. The product rule tells us that

$$0 = v(f\chi) = v(f)\chi(p) + v(\chi)f(p) = v(f).$$

Step 2: *Let (U, φ) be a chart around p, with image $\widetilde{U} = \varphi(U)$. Then there is unique linear map $\widetilde{v} :\ C^\infty(\widetilde{U}) \to \mathbb{R}$, again satisfying the product rule, such that $\widetilde{v}(\widetilde{f}) = v(f)$ whenever $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p} = \varphi(p)$.*

We want to define $\widetilde{v}$ by putting $\widetilde{v}(\widetilde{f}) = v(f)$, for any choice of function f such that $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$. (Note that such a function f always exists, for any given $\widetilde{f}$.) If g is another function such that $\widetilde{f}$ agrees with $g \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$, it follows from Step 1 that $v(f) = v(g)$. This shows that $\widetilde{v}$ is well-defined; the product rule for v implies the product rule for $\widetilde{v}$.

Step 3: *In a chart (U, φ) around p, the map $v:\ C^\infty(M) \to \mathbb{R}$ is of the form (5.2).*

Since the condition (5.2) does not depend on the choice of chart around p, we may assume that $\widetilde{p} = \varphi(p) = \mathbf{0}$, and that $\widetilde{U}$ is an open ball around $\mathbf{0}$. Define $\widetilde{v}$ as in Step 2. Given $f \in C^\infty(M)$, let $\tilde{f} = f \circ \varphi^{-1}$. By Hadamard's Lemma 5.8, we have that

$$\tilde{f}(\mathbf{u}) = \tilde{f}(\mathbf{0}) + \sum_{i=1}^{m} u^i h_i(\mathbf{u})$$

where $h_i \in C^\infty(\widetilde{U})$ with $h_i(\mathbf{0}) = \frac{\partial \widetilde{f}}{\partial u^i}(\mathbf{0})$. Using that $\widetilde{v}$ satisfies the product rule, and in particular that it vanishes on constants, we obtain

$$v(f) = \widetilde{v}(\widetilde{f}) = \sum_{i=1}^{m} \widetilde{v}(u^i)\, h_i(\mathbf{0}) = \sum_{i=1}^{m} a^i\, \frac{\partial \widetilde{f}}{\partial u^i}(\mathbf{0})$$

where we put $a^i = \widetilde{v}(u^i)$. $\square$

To summarize, we have the following alternative definition of tangent spaces:

Definition 5.9 (Tangent spaces – third definition). *The tangent space T_pM is the space of linear maps $C^\infty(M) \to \mathbb{R}$ satisfying the product rule,*

$$v(fg) = f(p)v(g) + v(f)\,g(p)$$

for all $f, g \in C^\infty(M)$.

At first sight, this characterization may seem a bit less intuitive than the definition as directional derivatives along curves. But it has the advantage of being less redundant – a tangent vector may be represented by many curves. Furthermore, it is immediate from this third definition (just as for the second definition, in terms of coordinates) that T_pM is a linear subspace of the vector space $L(C^\infty(M), \mathbb{R})$. (The fact that $\dim T_pM = \dim M$ is less obvious, though – for this the second definition is best.)

The following remark gives yet another characterization of the tangent space. Please read it only if you like it abstract – *otherwise skip this*!

Remark 5.10 (A fourth definition). There is a fourth definition of T_pM, as follows. For any $p \in M$, let $C_p^\infty(M)$ denotes the subspace of functions vanishing at p, and let $C_p^\infty(M)^2$ consist of finite sums $\sum_i f_i\, g_i$ where $f_i, g_i \in C_p^\infty(M)$. We have a direct sum decomposition

$$C^\infty(M) = \mathbb{R} \oplus C_p^\infty(M),$$

where $\mathbb{R}$ is regarded as the constant functions. Since any tangent vector $v:\ C^\infty(M) \to \mathbb{R}$ vanishes on constants, v is effectively a map $v:\ C^\infty_p(M) \to \mathbb{R}$. By the product rule, v vanishes on the subspace $C^\infty_p(M)^2 \subseteq C^\infty_p(M)$. Thus v descends to a linear map $C^\infty_p(M)/C^\infty_p(M)^2 \to \mathbb{R}$, i.e. an element of the dual space $(C^\infty_p(M)/C^\infty_p(M)^2)^*$. The map

$$T_pM \to (C^\infty_p(M)/C^\infty_p(M)^2)^*$$

just defined is an *isomorphism*, and can therefore be used as a definition of T_pM. This may appear very fancy on first sight, but really it just says that a tangent vector is a linear functional on $C^\infty(M)$ that vanishes on constants and depends only on the first order Taylor expansion of the function at p. Furthermore, this viewpoint lends itself to generalizations which are relevant to algebraic geometry and non-commutative geometry: The 'vanishing ideals' $C^\infty_p(M)$ are the maximal ideals in the algebra of smooth functions, with $C^\infty_p(M)^2$ their second power (in the sense of products of ideals). Thus, for any maximal ideal $\mathscr{I}$ in a commutative algebra $\mathscr{A}$ one may regard $(\mathscr{I}/\mathscr{I}^2)^*$ as a 'tangent space'.

After this lengthy discussion of tangent spaces, observe that the 'velocity vectors' of curves are naturally elements of the tangent space. Indeed, let $J \subseteq \mathbb{R}$ be an open interval, and $\gamma \in C^\infty(J,M)$ a smooth curve. Then for any $t_0 \in J$, the tangent (or *velocity*) vector

$$\dot\gamma(t_0) \in T_{\gamma(t_0)}M.$$

at time t_0 is given in terms of its action on functions by

$$(\dot\gamma(t_0))(f) = \frac{d}{dt}\Big|_{t=t_0} f(\gamma(t)) \tag{5.6}$$

We will also use the notation $\dfrac{d\gamma}{dt}(t_0)$ or $\dfrac{d\gamma}{dt}\Big|_{t_0}$ to denote this velocity vector.

5.2 Tangent maps

5.2.1 Definition of the tangent map, basic properties

For smooth maps $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ of Euclidean spaces, and any given $p \in U$, we considered the derivative to be the linear map

$$D_pF:\ \mathbb{R}^m \to \mathbb{R}^n,\ \mathbf{a} \mapsto \frac{d}{dt}\Big|_{t=0} F(p+t\mathbf{a}).$$

(See Definition 4.15.) The corresponding matrix is the Jacobian matrix of partial derivatives of F. The following definition generalizes the derivative to smooth maps between manifolds.

Definition 5.11. *Let M,N be manifolds and $F \in C^\infty(M,N)$. For any $p \in M$, we define the* tangent map *to be the linear map*

$$T_pF: \ T_pM \to T_{F(p)}N$$

given by

$$\big(T_pF(v)\big)(g) = v(g \circ F)$$

for $v \in T_pM$ and $g \in C^\infty(N)$.

One needs to verify that the right-hand side does indeed define a tangent vector:

67 (answer on page ??). Show that for all $v \in T_pM$, the map $g \mapsto v(g \circ F)$ satisfies the product rule at $q = F(p)$, hence defines an element of T_qN.

Proposition 5.12. *If $v \in T_pM$ is represented by a curve $\gamma: \ J \to M$, then $(T_pF)(v)$ is represented by the curve $F \circ \gamma$.*

Proof. Let $\gamma : J \to M$ be a smooth curve passing through p at $t = 0$, such that

$$v(g) = \left.\frac{d}{dt}\right|_{t=0} g(\gamma(t))$$

for any $g \in C^\infty(M)$. (I.e., $v = \dot{\gamma}(0)$ in the notation of (5.6).) Then $F \circ \gamma: \ J \to N$ is a smooth curve passing through $F(p) = q$ at $t = 0$. By definition, for any $h \in C^\infty(N)$:

$$(T_pF(v))(h) = v(h \circ F) = \left.\frac{d}{dt}\right|_{t=0} (h \circ F)(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} h(F \circ \gamma(t)).$$

That is, $T_pF(v)$ is represented by the curve $F \circ \gamma$. $\square$

Remark 5.13 (Pull-backs, push-forwards). For smooth maps $F \in C^\infty(M,N)$, one can consider various 'pull-backs' of objects on N to objects on M, and 'push-forwards' of objects on M to objects on N. Pull-backs are generally denoted by F^*, push-forwards by F_*. For example, functions on N can be pulled back to functions on M:

$$g \in C^\infty(N) \ \ \rightsquigarrow \ \ F^*g = g \circ F \in C^\infty(M).$$

Curves on M can be pushed forward to curves on N:

$$\gamma: \ J \to M \ \ \rightsquigarrow \ \ F_*\gamma = F \circ \gamma: \ J \to N.$$

Tangent vectors to M can also be pushed forward to tangent vectors to N:

$$v \in T_pM \ \ \rightsquigarrow \ \ F_*(v) = (T_pF)(v).$$

The definition of the tangent map can be phrased in these terms as $(F_*v)(g) = v(F^*g)$. Note also that if v is represented by the curve γ, then F_*v is represented by the curve $F_*\gamma$.

Proposition 5.14 (Chain rule). *Let M,N,Q be manifolds. Under composition of maps $F \in C^\infty(M,N)$ and $F' \in C^\infty(N,Q)$,*

$$T_p(F' \circ F) = T_{F(p)}F' \circ T_pF.$$

Proof. Observe that $T_p(F' \circ F)$ is determined by its action on tangent vectors $v \in T_pM$, and the resulting tangent vector $(T_p(F' \circ F))(v) \in T_{F'(F(p))}Q$ is determined by its action on functions $g \in C^\infty(Q)$. We have, using the definitions,

$$\begin{aligned}\big(T_p(F' \circ F)(v)\big)(g) &= v(g \circ (F' \circ F))\\ &= v((g \circ F') \circ F)\\ &= \big((T_pF)(v)\big)(g \circ F')\\ &= \Big((T_{F(p)}F')\big((T_pF)(v)\big)\Big)(g).\end{aligned}$$

68 (answer on page ??). Give a second proof of Proposition 5.14, using the characterization of tangent vectors as velocity vectors of curves (see Proposition 5.12).

69 (answer on page ??).

a) Show that the tangent map of the identity map $\mathrm{id}_M : M \to M$ at $p \in M$ is the identity map on the tangent space:
$$T_p\,\mathrm{id}_M = \mathrm{id}_{T_pM}.$$
b) Show that if $F \in C^\infty(M,N)$ is a diffeomorphism, then T_pF is a linear isomorphism, with inverse
$$(T_pF)^{-1} = (T_{F(p)}F^{-1}).$$
c) Suppose that $F \in C^\infty(M,N)$ is a *constant map*, that is, $F(M) = \{q\}$ for some element $q \in N$. Show that T_pF is the zero map, for all $p \in M$.

5.2.2 Coordinate description of the tangent map

To get a better understanding of the tangent map, let us first consider the special case where $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For $p \in U$, the tangent space T_pU is canonically identified with $\mathbb{R}^m$, using the basis

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p \in T_pU$$

of the tangent space (cf. Remark 5.5). Similarly, $T_{F(p)}V \cong \mathbb{R}^n$, using the basis given by partial derivatives $\frac{\partial}{\partial y^j}\Big|_{F(p)}$. Using these identifications, the tangent map becomes a linear map $T_pF : \mathbb{R}^m \to \mathbb{R}^n$, i.e. it is given by an $n \times m$-matrix. This matrix is exactly the Jacobian:

Proposition 5.15. *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For all $p \in M$, the tangent map T_pF coincides with the derivative (i.e., Jacobian matrix) D_pF of F at p.*

Proof. For $g \in C^\infty(V)$, we calculate

$$\begin{aligned}\Big((T_pF)\big(\frac{\partial}{\partial x^i}\Big|_p\big)\Big)(g) &= \frac{\partial}{\partial x^i}\Big|_p (g\circ F)\\ &= \sum_{j=1}^n \frac{\partial g}{\partial y^j}\Big|_{F(p)} \frac{\partial F^j}{\partial x^i}\Big|_p\\ &= \Big(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}\Big)(g).\end{aligned}$$

This shows

$$(T_pF)\big(\frac{\partial}{\partial x^i}\Big|_p\big) = \sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

Hence, in terms of the given bases of T_pU and $T_{F(p)}V$, the matrix of the linear map T_pF has entries $\frac{\partial F^j}{\partial x^i}\Big|_p$. □

Remark 5.16. For $F \in C^\infty(U,V)$, it is common to write $y = F(x)$, and accordingly write $\big(\frac{\partial y^j}{\partial x^i}\big)$ for the entries of the Jacobian matrix. In these terms, the derivative reads as

$$T_pF\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_j \frac{\partial y^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

This suggestive formula is often used for explicit calculations.

70 (answer on page ??). Consider $\mathbb{R}^2$ with standard coordinates x,y. On the open subset $\mathbb{R}^2 \setminus \{(x,0) :\ x \le 0\}$ introduce *polar coordinates* r,θ by

$$x = r\cos\theta\ ,\ y = r\sin\theta;$$

here $0 < r < \infty$ and $-\pi < \theta < \pi$. Express the tangent vectors

$$\frac{\partial}{\partial r}\Big|_p\ ,\ \frac{\partial}{\partial \theta}\Big|_p$$

as a combination of the tangent vectors

$$\frac{\partial}{\partial x}\Big|_p\ ,\ \frac{\partial}{\partial y}\Big|_p.$$

For a general smooth map $F \in C^\infty(M,N)$, we obtain a similar description once we pick coordinate charts. Given $p \in M$, choose charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$. Let $\widetilde{U} = \varphi(U)$, $\widetilde{V} = \psi(V)$, and put

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \ \widetilde{U} \to \widetilde{V}.$$

Since the coordinate map $\varphi : \ U \to \mathbb{R}^m$ is a diffeomorphism onto $\widetilde{U}$, it gives an isomorphism (cf. 69)

$$T_p\varphi : \ T_pU \to T_{\varphi(p)}\widetilde{U} = \mathbb{R}^m.$$

Similarly, $T_{F(p)}\psi$ gives an isomorphism of $T_{F(p)}V$ with $\mathbb{R}^n$. Note also that since $U \subseteq M$, $V \subseteq N$ are open, we have that $T_pU = T_pM$, $T_{F(p)}V = T_{F(p)}N$. We obtain,

$$T_{\varphi(p)}\widetilde{F} = T_{F(p)}\psi \circ T_pF \circ (T_p\varphi)^{-1}.$$

which may be depicted in a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^m \arrow[rr, "D_{\varphi(p)}\widetilde{F}"] & & \mathbb{R}^n \\
T_pM = T_pU \arrow[u, "T_p\varphi", "\cong"'] \arrow[rr, "T_pF"'] & & T_{F(p)}V = T_{F(p)}N \arrow[u, "\cong", "T_{F(p)}\psi"']
\end{tikzcd}$$

So, the choice of coordinates identifies the tangent spaces T_pM, $T_{F(p)}N$ with $\mathbb{R}^m, \mathbb{R}^n$ respectively, and the tangent map T_pF with the derivative of the coordinate expression of F (equivalently, the Jacobian matrix).
Now that we have recognized T_pF as the derivative expressed in a coordinate-free way, we may liberate some of our earlier definitions from coordinates:

Definition 5.17. *Let $F \in C^\infty(M,N)$.*

- *The* rank of F at $p \in M$, *denoted* $\mathrm{rank}_p(F)$*, is the rank of the linear map T_pF.*
- *F has* maximal rank at p *if* $\mathrm{rank}_p(F) = \min(\dim M, \dim N)$.
- *F is a* submersion *if T_pF is surjective for all $p \in M$,*
- *F is an* immersion *if T_pF is injective for all $p \in M$,*
- *F is a* local diffeomorphism *if T_pF is an isomorphism for all $p \in M$.*
- *$p \in M$ is a* critical point *of F is T_pF does not have maximal rank at p.*
- *$q \in N$ is a* regular value *of F if T_pF is surjective for all $p \in F^{-1}(q)$ (in particular, if $q \notin F(M)$).*
- *$q \in N$ is a* singular value *(sometimes called* critical value*) if it is not a regular value.*

71 (answer on page ??). To illustrate the merits of the coordinate free definitions, give simple proofs of the facts that the compositions of two submersions is again a submersion, and that the composition of two immersions is an immersion.

5.2.3 Tangent spaces of submanifolds

Suppose $S \subseteq M$ is a submanifold, and $p \in S$. Then the tangent space T_pS is canonically identified as a subspace of T_pM. Indeed, since the inclusion $i:\ S \hookrightarrow M$ is an immersion, the tangent map is an injective linear map,

$$T_p i:\ T_pS \to T_pM,$$

and we identify T_pS with the subspace given as the image of this map*. As a special case, we see that whenever M is realized as a submanifold of $\mathbb{R}^n$, then its tangent spaces T_pM may be viewed as subspaces of $T_p\mathbb{R}^n = \mathbb{R}^n$.

Proposition 5.18. *Let $F \in C^\infty(M,N)$ be a smooth map, and let $S = F^{-1}(q)$ be a submanifold given as the fiber of some regular value $q \in N$. For all $p \in S$,*

$$T_pS = \ker(T_pF),$$

as subspaces of T_pM.

Proof. Let $m = \dim M,\ n = \dim N$. Since T_pF is surjective, its kernel has dimension $m-n$. By the regular value theorem, this is also the dimension of S, hence of T_pS. It is therefore enough to show that $T_pS \subseteq \ker(T_pF)$. Letting $i:\ S \to M$ be the inclusion, we have to show that

$$T_pF \circ T_p i = T_p(F \circ i)$$

is the zero map. But $F \circ i$ is a *constant map*, taking all points of S to the constant value $q \in N$. The tangent map to a constant map is just zero (✐ 69). Hence $T_p(F \circ i) = 0$. □

As a special case, we can apply this result to smooth maps between open subsets of Euclidean spaces, where the tangent maps are directly given by the derivative (Jacobian matrix). Thus, suppose $V \subseteq \mathbb{R}^n$ is open, and $q \in \mathbb{R}^k$ is a regular value of $F \in C^\infty(V,\mathbb{R}^k)$, defining an embedded submanifold $M = F^{-1}(q)$. Then the tangent spaces $T_pM \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$ are given as

$$T_pM = \ker(T_pF) = \ker(D_pF). \tag{5.7}$$

Example 5.19. Recall that at the beginning of the chapter we have calculated T_pS^n directly from the curves definition of the tangent space. Alternatively, we may use (5.7): Regard S^n as the regular level set $F^{-1}(1)$ of the function $F:\ \mathbb{R}^{n+1} \to \mathbb{R},\ \mathbf{x} \mapsto ||\mathbf{x}||^2$. Then, for all $p \in S^n$, and all $\mathbf{a} \in \mathbb{R}^{n+1}$,

$$(D_pF)(\mathbf{a}) = \frac{d}{dt}\bigg|_{t=0} F(p+t\mathbf{a}) = \frac{d}{dt}\bigg|_{t=0} (p+t\mathbf{a})\cdot(p+t\mathbf{a}) = 2p\cdot\mathbf{a},$$

hence $T_pS^n = \{\mathbf{a} \in \mathbb{R}^{n+1} \,|\, \mathbf{a}\cdot p = 0\} = \mathrm{span}(p)^\perp$.

* Hopefully, the identifications are not getting too confusing: S gets identified with $i(S) \subseteq M$, hence also $p \in S$ with its image $i(p)$ in M, and T_pS gets identified with $(T_p i)(T_pS) \subseteq T_pM$.

As another typical application, suppose that $S \subseteq M$ is a submanifold, and $f \in C^\infty(S)$ is a smooth function given as the restriction $f = h|_S$ of a smooth function $h \in C^\infty(M)$. Consider the problem of finding the critical points $p \in S$ of f. Since f is a scalar function, $T_p f$ fails to have maximal rank if and only if it is zero:

$$\mathrm{Crit}(f) = \{p \in S|\ T_p f = 0\}.$$

Letting $i\colon\ S \to M$ be the inclusion, we have $f = h|_S = h \circ i$, hence $T_p f = T_p h \circ T_p i$. It follows that $T_p f = 0$ if and only if $T_p h$ vanishes on the range of $T_p i$, that is on $T_p S$:

$$\mathrm{Crit}(f) = \{p \in S|\ T_p S \subseteq \ker(T_p h)\}.$$

If $M = \mathbb{R}^m$, then $T_p h$ is just the Jacobian $D_p h$, whose kernel is sometimes relatively easy to compute – in any case this approach tends to be faster than a calculation in charts for S.

Example 5.20. Let $S \subseteq \mathbb{R}^3$ be a surface, and $f \in C^\infty(S)$ the 'height function' given by $f(x,y,z) = z$. To find $\mathrm{Crit}(f)$, regard f as the restriction of $h \in C^\infty(\mathbb{R}^3),\ h(x,y,z) = z$. The tangent map is

$$T_p h = D_p h = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

as a linear map $\mathbb{R}^3 \to \mathbb{R}$. Hence, $\ker T_p h$ is the xy-plane. On the other hand, $T_p S$ for $p \in S$ is 2-dimensional, hence the condition $T_p S \subseteq \ker T_p h$ is equivalent to $T_p S = \ker T_p h$. We conclude that the critical points of f are exactly those points of S where the tangent plane is 'horizontal', i.e. equal to the xy-plane.

72 (answer on page ??). Consider $f = h|_{S^2}$, where

$$h\colon\ \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto xy.$$

a) Find $T_p h = D_p h\colon\ \mathbb{R}^3 \to \mathbb{R}$, and compute its kernel $\ker T_p h$.
b) Find $\mathrm{Crit}(f)$ as the set of all $p \in S^2$ such that $T_p S^2 \subseteq \ker(T_p h)$. How many critical points are there?

Example 5.21. We had discussed various *matrix Lie groups* G as examples of manifolds (cf. Example 4.29). By definition, these are submanifolds $G \subseteq \mathrm{Mat}_\mathbb{R}(n)$ (for some $n \in \mathbb{N}$), consisting of invertible matrices with the properties

$$I \in G,\ \ A,B \in G \Rightarrow AB \in G,\ \ A \in G \Rightarrow A^{-1} \in G.$$

The tangent space to the identity (group unit) for such matrix Lie groups G turns out to be important; it is commonly denoted by lower case fraktur letter:

$$\mathfrak{g} = T_I G \subseteq \mathrm{Mat}_\mathbb{R}(n).$$

One calls $\mathfrak{g}$ the *Lie algebra* of G. It is a key fact (see 74) that $\mathfrak{g}$ is closed under commutation of matrices:

$$X_1, X_2 \in \mathfrak{g} \;\;\Rightarrow\;\; [X_1, X_2] = X_1X_2 - X_2X_1 \in \mathfrak{g}.$$

A vector subspace $\mathfrak{g} \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ that is closed under matrix commutators is called a *matrix Lie algebra*.

Some concrete examples:

a) The matrix Lie group

$$\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) \neq 0\}$$

of *all* invertible matrices is an open subset of $\mathrm{Mat}_{\mathbb{R}}(n)$, hence

$$\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}_{\mathbb{R}}(n)$$

is the entire space of matrices.

b) For the group $\mathrm{O}(n)$, consisting of matrices with $F(A) := A^\top A = I$, we found in Example 4.29 that $T_AF(X) = X^\top A + A^\top X$. In particular, $T_IF(X) = X^\top + X$. We read off the Lie algebra $\mathfrak{o}(n)$ as the kernel of this map:

$$\mathfrak{o}(n) = \{X \in \mathrm{Mat}_{\mathbb{R}}(n) |\ X^\top = -X\}.$$

73 (answer on page ??). Show that for every $X \in \mathrm{Mat}_{\mathbb{R}}(n)$,

$$\frac{d}{dt}\bigg|_{t=0} \det(I + tX) = \mathrm{tr}(X).$$

Use this result to compute the Lie algebra $\mathrm{SL}(n,\mathbb{R})$ of the special linear group

$$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) = 1\}.$$

The following 74 develops some important properties of matrix Lie groups.

74 (answer on page ??).

a) Show (using for example the curves definition of the tangent space) that the tangent space at general elements $A \in G$ can be described by left translation

$$T_AG = \{AX |\ X \in \mathfrak{g}\}$$

or also by right translation $T_AG = \{XA |\ X \in \mathfrak{g}\}$.

b) Show that $A \in G,\ X \in \mathfrak{g} \;\Rightarrow\; AXA^{-1} \in \mathfrak{g}$.

c) Show that $X, Y \in \mathfrak{g} \;\Rightarrow\; XY - YX \in \mathfrak{g}$. (Hint: Choose a curve $\gamma(t)$ in G representing Y.)

5.2.4 Example: Steiner's surface revisited

As we discussed in Section 4.3.3, Steiner's 'Roman surface' is the image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3, \ \ (x:y:z) \mapsto \frac{1}{x^2+y^2+z^2}(yz,\ xz,\ xy). \tag{5.8}$$

(We changed notation from α, β, γ to x, y, z.) What are the critical points of this map? (Recall that if $p \in \mathbb{R}\mathrm{P}^2$ is not a critical point, then the map restricts to an immersion on an open neighborhood of p.) To investigate this question, one can express the map in local charts, and compute the resulting Jacobian matrix. While this approach is perfectly fine, the resulting expressions will become rather complicated. A simpler approach is to consider the composition with the local diffeomorphism $\pi : S^2 \to \mathbb{R}\mathrm{P}^2$, given as

$$S^2 \to \mathbb{R}^3, \ \ (x,y,z) \mapsto (yz,\ xz,\ xy). \tag{5.9}$$

Since π is a surjective local diffeomorphism, the critical points of (5.8) are the images of the critical points of (5.9). In turn, this map is the restriction $F|_{S^2}$ of the map

$$F:\ \mathbb{R}^3 \to \mathbb{R}^3, \ \ (x,y,z) \mapsto (yz,\ xz,\ xy). \tag{5.10}$$

We have $T_p(F|_{S^2}) = T_pF|_{T_pS^2}$, hence $\ker(T_p(F|_{S^2})) = \ker(T_pF) \cap T_pS^2$. We are interested in points $p \in S^2$ where this intersection is non-zero.

75 (answer on page ??).

a) Compute $T_pF = D_pF$, and find its determinant. Conclude that the kernel is empty except when one of the coordinates is 0.
b) Suppose $p = (x,y,z)$ with $x = 0$. Find the kernel of the tangent map at p, and $\ker(T_pF) \cap T_pS$. Repeat with the cases $y = 0$ and $z = 0$.
c) What are the critical points of the map $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$?

If you have completed 75, you have found that the map (5.8) has 6 critical points. It is thus an immersion away from those points.

6

Vector fields

6.1 Vector fields as derivations

A vector field on a manifold may be regarded as a family of tangent vectors $X_p \in T_pM$ for $p \in M$, depending smoothly on the base points $p \in M$. One way of making precise what is meant by 'depending smoothly' is the following.

Definition 6.1 (Vector fields – first definition). *A collection of tangent vectors $X = \{X_p\}$ for $p \in M$ defines a vector field if and only if for all functions $f \in C^\infty(M)$ the function $p \mapsto X_p(f)$ is smooth. The space of all vector fields on M is denoted $\mathfrak{X}(M)$.*

76 (answer on page ??). Verify the (implicit) claim that the set $\mathfrak{X}(M)$ of all vector fields on M is a vector space.

We hence obtain a linear map, denoted by the same letter,

$$X: \ C^\infty(M) \to C^\infty(M)$$

such that

$$\big(X(f)\big)(p) = X_p(f). \tag{6.1}$$

Since each individual tangent vector X_p satisfy a product rule (5.3), it follows that X itself satisfies a product rule. We can use this as an alternative definition, realizing $\mathfrak{X}(M)$ as a subspace of the space $L(C^\infty(M), C^\infty(M))$:

Definition 6.2 (Vector fields – second definition). *A vector field on M is a linear map*

$$X: \ C^\infty(M) \to C^\infty(M)$$

satisfying the product rule,

$$X(fg) = X(f)g + fX(g) \tag{6.2}$$

for $f, g \in C^\infty(M)$.

77 (answer on page ??). Explain in more detail why these two definitions of vector fields are equivalent.

Remark 6.3. The condition (6.2) says that X is a *derivation* of the algebra $C^\infty(M)$ of smooth functions. More generally, a derivation of an algebra $\mathscr{A}$ is a linear map $D\colon\ \mathscr{A} \to \mathscr{A}$ such that for any $a_1, a_2 \in \mathscr{A}$

$$D(a_1 a_2) = D(a_1)\, a_2 + a_1\, D(a_2).$$

(Appendix A.3.3 reviews some facts about derivations.)
Vector fields can be multiplied by smooth functions:

$$C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ \ (h, X) \mapsto hX$$

where $(hX)(f) = hX(f)$. In algebraic terminology, this makes the space of vector fields into a *module* over the algebra of smooth functions. (See A.3.4 in the appendix.)

78 (answer on page ??). Explain why hX is again a vector field.

We can also express the smoothness of the collection of tangent vectors $\{X_p\}$ in terms of coordinate charts (U, φ). Recall that for any $p \in U$, and all $f \in C^\infty(M)$, the tangent vector X_p is expressed as

$$X_p(f) = \sum_{i=1}^m a^i \frac{\partial}{\partial u^i}\Big|_{\mathbf{u}=\varphi(p)} (f \circ \varphi^{-1}).$$

The vector $\mathbf{a} = (a^1, \dots, a^m) \in \mathbb{R}^m$ represents X_p in the chart; i.e., $(T_p\varphi)(X_p) = \mathbf{a}$ under the identification $T_{\varphi(p)}\varphi(U) = \mathbb{R}^m$. As p varies in U, the vector $\mathbf{a}$ becomes a function of $p \in U$, or equivalently of $\mathbf{u} = \varphi(p)$.

Proposition 6.4. *The collection of tangent vectors $\{X_p \in T_pM,\ p \in M\}$ defines a vector field X such that $X(f)|_p = X_p(f)$, if and only if for all charts (U, φ), the functions $a^i\colon\ \varphi(U) \to \mathbb{R}$ defined by*

$$X_{\varphi^{-1}(\mathbf{u})}(f) = \sum_{i=1}^m a^i(\mathbf{u})\ \frac{\partial}{\partial u^i}\Big|_{\mathbf{u}} (f \circ \varphi^{-1}),$$

are smooth.

Proof. "$\Leftarrow$" Suppose the coefficient functions a^i are smooth, for all charts (U,φ). Then it follows that for all $f \in C^\infty(M)$, the function

$$X(f)\circ\varphi^{-1}:\ \varphi(U)\to\mathbb{R},\ \ \mathbf{u}\mapsto X_{\varphi^{-1}(\mathbf{u})}(f)$$

is smooth. Consequently $X(f)|_U$ is smooth, and since this is true for all charts, $X(f)$ is smooth.

"$\Rightarrow$" Let X be a vector field, defining a collection of tangent vectors X_p. Given a chart (U,φ), we want to show that the coefficient functions a^i are smooth near any given point $\widetilde{p}=\varphi(p)\in\widetilde{U}=\varphi(U)$. Let $g\in C^\infty(U)$ be function whose local coordinate expression $\widetilde{g}=g\circ\varphi^{-1}$ is the coordinate function $\mathbf{u}\mapsto u^i$. This function may not directly extend from U to M, but we may choose $f\in C^\infty(M)$ such that

$$f|_{U_1}=g|_{U_1}$$

over a possibly smaller neighborhood $U_1\subseteq U$ of p. (See 79 below.) Then $f\circ\varphi^{-1}$ coincides with u^i on $\widetilde{U}_1=\varphi(U_1)$, and hence $X(f)\circ\varphi^{-1}=a^i$ on $\widetilde{U}_1$. In particular, the a^i are smooth on $\widetilde{U}_1$. $\square$

79 (answer on page ??). Explain how to construct the function f in the second part of the proof.

In particular, we see that vector fields on open subsets $U\subseteq\mathbb{R}^m$ are of the form

$$X=\sum_i a^i\frac{\partial}{\partial x^i}$$

where $a^i\in C^\infty(U)$. Under a diffeomorphism $F:\ U\to V,\ x\mapsto y=F(x)$, the coordinate vector fields transform with the Jacobian

$$TF\Big(\frac{\partial}{\partial x^i}\Big)=\sum_j\frac{\partial F^j}{\partial x^i}\Big|_{x=F^{-1}(y)}\ \frac{\partial}{\partial y^j}.$$

See Proposition 5.15; as in the remark following that proposition this 'change of coordinates' is often written

$$\frac{\partial}{\partial x^i}=\sum_j\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}.$$

Here one thinks of the x^i and y^j as coordinates on the same set (U and V are 'identified' via F), and one uses the simplified notation $y=y(x)$ instead of $y=F(x)$.

80 (answer on page ??). Consider $\mathbb{R}^3$ with coordinates x, y, z. Introduce new coordinates u, v, w by setting

$$x = e^u v, \quad y = e^v, \quad z = uv^2 w$$

valid on the region where $x \geq y > 1$.

a) Express u, v, w in terms of x, y, z.
b) Express the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$ as a combination of the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ with coefficients functions of x, y, z.

6.2 Lie brackets

Let M be a manifold. Given vector fields $X, Y : C^\infty(M) \to C^\infty(M)$, the composition $X \circ Y$ is not a vector field: For example, if $X = Y = \frac{\partial}{\partial x}$ as vector fields on $\mathbb{R}$, then $X \circ Y = \frac{\partial^2}{\partial x^2}$ is a second order derivative, which is not a vector field (it does not satisfy the Leibnitz rule). However, the commutator turns out to be a vector field:

Theorem 6.5. *For any two vector fields $X, Y \in \mathfrak{X}(M)$ (regarded as derivations, as in Definition 6.2), the commutator*

$$[X, Y] := X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

is again a vector field.

Proof. We need to show that $[X, Y]$ is a derivation of the algebra $C^\infty(M)$. Let $f, g \in C^\infty(M)$. Then

$$\begin{aligned}(X \circ Y)(fg) &= X\big(Y(f)g + fY(g)\big) \\ &= (X \circ Y)(f)\,g + f\,(X \circ Y)(g) + Y(f)X(g) + X(f)Y(g).\end{aligned}$$

Similarly,

$$(Y \circ X)(fg) = (Y \circ X)(f)\,g + f\,(Y \circ X)(g) + X(f)Y(g) + Y(f)X(g).$$

Subtracting the latter equation from the former, several terms cancel, and we obtain

$$[X, Y](fg) = [X, Y](f)\,g + f\,[X, Y](g)$$

as desired.

Remark 6.6. A similar calculation applies to derivations of algebras in general: The commutator of two derivations is again a derivations.

Definition 6.7. *The vector field*

$$[X, Y] := X \circ Y - Y \circ X$$

is called the Lie bracket *of $X, Y \in \mathfrak{X}(M)$. If $[X, Y] = 0$ we say that the vector fields* X, Y commute.

The Lie brackets are named after Sophus Lie (1842-1899); the space of vector fields with its Lie bracket is a special case of a *Lie algebra*. (The $[\cdot,\cdot]$ is a common notation for the commutator in an algebra, see Appendix A.3.3.)

It is instructive to see how the Lie bracket of vector fields works out in local coordinates. For open subsets $U \subseteq \mathbb{R}^m$, if

$$X = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^{m} b^i \frac{\partial}{\partial x^i},$$

with coefficient functions $a^i, b^i \in C^\infty(U)$, the composition $X \circ Y$ is a second order differential operator, calculated by the product rule:

$$X \circ Y = \sum_{i=1}^{m}\sum_{j=1}^{m} a^j \frac{\partial b^i}{\partial x^j}\frac{\partial}{\partial x^i} + \sum_{i=1}^{m}\sum_{j=1}^{m} a^i b^j \frac{\partial^2}{\partial x^i \partial x^j}$$

(an equality of operators acting on $C^\infty(U)$). Subtracting a similar expression for $Y \circ X$, the terms involving second derivatives cancel, and we obtain

$$[X,Y] = \sum_{i=1}^{m}\sum_{j=1}^{m} \Big(a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \Big) \frac{\partial}{\partial x^i}.$$

This calculation applies to general manifolds, by taking local coordinates.

Note: When calculating Lie brackets $X \circ Y - Y \circ X$ of vector fields X,Y in local coordinates, it is not necessary to work out the second order derivatives – we know in advance that these are going to cancel out!

The following fact is often used in calculations.

81 (answer on page ??). Let $X,Y \in \mathfrak{X}(M)$, and $f,g \in C^\infty(M)$. Show that

$$[X, fY] = (Xf)Y + f\,[X,Y].$$

Derive a similar formula for $[fX, gY]$.

The geometric significance of the Lie bracket will become clear later. At this stage, let us just note that since the Lie bracket of two vector fields is defined in a coordinate-free way, its vanishing or non-vanishing does not depend on choices of coordinates. For example, since the Lie brackets between coordinate vector fields vanish,

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0, \;\; 0 \le i < j \le m,$$

two non-zero vector fields X,Y can only be coordinate vector fields in suitable coordinates if their Lie bracket is zero, i.e. if X,Y commute.

Example 6.8. Consider the vector fields on $\mathbb{R}^2$ given by

$$X = \frac{\partial}{\partial x}, \;\; Y = (1+x^2)\frac{\partial}{\partial y}.$$

Does there exists a change of coordinates $(u,v) = \varphi(x,y)$ (at least locally, near any given point) such that in the new coordinates, these vector fields are the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$? The answer is no: Since

$$[X,Y] = 2x\frac{\partial}{\partial y},$$

is non-vanishing in x,y coordinates, there cannot be a change of coordinates to make it vanish in u,v coordinates.

Example 6.9. Consider the same problem for the vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \;\; Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

This time, we may verify that X,Y commute: $[X,Y] = 0$. Can we make a coordinate change so that X,Y become the coordinate vector fields? Note that we will have to remove the origin $p = \mathbf{0}$, since X,Y vanish there. Near points of $\mathbb{R}^2 \backslash \{\mathbf{0}\}$, it is convenient to introduce polar coordinates

$$x = r\cos\theta, \;\; y = r\sin\theta$$

(with $r > 0$ and θ varying in an open interval of length at most 2π). We have

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{1}{r}Y, \quad \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = X$$

Hence

$$X = \frac{\partial}{\partial \theta}, \; Y = r\frac{\partial}{\partial r}.$$

To get this into the desired form, we make another change of coordinates $\rho = \rho(r)$ in such a way that Y becomes $\frac{\partial}{\partial \rho}$. Since

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = \rho'(r)\frac{\partial}{\partial \rho}$$

we want $\rho'(r) = \frac{1}{r}$, thus $\rho = \ln(r)$, or $r = e^\rho$. Hence, the desired change of coordinates is

$$x = e^\rho \cos\theta, \;\; y = e^\rho \sin\theta.$$

82 (answer on page ??). Consider the following two vector fields on $\mathbb{R}^2$, on the open subset where $xy > 0$,

$$X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad Y = 2\sqrt{xy}\frac{\partial}{\partial x}$$

a) Compute their Lie bracket $[X,Y]$.
b) Can you find coordinates u,v in which X,Y becomes the coordinate vector fields?

Definition 6.10. *Let $S \subseteq M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is called* tangent to S *if for all $p \in S$, the tangent vector X_p lies in $T_pS \subseteq T_pM$. (Thus X restricts to a vector field $X|_S \in \mathfrak{X}(S)$.)*

Proposition 6.11. *If two vector fields $X,Y \in \mathfrak{X}(M)$ are tangent to a submanifold $S \subseteq M$, then their Lie bracket $[X,Y]$ is again tangent to S.*

Proposition 6.11 can be proved by using the coordinate expressions of X,Y in submanifold charts. But we will postpone the proof for now since there is a much shorter, coordinate-independent proof (see 87 in the next section).

83 (answer on page ??).
a) Show that the three vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

on $\mathbb{R}^3$ are tangent to the 2-sphere S^2.
b) Show that the brackets $[X,Y]$, $[Y,Z]$, $[Z,X]$ are again tangent to the 2-sphere.

As mentioned above, vector fields with the Lie bracket are a special case of *Lie algebras*. More generally, a Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto [X,Y]$ which is skew-symmetric (i.e., $[X,Y] = -[Y,X]$) and satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \tag{6.3}$$

for all $X,Y,Z \in \mathfrak{g}$. Spaces $\mathfrak{X}(M)$ of vector fields on manifolds provide one class of examples, the matrix Lie algebras $\mathfrak{g} \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ are another class.

84 (answer on page ??). Verify that the Lie bracket on vector fields on a manifold M satisfies the Jacobi identity, thus $\mathfrak{g} = \mathfrak{X}(M)$ is a Lie algebra. If you like it more abstract, verify more generally that the space of derivations of any algebra $\mathscr{A}$ is a Lie algebra $\mathfrak{g} = \mathrm{Der}(\mathscr{A})$.

6.3 Related vector fields

Given a smooth map $F \in C^\infty(M,N)$, the tangent map may be used to map individual tangent vectors at points of M to tangent vectors at points of N. However, this point-wise 'push-forward' operation of tangent vectors does not give rise to a push-forward operation of vector *fields* X on M to vector fields Y on N, unless F is a diffeomorphism.

85 (answer on page ??). Let $F \in C^\infty(M,N)$ and $X \in \mathfrak{X}(M)$. We would like to define a vector field $F_*X \in \mathfrak{X}(N)$ such that

$$(F_*X)_{F(p)} = T_pF(X_p)$$

for all $p \in M$. What's the problem with this 'definition' when
a) F is not surjective?
b) F is not injective?
c) F is bijective, but not a diffeomorphism?

The following may be seen as a 'workaround', which turns out to be extremely useful.

Definition 6.12. *Let $F \in C^\infty(M,N)$ be a smooth map. Vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called F-related, written as*

$$X \sim_F Y,$$

if $T_pF(X_p) = Y_{F(p)}$ for all $p \in M$.

As an immediate consequence of the definition, under composition of smooth maps between manifolds,

$$X \sim_F Y,\;\; Y \sim_{F'} Z \;\;\Rightarrow\;\; X \sim_{F'\circ F} Z. \tag{6.4}$$

Example 6.13. Suppose $F:\ M \to N$ is a submersion, and $X \in \mathfrak{X}(M)$. Then $X \sim_F 0$ if and only if $T_pF(X_p) = 0$ for all $p \in M$. That is, $X_p \in \ker(T_pF)$. Since F is a submersion, $\ker(T_pF)$ is just the tangent space to the fiber $S = F^{-1}(q)$, where $q = F(p)$. (See Proposition 5.18.) We conclude that

$$X \sim_F 0 \;\Leftrightarrow\; X \text{ is tangent to the fibers of } F.$$

More generally, $X \sim_F Y$ is the statement that X is a *lift* of the vector field Y; if the submersion F is surjective it is justified to write this as $Y = F_*X$ since Y is uniquely determined by X.

Example 6.14. Let $\pi:\ S^n \to \mathbb{R}\mathrm{P}^n$ be the usual quotient map for the projective space, and $X \in \mathfrak{X}(S^n)$. Then $X \sim_\pi Y$ for some $Y \in \mathfrak{X}(\mathbb{R}\mathrm{P}^n)$ if and only if X is invariant under the transformation $F:\ S^n \to S^n,\ x \mapsto -x$ (that is, $TF \circ X = X \circ F$), and with Y the induced vector field on the quotient.

The following 86 examines the notion of related vector fields in the case of embeddings:

86 (answer on page ??). Suppose $S \subseteq M$ is an embedded submanifold, and $i: S \hookrightarrow M$ the inclusion map. For vector fields $X \in \mathfrak{X}(S)$ and $Y \in \mathfrak{X}(M)$, show:

$$X \sim_i Y \Leftrightarrow Y \text{ is tangent to } S\text{, with } X \text{ as its restriction}$$
$$0 \sim_i Y \Leftrightarrow Y \text{ vanishes along the submanifold } S.$$

The F-relation of vector fields has a simple interpretation in terms of the 'vector fields as derivations'

Proposition 6.15. *One has $X \sim_F Y$ if and only if for all $g \in C^\infty(N)$,*

$$X(g \circ F) = Y(g) \circ F.$$

Proof. The condition $X(g \circ F) = Y(g) \circ F$ says that

$$(T_pF(X_p))(g) = Y_{F(p)}(g)$$

for all $p \in M$. □

In terms of the pull-back notation, with $F^*g = g \circ F$ for $g \in C^\infty(N)$, the proposition amounts to $X \circ F^* = F^* \circ Y$, which can be depicted as the commutative diagram:

$$\begin{tikzcd} C^\infty(M) \arrow[r, "X"'] & C^\infty(M) \\ C^\infty(N) \arrow[u, "F^*"'] \arrow[r, "Y"'] & C^\infty(N) \arrow[u, "F^*"'] \end{tikzcd} \tag{6.5}$$

The key fact concerning related vector fields is the following.

Theorem 6.16. *Let $F \in C^\infty(M,N)$, and let vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be given. Then*

$$X_1 \sim_F Y_1, \quad X_2 \sim_F Y_2 \Rightarrow \quad [X_1, X_2] \sim_F [Y_1, Y_2].$$

Proof. Let $g \in C^\infty(N)$ be arbitrary. Then, since $X_1 \sim_F Y_1$ and $X_2 \sim_F Y_2$, we have

$$\begin{aligned}
[X_1, X_2](g \circ F) &= X_1(X_2(g \circ F)) - X_2(X_1(g \circ F)) \\
&= X_1(Y_2(g) \circ F) - X_2(Y_1(g) \circ F) \\
&= Y_1(Y_2(g)) \circ F - Y_2(Y_1(g)) \circ F \\
&= [Y_1, Y_2](g) \circ F. \square
\end{aligned}$$

87 (answer on page ??). Prove Proposition 6.11 from page 105: If two vector fields Y_1, Y_2 are tangent to a submanifold $S \subseteq M$ then their Lie bracket $[Y_1, Y_2]$ is again tangent to S, and the Lie bracket of their restriction is the restriction of the Lie brackets.

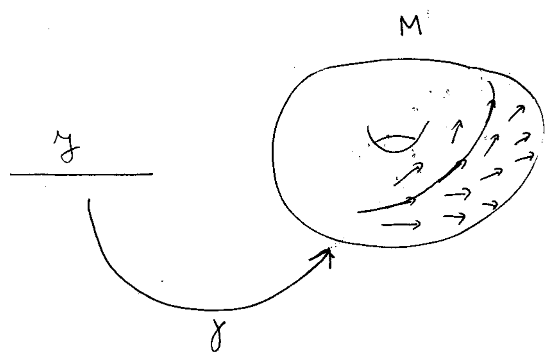
6.4 Flows of vector fields

6.4.1 Solution curves

Let $\gamma: J \to M$ be a curve, with $J \subseteq \mathbb{R}$ an open interval. In (5.6) we defined the velocity vector at time $t \in J$
$$\dot{\gamma}(t) \in T_{\gamma(t)}M$$
in terms of its action on functions as
$$\big(\dot{\gamma}(t)\big)(f) = \frac{d}{dt} f\big(\gamma(t)\big).$$
The curve representing this tangent vector for a given t, in the sense of Definition 5.2, is the shifted curve $\tau \mapsto \gamma(t+\tau)$. Equivalently, one may think of the velocity vector as the image of the coordinate vector $\frac{\partial}{\partial t}\Big|_t \in T_t J \cong \mathbb{R}$ under the tangent map $T_t\gamma$:
$$\dot{\gamma}(t) = (T_t\gamma)\Big(\frac{\partial}{\partial t}\Big|_t \Big).$$

Definition 6.17. *Suppose $X \in \mathfrak{X}(M)$ is a vector field on a manifold M. A smooth curve $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an open interval, is called a* solution curve *to X if*
$$\dot{\gamma}(t) = X_{\gamma(t)} \tag{6.6}$$
for all $t \in J$.

Geometrically, Equation (6.6) means that the solution curve γ is at all times t tangent to the given vector field, with a 'speed' as prescribed by the vector field.



We can also restate the definition of solution curves in terms of related vector fields:
$$\frac{\partial}{\partial t} \sim_\gamma X. \tag{6.7}$$
This last characterization has as a direct consequence:

Proposition 6.18. *Suppose $F \in C^\infty(M,N)$ is a smooth map of manifolds, and let $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ be vector fields with $X \sim_F Y$. If $\gamma\colon J \to M$ is a solution curve for X, then $F \circ \gamma\colon J \to N$ is a solution curve for Y.*

Proof. By (6.4),

$$\frac{\partial}{\partial t} \sim_\gamma X,\;\; X \sim_F Y \;\;\Rightarrow\;\; \frac{\partial}{\partial t} \sim_{F\circ\gamma} Y.$$

Alternatively, the claim follows from Proposition 5.12. □

Given a vector field $X \in \mathfrak{X}(M)$ and a point $p \in M$, one may ask about the existence of a solution curve $\gamma\colon J \to M$, for some interval J around 0, with the given initial conditions $\gamma(0) = p$, $\dot\gamma(0) = X_p$. Furthermore, one may ask whether such a solution is unique, i.e. whether any two solutions agree on their common domain of definition. We shall discuss this problem first for open subsets of Euclidean spaces.

6.4.2 Existence and uniqueness for open subsets of $\mathbb{R}^m$

Consider first the case that $M = U \subseteq \mathbb{R}^m$. Here curves $\gamma(t)$ are of the form

$$\gamma(t) = \mathbf{x}(t) = (x^1(t), \ldots, x^m(t)),$$

hence

$$\dot\gamma(t)(f) = \frac{d}{dt} f(\mathbf{x}(t)) = \sum_{i=1}^m \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\partial f}{\partial x^i}(\mathbf{x}(t)).$$

That is

$$\dot\gamma(t) = \sum_{i=1}^m \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\partial}{\partial x^i}\Big|_{\mathbf{x}(t)}.$$

On the other hand, vector fields have the form $X = \sum_{i=1}^m a^i(\mathbf{x}) \frac{\partial}{\partial x^i}$. Hence (6.6) becomes the system of first order ordinary differential equations,

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = a^i(\mathbf{x}(t)), \quad i = 1, \ldots, m; \tag{6.8}$$

the initial condition $\gamma(0) = p$ takes the form

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{6.9}$$

where $\mathbf{x}_0 = (x_0^1, \ldots, x_0^m) \in U$ is the coordinate vector for p.

Example 6.19. Consider the case of a *constant* vector field $X = \sum a^i \frac{\partial}{\partial x^i}$, with $\mathbf{a} = (a^1, \ldots, a^m) \in \mathbb{R}^m$ regarded as a constant function of $\mathbf{x}$. Then the solutions of (6.8) with initial condition $\mathbf{x}(0) = \mathbf{x}_0 = (x_0^1, \ldots, x_0^m)$ are given by $x^i(t) = x_0^i + a^i t$ for $i = 1, \ldots, m$. That is, the solutions curves are affine lines,

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{a}.$$

As a special case, the solution curves for the coordinate vector field $\frac{\partial}{\partial x^j}$, with initial condition $\mathbf{x}(0) = \mathbf{x}_0 = (x_0^1, \ldots, x_0^m)$ are

$$x^i(t) = \begin{cases} x_0^i & i \neq j, \\ x_0^j + t & i = j \end{cases}.$$

Example 6.20. Consider the following vector field on $\mathbb{R}^m$,

$$X = \sum_{i=1}^m x^i \frac{\partial}{\partial x^i}.$$

Here $a^i(x) = x^i$, hence (6.8) reads as $\frac{dx^i}{dt} = x^i$, with solutions $x^i(t) = c_i e^t$, for arbitrary constants c_i. Such a solution satisfies the initial condition (6.9) if and only if $c^i = x_0^i$; hence the we obtain

$$\mathbf{x}(t) = e^t\, \mathbf{x}_0$$

as the solution of the initial value problem.

88 (answer on page ??). Consider the vector field on $\mathbb{R}^2$,

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Find the solution curve for any given initial condition $(x_0, y_0) \in \mathbb{R}^2$. Draw a picture of the vector field.

Explicit solutions of the initial value problem (6.8), (6.9) can only be found for certain classes of such equations, studied in the theory of ODE's. Even without finding explicit solutions, ODE theory gives a wealth of information on qualitative behavior of solutions. The first general result says that a solution to the initial value problem always exists, and moreover that there is a unique maximal solution:

Theorem 6.21 (Existence and uniqueness theorem for ODEs). *Let $U \subseteq \mathbb{R}^m$ be an open subset, and $\mathbf{a} \in C^\infty(U, \mathbb{R}^m)$. For any given $\mathbf{x}_0 \in U$, there is an open interval $J_{\mathbf{x}_0} \subseteq \mathbb{R}$ around 0, and a solution $\mathbf{x} : J_{\mathbf{x}_0} \to U$ of the system of ODEs*

$$\frac{dx^i}{dt} = a^i(\mathbf{x}(t)), \quad i = 1, \ldots, m$$

with initial condition $\mathbf{x}(0) = \mathbf{x}_0$, and which is maximal in the sense that any other solution to this initial value problem is obtained by restriction to some subinterval of $J_{\mathbf{x}_0}$.

Thus, $J_{\mathbf{x}_0}$ is the maximal open interval on which the solution is defined. The solution depends smoothly on initial conditions, in the following sense. For any given $\mathbf{x}_0$, let $\Phi(t, \mathbf{x}_0)$ be the solution $\mathbf{x}(t)$ of the initial value problem with initial condition $\mathbf{x}_0$.

Since we are interested in $\mathbf{x}_0$ as a 'variable' rather than a 'constant', we will write $\mathbf{x}$ in place of $\mathbf{x}_0$; thus $t \mapsto \Phi(t,\mathbf{x})$ is the solution curve that was earlier denoted $\mathbf{x}(t)$. Then, $\Phi(t,\mathbf{x})$ is a function defined for $t \in J_{\mathbf{x}}$, which is smooth as a function of t, and is characterized by the equations

$$\frac{d}{dt}\Phi(t,\mathbf{x}) = \mathbf{a}\big(\Phi(t,\mathbf{x})\big), \quad \Phi(0,\mathbf{x}) = \mathbf{x}. \tag{6.10}$$

Theorem 6.22 (Dependence on initial conditions for ODEs). *For $\mathbf{a} \in C^\infty(U,\mathbb{R}^m)$ as above, the set*

$$\mathscr{J} = \{(t,\mathbf{x}) \in \mathbb{R} \times U \,|\, t \in J_{\mathbf{x}}\}.$$

is an open neighborhood of $\{0\} \times U$ in $\mathbb{R} \times U$, and the map

$$\Phi: \ \mathscr{J} \to U, \ (t,\mathbf{x}) \mapsto \Phi(t,\mathbf{x})$$

is smooth.

In general, the interval $J_{\mathbf{x}}$ for given $\mathbf{x}$ may be strictly smaller than $\mathbb{R}$, because a solution might "escape to infinity in finite time," as illustrated in the following example. (This language regards t as a time-parameter for the solution curve $\mathbf{x}(t)$.)

Example 6.23. Consider the ODE in one variable,

$$\frac{dx}{dt} = x^2.$$

The initial value problem $x(t_0) = x_0$ for this ODE is solved by the method of *separation of variables*: One formally writes $\mathrm{d}t = x^{-2}\mathrm{d}x$, and then integrates both sides to obtain $t - t_0 = \int_{x_0}^{x} u^{-2}\mathrm{d}u = -x^{-1} + x_0^{-1}$. In our case, $t_0 = 0$. Solving for x, we obtain

$$x(t) = \frac{x_0}{1 - t x_0}.$$

(This solution is also correct for $x_0 = 0$, even though the calculation did not apply to this case.) Note that the solution is only defined for $1 - t x_0 \neq 0$, and since we start at $t_0 = 0$ we must have $1 - t x_0 > 0$. Hence, the domain of definition of the solution curve $x(t)$ with initial condition x_0 is $J_{x_0} = \{t \in \mathbb{R} |\, t x_0 < 1\}$. We read off that

$$\Phi(t,x) = \frac{x}{1 - tx}.$$

with domain of definition $\mathscr{J} = \{(t,x) |\, tx < 1\}$.

89 (answer on page ??). For each of the following ODEs: find the solution curves with initial condition $x(t) = x_0 \in U$; find J_{x_0}, $\mathscr{J}$, and $\Phi(t,x)$.

a) $\dfrac{dx}{dt} = 1$ on $U = (0,1) \subseteq \mathbb{R}$.

b) $\dfrac{dx}{dt} = 1 + x^2$ on $U = \mathbb{R}$.

6.4.3 Existence and uniqueness for vector fields on manifolds

For general vector fields $X \in \mathfrak{X}(M)$ on manifolds, Equation (6.6) becomes (6.8) after introducing local coordinates. In detail: Let (U, φ) be a coordinate chart. In the chart, X becomes the vector field

$$\varphi_*(X) = \sum_{i=1}^{m} a^j(\mathbf{x}) \frac{\partial}{\partial x^i}$$

and $\varphi(\gamma(t)) = \mathbf{x}(t)$ with

$$\frac{dx^i}{dt} = a^i(\mathbf{x}(t)).$$

If $\mathbf{a} = (a^1, \ldots, a^m): \ \varphi(U) \to \mathbb{R}^m$ corresponds to X in a local chart (U, φ), then any solution curve $\mathbf{x}: \ J \to \varphi(U)$ for $\mathbf{a}$ defines a solution curve $\gamma(t) = \varphi^{-1}(\mathbf{x}(t))$ for X. The existence and uniqueness theorem for ODEs extends to manifolds, as follows:

Theorem 6.24 (Solutions of vector fields on manifolds). *Let $X \in \mathfrak{X}(M)$ be a vector field on a manifold M. For any given $p \in M$, there is an open interval $J_p \subseteq \mathbb{R}$ around 0, and a unique solution $\gamma: \ J_p \to M$ of the initial value problem*

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \gamma(0) = p, \tag{6.11}$$

which is maximal in the sense that any other solution is obtained by restriction to a subinterval. The set

$$\mathscr{J} = \{(t, p) \in \mathbb{R} \times M |\ t \in J_p\}$$

is an open neighborhood of $\{0\} \times M$, and the map

$$\Phi: \ \mathscr{J} \to M, \ (t, p) \mapsto \Phi(t, p)$$

such that $\gamma(t) = \Phi(t, p)$ solves the initial value problem (6.11), is smooth.

Proof. Existence and uniqueness of solutions for small times t follows from the existence and uniqueness theorem for ODEs, by considering the vector field in local charts. To prove uniqueness even for large times t, let $\gamma: \ J \to M$ be a maximal solution of (6.11) (i.e., a solution that cannot be extended to a larger open interval), and let $\gamma_1: \ J_1 \to M$ be another solution of the same initial value problem. Suppose that $\gamma_1(t) \neq \gamma(t)$ for some $t \in J \cap J_1, \ t > 0$.
Then we can define

$$b = \inf\{t \in J \cap J_1 |\ t > 0, \ \gamma_1(t) \neq \gamma(t)\}.$$

By the uniqueness for small t, we have $b > 0$. We will get a contradiction in each of the following cases:
Case 1: $\gamma_1(b) = \gamma(b) =: q$. Then both $\lambda_1(s) = \gamma_1(b+s)$ and $\lambda(s) = \gamma(b+s)$ are solutions to the initial value problem

$$\lambda(0) = q, \quad \dot{\lambda}(s) = X_{\lambda(s)};$$

hence they have to agree for small $|s|$, and consequently $\gamma_1(t), \gamma(t)$ have to agree for t close to b. This contradicts the definition of b.
Case 2: $\gamma_1(b) \neq \gamma(b)$. Using the Hausdorff property of M, we can choose disjoint open neighborhoods U of $\gamma(b)$ and U_1 of $\gamma(b_1)$. For $t = b - \varepsilon$ with $\varepsilon > 0$ sufficiently small, $\gamma(t) \in U$ while $\gamma_1(t) \in U_1$. But this is impossible since $\gamma(t) = \gamma_1(t)$ for $0 \leq t < b$.
These contradictions show that $\gamma_1(t) = \gamma(t)$ for $t \in J_1 \cap J, t > 0$. Similarly $\gamma_1(t) = \gamma(t)$ for $t \in J_1 \cap J, t < 0$. Hence $\gamma_1(t) = \gamma(t)$ for all $t \in J_1 \cap J$. Since γ is a maximal solution, it follows that $J_1 \subseteq J$, with $\gamma_1 = \gamma|_{J_1}$.
The result for ODEs about the smooth dependence on initial conditions shows, by working in local coordinate charts, that $\mathscr{J}$ contains an open neighborhood of $\{0\} \times M$, on which Φ is given by a smooth map. The fact that $\mathscr{J}$ itself is open, and the map Φ is smooth everywhere, follows by the 'flow property' to be discussed below. (We omit the details of this part of the proof.) □

Note that the uniqueness part uses the Hausdorff property from the definition of manifolds. Indeed, the uniqueness part may fail for non-Hausdorff manifolds.

Example 6.25. An example is the non-Hausdorff manifold from Example 2.15, $M = \widetilde{M}/\sim$ where

$$\widetilde{M} = (\mathbb{R} \times \{1\}) \cup (\mathbb{R} \times \{-1\})$$

is a disjoint union of two copies of the real line (thought of as embedded in $\mathbb{R}^2$), and where $\sim$ glues the two copies along the strictly negative real axis. Let $\pi : \widetilde{M} \to M$ be the quotient map. The vector field $\widetilde{X}$ on $\widetilde{M}$ given as $\frac{\partial}{\partial x}$ on both copies descends to a vector field X on M, i.e.,

$$\widetilde{X} \sim_\pi X.$$

The solution curves of the initial value problem for X, with $\gamma(0) = p$ on the negative real axis, are not unique: The solution curves move from the left to the right, but may continue on the 'upper branch' or on the 'lower branch' of M. Concretely, the curves $\widetilde{\gamma}_+, \widetilde{\gamma}_- : \mathbb{R} \to \widetilde{M}$, given as

$$\widetilde{\gamma}_\pm(t) = (t - 1, \pm 1),$$

are each a solution curve of $\widetilde{X}$, hence their images under π are solution curves $\gamma_\pm$ of X. (See Proposition 6.18.) These have the same initial condition, and coincide for $t < 1$ but are different for $t \geq 1$.

6.4.4 Flows

Given a vector field X, the map $\Phi : \mathscr{J} \to M$ is called the *flow* of X. For any given p, the curve $\gamma(t) = \Phi(t, p)$ is a solution curve. But one can also fix t and consider the time-t flow,

$$\Phi_t(p) := \Phi(t, p)$$

as a function of p. It is a smooth map $\Phi_t : U_t \to M$, defined on the open subset

$$U_t = \{p \in M |\ (t, p) \in \mathscr{J}\}.$$

Note that $\Phi_0 = \mathrm{id}_M$.

Example 6.26. In Example 6.23 we computed the set $\mathscr{J}$ and the flow for the vector field $X = x^2 \frac{\partial}{\partial x}$ on the real line $\mathbb{R}$. We have found that $\mathscr{J}$ is described by the equation $tx < 1$. Hence, the domain of definition of $\Phi_t(x) = x/(1-tx)$ is

$$U_t = \{x \in \mathbb{R} |\ tx < 1\}.$$

Intuitively, $\Phi_t(p)$ is obtained from the initial point $p \in M$ by flowing for time t along the vector field X. One expects that first flowing for time s, and then flowing for time t, should be the same as flowing for time $t+s$. Indeed one has the following *flow property*.

Theorem 6.27 (Flow property). *Let $X \in \mathfrak{X}(M)$, with flow $\Phi : \mathscr{J} \to M$. Let $(s,p) \in \mathscr{J}$, and $t \in \mathbb{R}$. Then*

$$(t, \Phi_s(p)) \in \mathscr{J} \Leftrightarrow (t+s, p) \in \mathscr{J},$$

and in this case

$$\Phi_t(\Phi_s(p)) = \Phi_{t+s}(p).$$

Proof. We claim that, for fixed s, both

$$t \mapsto \Phi_t(\Phi_s(p)), \quad t \mapsto \Phi_{t+s}(p)$$

are maximal solution curves of X, for the same initial condition $q = \Phi_s(p)$. This is clear for the first curve, and follows for the second curve by the calculation, for $f \in C^\infty(M)$,

$$\frac{d}{dt} f\big(\Phi_{t+s}(p)\big) = \frac{d}{d\tau}\Big|_{\tau=t+s} f\big(\Phi_\tau(p)\big) = X_{\Phi_\tau(p)}(f)\Big|_{\tau=t+s} = X_{\Phi_{t+s}(p)}(f).$$

Hence, by the uniqueness part of Theorem 6.24 the two curves must coincide. The domain of definition of $t \mapsto \Phi_{t+s}(p)$ is the interval $J_p \subseteq \mathbb{R}$, shifted by s. That is, $t \in J_{\Phi(s,p)}$ if and only if $t+s \in J_p$. $\square$

Corollary 6.28. *For all $t \in \mathbb{R}$, the map Φ_t is a diffeomorphism from its domain U_t onto its image $\Phi_t(U_t)$.*

Proof. Let $p,q \in M$ with $\Phi_t(p) = q$. Thus, $(t,p) \in \mathscr{J}$. Since we always have $(0,p) \in \mathscr{J}$, the theorem shows $(-t,q) \in \mathscr{J}$; furthermore, $\Phi_{-t}(q) = \Phi_{-t}(\Phi_t(p)) = \Phi_0(p) = p$. We conclude that Φ_t takes values in U_{-t}, and Φ_{-t} is an inverse map.

Example 6.29. Let us illustrate the flow property for various vector fields on $\mathbb{R}$.

a) The flow property is evident for $\frac{\partial}{\partial x}$ with flow $\Phi_t(x) = x + t$, defined for all t.
b) The vector field $x\frac{\partial}{\partial x}$ has flow $\Phi_t(x) = e^t x$, defined for all t. The flow property holds:
$$\Phi_t(\Phi_s(x)) = e^t \Phi_s(x) = e^t e^s x = e^{t+s} x = \Phi_{t+s}(x).$$
c) The vector field $x^2 \frac{\partial}{\partial x}$ has flow $\Phi_t(x) = x/(1-tx)$, defined for $1 - tx > 0$. We can explicitly verify the flow property:
$$\Phi_t(\Phi_s(x)) = \frac{\Phi_s(x)}{1 - t\Phi_s(x)} = \frac{\frac{x}{1-sx}}{1 - t\frac{x}{1-sx}} = \frac{x}{1-(t+s)x} = \Phi_{t+s}(x).$$

6.4.5 Complete vector fields

Let X be a vector field, and $\mathscr{J} = \mathscr{J}^X \subseteq \mathbb{R} \times M$ be the domain of definition for the flow $\Phi = \Phi^X$.

Definition 6.30. *A vector field $X \in \mathfrak{X}(M)$ is called complete if $\mathscr{J}^X = \mathbb{R} \times M$.*

Thus X is complete if and only if all solution curves exist for all time.

Example 6.31. The vector field $x\frac{\partial}{\partial x}$ on $M = \mathbb{R}$ is complete, but $x^2\frac{\partial}{\partial x}$ is incomplete.

A vector field may fail to be complete if a solution curve escapes to infinity in finite time. This suggests that a vector fields X that vanishes outside a compact set must be complete, because the solution curves are 'trapped' and cannot escape to infinity. Similarly to the definition of support of a function (see Definition 3.3), we define the support of a vector field X to be the smallest closed subset

$$\operatorname{supp}(X) \subseteq M$$

with the property that $X_p = 0$ for $p \notin \operatorname{supp}(X)$. We say that X has compact support if $\operatorname{supp}(X)$ is compact.

Proposition 6.32. *Every vector field with compact support is complete. In particular, every vector field on a compact manifold is complete.*

Proof. Note that if X vanishes at some point $p \in M$, then the unique solution curve through p is the constant curve $\gamma(t) = p$, defined for all $t \in \mathbb{R}$. Thus, if a solution curve lies in $M \setminus \operatorname{supp}(X)$ for *some* t, then it must be constant, and in particular lie in $M \setminus \operatorname{supp}(X)$ for *all* t in its domain. Hence, if a solution curve lies in $\operatorname{supp}(X)$ for some t, then it must lie in $\operatorname{supp}(X)$ for all t in its domain of definition.
Let $U_\varepsilon \subseteq M$ be the set of all p such that the solution curve γ with initial condition $\gamma(0) = p$ exists for $|t| < \varepsilon$ (that is, $(-\varepsilon, \varepsilon) \subseteq J_p$). By smooth dependence on initial conditions (see Theorem 6.24), U_ε is open. The collection of all U_ε with $\varepsilon > 0$ covers $\operatorname{supp}(X)$, since every solution curve exists for sufficiently small time. Since $\operatorname{supp}(X)$ is compact, there exists a finite subcover $U_{\varepsilon_1}, \ldots, U_{\varepsilon_k}$. Let ε be the smallest of $\varepsilon_1, \ldots, \varepsilon_k$. Then $U_{\varepsilon_i} \subseteq U_\varepsilon$, for all i, and hence $\operatorname{supp}(X) \subseteq U_\varepsilon$. Hence, for any $p \in \operatorname{supp}(X)$ we have

$$(-\varepsilon, \varepsilon) \subseteq J_p,$$

that is, any solution curve $\gamma(t)$ starting in $\operatorname{supp}(X)$ exists for times $|t| < \varepsilon$. For solution curves starting in $M \setminus \operatorname{supp}(X)$, this is true as well. By 90 below (applied to $\delta = \varepsilon/2$, say), we are done. $\square$

90 (answer on page ??). Let $X \in \mathfrak{X}(M)$ be a vector field, and suppose $\delta > 0$ is such that every solution curve exists at least for times t with $|t| \leq \delta$. Use the 'flow property' to argue that X is complete.

Theorem 6.33. *If X is a complete vector field, the flow Φ_t defines a 1-parameter group of diffeomorphisms. That is, each Φ_t is a diffeomorphism and*

$$\Phi_0 = \mathrm{id}_M, \quad \Phi_t \circ \Phi_s = \Phi_{t+s}.$$

Conversely, if Φ_t is a 1-parameter group of diffeomorphisms such that the map $(t,p) \mapsto \Phi_t(p)$ is smooth, the equation

$$X_p(f) = \frac{d}{dt}\Big|_{t=0} f(\Phi_t(p))$$

defines a complete vector field X on M, with flow Φ_t.

Proof. It remains to show the second statement. Given Φ_t, the linear map

$$C^\infty(M) \to C^\infty(M), \quad f \mapsto \frac{d}{dt}\Big|_{t=0} f(\Phi_t(p))$$

satisfies the product rule (6.2), hence it is a vector field X. Given $p \in M$ the curve $\Phi_t(p)$ is an integral curve of X since

$$\frac{d}{dt}\Phi_t(p) = \frac{d}{ds}\Big|_{s=0}\Phi_{t+s}(p) = \frac{d}{ds}\Big|_{s=0}\Phi_s(\Phi_t(p)) = X_{\Phi_t(p)}.$$

□

Remark 6.34. In terms of pull-backs, the relation between the vector field and its flow reads as

$$\frac{d}{dt}\Phi_t^*(f) = \Phi_t^* \frac{d}{ds}\Big|_{s=0}\Phi_s^*(f) = \Phi_t^* X(f).$$

(To make sense of the derivative, you should think of both sides as evaluated at a point of M.) This identity of linear operators on $C^\infty(M)$

$$\frac{d}{dt}\Phi_t^* = \Phi_t^* \circ X$$

may be viewed as the definition of the flow. (To make sense of the derivative, you should think of both sides as applied to a function, and evaluated at a point.)

Example 6.35. Given a square matrix $A \in \mathrm{Mat}_{\mathbb{R}}(m)$ let

$$\Phi_t : \mathbb{R}^m \to \mathbb{R}^m, \ \mathbf{x} \mapsto e^{tA}\mathbf{x} = \Big(\sum_{N=0}^{\infty} \frac{t^N}{N!} A^N\Big)\mathbf{x}$$

(using the exponential map of matrices). Since $e^{(t+s)A} = e^{tA}e^{sA}$, and since $(t,\mathbf{x}) \mapsto e^{tA}\mathbf{x}$ is a smooth map, Φ_t defines a flow. What is the corresponding vector field X? For any function $f \in C^\infty(\mathbb{R}^m)$ we calculate,

$$\begin{aligned} X(f)(\mathbf{x}) &= \frac{d}{dt}\Big|_{t=0} f(e^{tA}\mathbf{x}) \\ &= \sum_j \frac{\partial f}{\partial x^j}(A\mathbf{x})^j \\ &= \sum_{ij} A_i{}^j x^i \frac{\partial f}{\partial x^j}. \end{aligned}$$

This shows that

$$X = \sum_{ij} A_i{}^j x^i \frac{\partial}{\partial x^j}. \tag{6.12}$$

As a special case, taking A to be the identity matrix, we get the *Euler vector field* $X = \sum_i x^i \frac{\partial}{\partial x^i}$*, with its corresponding flow* $\Phi_t(\mathbf{x}) = e^t\mathbf{x}$ *(cf. Problem* **??** *at the end of this chapter).*

To conclude this section, we characterize related vector fields in terms of their flows:

Proposition 6.36. *Let* $F \in C^\infty(M,N)$*, and let* $X \in \mathfrak{X}(M)$*,* $Y \in \mathfrak{X}(N)$ *be complete vector fields, with flows* Φ_t^X*,* Φ_t^Y*. Then*

$$X \sim_F Y \quad \Leftrightarrow \quad F \circ \Phi_t^X = \Phi_t^Y \circ F \quad \textit{for all } t.$$

In short, vector fields are F*-related if and only if their flows are* F*-related.*

Proof. Suppose $F \circ \Phi_t^X = \Phi_t^Y \circ F$ for all t. For $g \in C^\infty(N)$, and $p \in M$, taking a t-derivative of

$$g(F(\Phi_t^X(p))) = g(\Phi_t^Y(F(p)))$$

at $t = 0$ on both sides, we get

$$\big(T_pF(X_p)\big)(g) = Y_{F(p)}(g)$$

i.e. $T_pF(X_p) = Y_{F(p)}$. Hence $X \sim_F Y$. Conversely, suppose $X \sim_F Y$. By Proposition 6.18, if $\gamma\colon J \to M$ is a solution curve for X, with initial condition $\gamma(0) = p$ then $F \circ \gamma\colon J \to M$ is a solution curve for Y, with initial condition $F(p)$. That is, $F(\Phi_t^X(p)) = \Phi_t^Y(F(p))$, or $F \circ \Phi_t^X = \Phi_t^Y \circ F$. $\square$

Remark 6.37. The proposition generalizes to possibly incomplete vector fields: The vector fields are related if and only if $F \circ \Phi^X = \Phi^Y \circ (\mathrm{id}_{\mathbb{R}} \times F)$.

Remark 6.38. Flows of incomplete vector fields X can be cumbersome to deal with, since one has to take into account the domain of definition of the flow, $\mathscr{J} \subseteq \mathbb{R} \times M$. However, if one is only interested in the short-time behavior of the flow, near a given point $p \in M$, one can make X complete by multiplying with a compactly supported function $\chi \in C^\infty(M)$ such that $\chi = 1$ on a neighborhood U of p. Indeed,

$$X' = \chi X$$

is compactly supported and therefore complete, and since $X'|_U = X|_U$ its integral curves with initial condition in U coincide with those of X for the time interval where they stay inside U.

6.5 Geometric interpretation of the Lie bracket

For any smooth map $F \in C^\infty(M,N)$ we defined the pull-back of smooth functions

$$F^*: C^\infty(N) \to C^\infty(M), \quad g \mapsto g \circ F.$$

If F is a *diffeomorphism*, then every function in $C^\infty(M)$ can be written as the pullback of a function in $C^\infty(N)$. Hence we can also pull back vector fields

$$F^*: \mathfrak{X}(N) \to \mathfrak{X}(M), \quad Y \mapsto F^*Y,$$

by requiring the the following diagram commutes (cf. diagram (6.5)):

$$\begin{tikzcd} C^\infty(M) \arrow[r, "F^*Y"'] & C^\infty(M) \\ C^\infty(N) \arrow[u, "F^*"'] \arrow[r, "Y"'] & C^\infty(N) \arrow[u, "F^*"'] \end{tikzcd}$$

i.e. by the condition $(F^*Y)(F^*g) = F^*(Y(g))$ for all functions $g \in C^\infty(N)$. That is, $F^*Y \sim_F Y$, or in more detail

$$(F^*Y)_p = (T_pF)^{-1}Y_{F(p)}$$

for all $p \in M$. By Theorem 6.16, we have $F^*[X,Y] = [F^*X, F^*Y]$.
Now, any complete vector field $X \in \mathfrak{X}(M)$ with flow Φ_t gives rise to pull-back maps

$$\Phi_t^*: C^\infty(M) \to C^\infty(M), \quad \Phi_t^*: \mathfrak{X}(M) \to \mathfrak{X}(M).$$

Definition 6.39. *The* Lie derivative of a function f *with respect to* X *is the function*

$$L_X(f) = \frac{d}{dt}\Big|_{t=0} \Phi_t^* f \in C^\infty(M). \tag{6.13}$$

The Lie derivative of a vector field Y *with respect to* X *is the vector field*

$$L_X(Y) = \frac{d}{dt}\Big|_{t=0} \Phi_t^* Y \in \mathfrak{X}(M). \tag{6.14}$$

Some explanations and comments:

a) In (6.13), the right hand side is interpreted in terms of evaluation at $p \in M$:

$$\Big(\frac{d}{dt}\Big|_{t=0} \Phi_t^* f\Big)(p) = \frac{d}{dt}\Big|_{t=0} \big((\Phi_t^* f)(p)\big) = \frac{d}{dt}\Big|_{t=0} f\big(\Phi_t(p)\big).$$

So, the Lie derivative of a function measures how the function changes along the flow of the vector field. Since $\gamma(t) = \Phi_t(p)$ is just the solution curve of X with initial condition $\gamma(0) = p$, we see that

$$L_X(f) = X(f).$$

b) Similarly, to interpret the right hand side of (6.14) one evaluates at $p \in M$ to obtain a family of tangent vectors in T_pM:

$$\Big(\frac{d}{dt}\Big|_{t=0} \Phi_t^* Y\Big)\Big|_p = \frac{d}{dt}\Big|_{t=0} (\Phi_t^* Y)_p.$$

Here

$$(\Phi_t^* Y)_p = (T_p \Phi_t^{-1})\, Y_{\Phi_t(p)};$$

that is, we use the inverse to the tangent map of the flow of X to move $Y_{\Phi_t(p)}$ to p. If Y were invariant under the flow of X, this would agree with Y_p; hence $(\Phi_t^* Y)_p - Y_p$ measures how Y fails to be Φ_t-invariant. $L_X(Y)$ is the infinitesimal version of this. That is, the Lie derivative measures infinitesimally how Y changes along the flow of X. As we will see below, the infinitesimal version actually implies the global version.

c) The definition of Lie derivative also works for incomplete vector fields, since it only involves derivatives at $t = 0$.

d) In (6.13) and (6.14), it was taken for granted that the right hand side does indeed define a smooth function, or respectively a vector field. For functions, this follows from $L_X f = X(f)$, for vector fields from Theorem 6.40 below.

e) From now on, we will usually drop the parentheses when writing Lie derivatives; e.g., $L_X Y$ in place of $L_X(Y)$.

Theorem 6.40. *For any $X, Y \in \mathfrak{X}(M)$, the Lie derivative $L_X Y$ is just the Lie bracket:*

$$L_X Y = [X, Y].$$

Proof. Let $\Phi_t = \Phi_t^X$ be the flow of X. For all $f \in C^\infty(M)$ we obtain, by taking the t-derivative at $t = 0$ of both sides of

$$\Phi_t^*(Y(f)) = (\Phi_t^* Y)(\Phi_t^* f)$$

that

$$\begin{aligned} X(Y(f)) &= \Big(\frac{d}{dt}\Big|_{t=0} \Phi_t^* Y\Big)(f) + Y\Big(\frac{d}{dt}\Big|_{t=0} \Phi_t^* f\Big) \\ &= (L_X Y)(f) + Y(X(f)). \end{aligned}$$

(Again, in this calculation you may take all the terms as evaluated at a point $p \in M$, so that the calculation really just involves derivatives of $\mathbb{R}$-valued functions of t.) That is, $L_X Y = X \circ Y - Y \circ X = [X, Y]$. $\square$

91 (answer on page ??). Justify the calculation above of the t-derivative at $t = 0$.

We see in particular that $L_X Y$ is skew-symmetric in X and Y – this was not obvious from the definition.
The result $[X,Y] = L_X Y$ gives an interpretation of the Lie bracket, as measuring infinitesimally how Y changes along the flow of X. The following result strengthens this interpretation of the Lie bracket.

Theorem 6.41. *Let X,Y be complete vector fields, with flows Φ_t, Ψ_s. Then*

$$\begin{aligned}[X,Y] = 0 &\Leftrightarrow \Phi_t^* Y = Y \quad \text{for all } t\\ &\Leftrightarrow \Psi_s^* X = X \quad \text{for all } s\\ &\Leftrightarrow \Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \quad \text{for all } s,t.\end{aligned}$$

Proof. The calculation

$$\frac{d}{dt}(\Phi_t)^* Y = (\Phi_t)^* L_X Y = (\Phi_t)^* [X,Y]$$

shows that $\Phi_t^* Y$ is independent of t if and only if $[X,Y] = 0$. Since $[Y,X] = -[X,Y]$, interchanging the roles of X,Y this is also equivalent to $\Psi_s^* X$ being independent of s. The property $\Phi_t^* Y = Y$ means that Y is Φ_t-related to itself, hence by Proposition 6.36 it takes the flow of Ψ_s to itself, that is

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t.$$

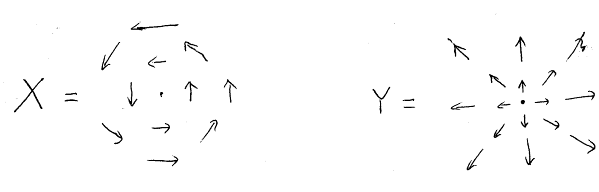
Conversely, if this equation holds then $\Phi_t^*(\Psi_s^* f) = \Psi_s^*(\Phi_t^* f)$ for all $f \in C^\infty(M)$. Differentiating with respect to s at $s = 0$, we obtain

$$\Phi_t^*(L_Y(f)) = L_Y(\Phi_t^* f).$$

Differentiating with respect to t at $t = 0$, we get $L_X(L_Y(f)) = L_Y(L_X(f))$, that is, $L_{[X,Y]} = [L_X, L_Y] = 0$ and therefore $[X,Y] = 0$. □

Example 6.42. Let $X = \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$. Then $[X,Y] = 0$ if and only if Y is invariant under translation in the y-direction.

Example 6.43. The vector fields $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ commute. This is verified by direct calculation but can also be 'seen' in the following picture



The flow of X is rotations around the origin, but Y is invariant under rotations. Likewise, the flow of Y is by dilations away from the origin, but X is invariant under dilations.

Similar, but less elegant, statements hold when X,Y are possibly incomplete. Let X be a vector field, with flow $\Phi: \mathscr{J}^X \to M$. For $t \in \mathbb{R}$ let $U_t \subseteq M$ be the open subset of all $q \in M$ such that $(t,q) \in \mathscr{J}^X$ (so that Φ_t restricts to a diffeomorphism from U_t onto its image). Given another vector field Y, we have that $[X,Y] = 0$ if and only if

$$Y|_{U_t} \sim_{\Phi_t} Y$$

for all $t \in \mathbb{R}$.

92 (answer on page ??). Give an example of a manifold M and vector fields X,Y with $[X,Y] = 0$, such that there exist $p \in M$ and $s,t \in \mathbb{R}$ with

$$\Phi_t(\Psi_s(p)) \neq \Psi_s(\Phi_t(p))$$

even though both sides are defined (in the sense that $(t, \Phi_s(p))$, (t,p) are in the domain of X, and $(s,p), (s,\Phi_t(p))$ are in the domain of Y.

What we can say for $[X,Y] = 0$ is that if $U \subseteq M$ is an open subset and $R \subseteq \mathbb{R}^2$ an open rectangle around $(0,0)$ such that $\Phi_s(\Psi_t(p))$, $\Phi_t(\Psi_s(p))$ are both defined for *all* $(s,t) \in R$ and all $p \in U$, then the two are equal.

Remark 6.44. Theorem 6.41 shows that the Lie bracket of vector fields X,Y vanishes if and only if the their flows commute. More precisely, the Lie bracket measures the *extent* to which the flows fail to commute. Indeed, for a function f we have that

$$\Phi_t^* f = f + tX(f) + \dots, \quad \Psi_s^* f = f + sY(f) + \dots$$

where the dots indicate higher order terms in the Taylor expansion. Using a short calculation, we find

$$(\Phi_t^* \Psi_s^* - \Psi_s^* \Phi_t^*) f = st\,[X,Y](f) + \dots$$

where the dots indicate terms cubic in s,t or higher.

The Jacobi identity (6.3) for vector fields

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

also has interpretations in terms of Lie derivatives and flows. Bringing the last two terms to the right hand side, and using skew-symmetry, the identity is equivalent to $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$. That is,

$$L_X[Y,Z] = [L_X Y, Z] + [Y, L_X Z].$$

This says (by definition) that L_X is a derivation of the Lie bracket. The Jacobi identity is now explained as the derivative at $t=0$ of the identity

$$\Phi_t^*[Y,Z] = [\Phi_t^* Y, \Phi_t^* Z], \tag{6.15}$$

where Φ_t is the flow of X (which we take to be complete, for simplicity).

93 (answer on page ??). Explain why the identity (6.15) holds.

6.6 Frobenius' theorem

We saw that for any vector field $X \in \mathfrak{X}(M)$, there are *solution curves* through any given point $p \in M$. The image of this curve is an (immersed) submanifold to which X is everywhere tangent. One might similarly ask about 'integral surfaces' for pairs of vector fields X,Y, and more generally 'integral submanifolds' for collections of vector fields. But the situation gets more complicated. To see what can go wrong recall that by Proposition 6.11, if two vector fields are tangent to a submanifold then so is their Lie bracket.

94 (answer on page ??). On $\mathbb{R}^3$, consider the vector fields

$$X = \frac{\partial}{\partial x},\ Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Show that there does *not* exist a surface $S \subseteq \mathbb{R}^3$ such that X,Y are everywhere tangent to S.

95 (answer on page ??). On $\mathbb{R}^3$, consider the vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ \ Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\ \ Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Show that for $p \neq (0,0,0)$, the vector fields X,Y,Z span a 2-dimensional subspace of $T_p\mathbb{R}^3$, and find a 2-dimensional surface S passing through p.

To formulate the 'integrability problem', we make the following definition.

Definition 6.45. *Suppose $X_1,\ldots,X_r$ are vector fields on the manifold M, such that the tangent vectors*

$$X_1|_p,\ldots,X_r|_p \in T_pM$$

are linearly independent for all $p \in M$. An r-dimensional submanifold $S \subseteq M$ is called an integral submanifold *if the vector fields $X_1,\ldots,X_r$ are all tangent to S.*

Remark 6.46. In practice, one is given these vector fields only locally, on an open neighborhood U of a given point $p \in M$. (In such a case, simply replace M with U in the definition.) For instance, in 95, once it is observed that X,Y,Z span a 2-dimensional of $T_p\mathbb{R}^3$, one would take X_1, X_2 to be two of the vector fields X,Y,Z which are linearly independent on a neighborhood U of p, and replace $\mathbb{R}^3$ with that neighborhood.

Let us *suppose* that there we are given $X_1,\dots,X_r$ as above, and there exists an r-dimensional integral submanifold S through every given point $p \in M$. By Proposition 6.11, since the X_i are tangent to S, all $[X_i,X_j]$ are tangent to S at p, and are hence a linear combination of $X_1|_p,\dots,X_r|_p$. It follows that

$$[X_i,X_j] = \sum_{k=1}^{r} c_{ij}^k X_k \tag{6.16}$$

for certain functions $c_{ij}^k \in C^\infty(M)$, where smoothness holds by 96 below. We refer to (6.16) as the *Frobenius condition*, named after F. G. Frobenius (1849-1917).

96 (answer on page ??). Show that if $X_1,\dots,X_r \in \mathfrak{X}(M)$ are linearly independent everywhere, and $Y \in \mathfrak{X}(M)$ is such that

$$Y|_p = \sum_{k=1}^{r} a_k(p) X_k|_p$$

for all $p \in M$, then the coefficients are smooth functions $a_k \in C^\infty(M)$.

97 (answer on page ??). Suppose that both $X_1',\dots,X_r'$ and $X_1,\dots,X_r$ are linearly independent at all points $p \in M$, spanning the same subspace of T_pM everywhere. Show that the first set of vector fields satisfies the Frobenius condition if and only if the second set does.

We shall see that the Frobenius condition is not only necessary but also sufficient for the existence of integral submanifolds. Note that Lemma 6.48 below is a special case of Frobenius' theorem; we will reduce the general case to this special case. We first prove a simpler version of Lemma 6.48, dealing with a single vector field.

The *critical set* of a vector field $X \in \mathfrak{X}(M)$ is the set of points $p \in M$ such that $X_p = 0$. The following result gives a local normal form for vector fields, away from their critical set.

Lemma 6.47 (Flow straightening lemma). *Let $X \in \mathfrak{X}(M)$ be a vector field, and $p \in M$ such that $X_p \neq 0$. Then there exists a coordinate chart (U,φ) about p, with corresponding local coordinates $x^1,\dots,x^m$, such that*

$$X|_U = \frac{\partial}{\partial x^1}.$$

In this Lemma, we are using the coordinate chart to identify U with the open subset $\varphi(U) \subseteq \mathbb{R}^m$ and thus think of $\frac{\partial}{\partial x^1}$ as a vector field on U. Avoiding the identification, one should write $X|_U \sim_\varphi \frac{\partial}{\partial x^1}$.

Proof. Choose a submanifold N of codimension 1, with $p \in N$, such that X is not tangent to N at p. (See ✎ 98 below.) The idea is to use the time variable t of the flow of X as the x^1-coordinate, and complete to a coordinate system near p by choosing coordinates $x^2, \ldots, x^n$ on N. Let us first assume that X is complete, and denote its flow by $\Phi:\ \mathbb{R} \times M \to M$. (We indicate at the end of the proof how to deal with the incomplete case.)

Claim: The restriction of the flow,

$$\Phi|_{\mathbb{R}\times N}:\ \mathbb{R} \times N \to M \tag{6.17}$$

has maximal rank at $(0,p)$.

Proof of claim: with the standard identification $T_0\mathbb{R} = \mathbb{R}$, the tangent map to Φ at $(0,p)$ is

$$T_{(0,p)}\Phi:\ \mathbb{R} \times T_pM \to T_pM,\ (s,v) \mapsto v + sX_p. \tag{6.18}$$

Indeed, $(T_{(0,p)}\Phi)(0,v) = v$, since $\Phi|_{0\times M}$ is the identity map of M; on the other hand, $(T_{(0,p)}\Phi)(s,0) = sX_p$ since $\Phi|_{\mathbb{R}\times\{p\}}:\ \mathbb{R} \to M$ is the integral curve of X through p. Hence (6.18) restricts to an isomorphism $\mathbb{R} \times T_pN \to T_pM$, proving the claim.

By the inverse function theorem, there exists a neighborhood of $(0,p)$ in $\mathbb{R} \times N$ on which $\Phi|_{\mathbb{R}\times N}$ restricts to a diffeomorphism. We may take this neighborhood to be of the form $(-\varepsilon,\varepsilon) \times V$, where V is the domain of a chart (V,ψ) around p in N. In conclusion,

$$\Phi|_{(-\varepsilon,\varepsilon)\times V}:\ (-\varepsilon,\varepsilon) \times V \to M \tag{6.19}$$

is a diffeomorphism onto its image $U \subseteq M$. This diffeomorphism takes $\frac{\partial}{\partial t}$ to $X|_U$, by the property

$$\frac{\partial}{\partial t} \sim_\Phi X, \tag{6.20}$$

of the flow. On the other hand, the coordinate map

$$(-\varepsilon,\varepsilon) \times V \to \mathbb{R}^m,\ (t,q) \mapsto (t, \psi(q))$$

takes $\frac{\partial}{\partial t}$ to the coordinate vector field $\frac{\partial}{\partial x^1}$. Hence, its composition with the map $U \to (-\varepsilon,\varepsilon) \times V$ inverse to (6.19) is the desired coordinate map $\varphi:\ U \to \mathbb{R}^m$, taking $X|_U$ to $\frac{\partial}{\partial x^1}$.

The case of possibly incomplete X is essentially the same, but working with the flow domain $\mathscr{J} \subseteq \mathbb{R} \times M$ in place of $\mathbb{R} \times M$. Choosing N as before, the intersection of the flow domain with $\mathbb{R} \times N$ is a codimension 1 submanifold of $\mathscr{J}$; as before we find a neighborhood of the form $(-\varepsilon,\varepsilon) \times V \subseteq \mathscr{J} \cap \mathbb{R} \times N$ over which Φ restricts to a diffeomorphism. The rest of the proof is unchanged. $\square$

98 (answer on page ??). Suppose $E \subseteq T_pM$ is a subspace of dimension k. Show that there exists a codimension k submanifold $N \subseteq M$, with $T_pM = T_pN \oplus E$.

The flow straightening lemma generalizes to collections of commuting vector fields. The proof uses the fact (Proposition 6.11) that if vector fields commute, their flow commute. The idea is to use these flows to build a coordinate system.

Lemma 6.48. *Let $p \in M$, and let $X_1, \ldots, X_r \in \mathfrak{X}(M)$ be vector fields, whose values at $p \in M$ are linearly independent, and with*

$$[X_i, X_j] = 0$$

for all $1 \leq i, j \leq r$. Then there exists a coordinate chart (U, φ) near p, with corresponding local coordinates $x^1, \ldots, x^m$, such that

$$X_1|_U = \frac{\partial}{\partial x^1}, \ \ldots, \ X_r|_U = \frac{\partial}{\partial x^r}.$$

Proof. Again, we will first make the (very strong) assumption that the vector fields X_i are all complete. We will explain at the end how to deal with the general case. Since the X_i commute, their flows Φ_i commute. Consider the map

$$\Phi: \ \mathbb{R}^k \times M \to M, \ \ (t_1, \ldots, t_r, q) \mapsto \big((\Phi_1)_{t_1} \circ \cdots \circ (\Phi_r)_{t_r}\big)(q).$$

Using that the Φ_i commute, we obtain that

$$\frac{\partial}{\partial t_i} \sim_\Phi X_i \tag{6.21}$$

for $i = 1, \ldots, k$. As a consequence, the tangent map to Φ at $(0, \ldots, 0, p)$ is

$$T_{(0,\ldots,0,p)}\Phi(s_1, \ldots, s_r, v) = v + s_1 X_1|_p + \cdots + s_r X_r|_p.$$

Choose a codimension k submanifold $N \subseteq M$, with $p \in N$, such that T_pN is the complement to the subspace spanned by $X_1|_p, \ldots, X_r|_p$. Then the restriction of Φ to $\mathbb{R}^k \times N$ has maximal rank at $(0, \ldots, 0, p)$, hence it is a diffeomorphism on some neighborhood of this point in $\mathbb{R}^k \times N$. We may take this neighborhood of the form $(-\varepsilon, \varepsilon)^k \times V$, with V the domain of a coordinate chart (V, ψ) around p in N. It then follows that

$$\Phi|_{(-\varepsilon,\varepsilon)^k \times V}: \ (-\varepsilon, \varepsilon)^k \times V \to M$$

is a diffeomorphism onto some open neighborhood $U \subseteq M$ of p. We take φ to be the its inverse, followed by the map

$$(-\varepsilon, \varepsilon)^k \times V \to \mathbb{R}^m, \ (t_1, \ldots, t_r, q) \mapsto (t_1, \ldots, t_r, \psi(q)).$$

In the incomplete case, we replace $\mathbb{R}^k \times M$ with the open subset $\mathscr{J} \subseteq \mathbb{R}^k \times M$ on which Φ is defined, in the sense that (t_r, q) is in the flow domain of X_r, $(t_{k-1}, (\Phi_r)_{t_r})$

is in the flow domain of X_{k-1}, and so on. Since the $(\Phi_i)_{t_i}$ commute locally, for t_i sufficiently small, Equation (6.21) holds over a possibly smaller open neighborhood $\mathscr{J}'$ of $\{0\} \times M$ inside $\mathbb{R}^k \times M$. Taking $(-\varepsilon, \varepsilon)^k \times V$ to be contained in $\mathscr{J}'$, the rest of the proof is as before. $\square$

We are now in a position to prove the following result of Frobenius.

Theorem 6.49 (Frobenius theorem). *Let $X_1, \ldots, X_r \in \mathfrak{X}(M)$ be vector fields such that $X_1|_p, \ldots, X_r|_p \in T_pM$ are linearly independent for all $p \in M$. The following are equivalent:*

a) There exists an integral submanifold through every $p \in M$.
b) The Lie brackets $[X_i, X_j]$ satisfy the Frobenius condition (6.16)*, for suitable functions $c_{ij}^k \in C^\infty(M)$.*

In fact, it is then possible to find a coordinate chart (U, φ) near any given point of M, with coordinates denoted $(x^1, \ldots, x^m)$, in such a way that the integral submanifolds are given by $x^{r+1} = \text{const}, \ldots, x^m = \text{const}$.

Proof. We have seen that the Frobenius condition (6.16) is necessary for the existence of integral submanifolds; it remains to show that it is also sufficient. Thus suppose (6.16) holds true, and consider $p \in M$.
By choosing a coordinate chart around p, we may assume that M is an open subset U of $\mathbb{R}^m$, with p the origin. Since $X_1|_p, \ldots, X_r|_p$ are linearly independent, they form part of a basis of $\mathbb{R}^n$, and by a linear change of coordinates we can assume that the $X_i|_p$ coincide with the first r coordinate vectors at the origin. Thus

$$X_i = \sum_{j=1}^{r} a_{ij}(\mathbf{x}) \frac{\partial}{\partial x^j} + \sum_{j=r+1}^{m} b_{ij}(\mathbf{x}) \frac{\partial}{\partial x^j}.$$

where $a_{ij}(0,0) = \delta_{ij}$ and $b_{ij}(0,0) = 0$. Since the matrix with entries a_i^j is invertible at the origin, it remains invertible for points near the origin. Hence we may define new vector fields $X_1', \ldots, X_r'$, on a possibly smaller neighborhood of the origin, such that

$$X_i = \sum_{j=1}^{r} a_{ij} X_j'.$$

By ✐ 97, the X_i' again satisfy the Frobenius condition, and clearly S is an integral submanifold for $\{X_i'\}$ if and only if it is an integral submanifold for $\{X_i\}$. The X_i have the form

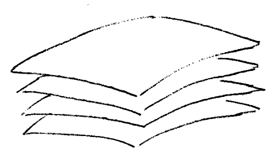
$$X_i' = \frac{\partial}{\partial x^i} + \sum_{j=r+1}^{m} b_{ij}'(\mathbf{x}) \frac{\partial}{\partial x^j}. \tag{6.22}$$

where $b_{ij}'(0,0) = 0$. We claim that the vector fields (6.22) commute. On the one hand, the Frobenius condition says that

$$[X_i', X_j'] = \sum_{k=1}^{r} (c')_{ij}^k X_k'$$

for some functions $(c')^k_{ij}$. Compare the coefficients of $\frac{\partial}{\partial x^k}$, $k = 1, \ldots, r$ in the pointwise basis given by coordinate vectors. On the right hand side, the coefficient is $(c')^k_{ij}$. On the left hand side, the coefficient is 0, since the Lie bracket between two vector fields of the form (6.22) lies in the pointwise span of $\frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x^m}$. This means $(c')^k_{ij} = 0$, and establishes that $[X'_i, X'_j] = 0$, as claimed. By Lemma 6.48, we may change the coordinates to arrange that $X'_1, \ldots, X'_r$ becomes the first r coordinate vector fields. In such coordinates, it is evident that the level set for the remaining coordinates are integral submanifolds. □

Thus, if $X_1, \ldots, X_r$ are pointwise linearly independent and satisfy the Frobenius condition, then any $p \in M$ has an open neighborhood with a 'nice' decomposition into r-dimensional integral submanifolds.



It is an example of a *foliation*.

There is a more general (and also more elegant) version of Frobenius' theorem. Suppose $r \le m$ is given, and

$$\mathscr{E} \subseteq \mathfrak{X}(M)$$

is a subspace of the space of vector fields, with the following properties:

- For all $p \in M$, the subspace $E_p = \{X_p \mid X \in \mathscr{E}\}$ is an r-dimensional subspace of T_pM.
- If $X \in \mathfrak{X}(M)$ is a vector field with the property $X_p \in E_p$ for all $p \in M$, then $X \in \mathscr{E}$.

One calls $\mathscr{E}$ a *rank r distribution*. (Using the terminology of vector bundles, developed in Chapter **??**, the definition says that $\mathscr{E}$ is the space of sections of a rank r subbundle $E \subseteq TM$.) Note that if $X_1, \ldots, X_r$ are pointwise linearly independent vector fields, as in the statement of Theorem 6.49, then one obtains a rank r distribution by letting $\mathscr{E}$ be the set of all $\sum_{i=1}^r f_i X_i$ with $f_i \in C^\infty(M)$. However, the new setting is more general, since it also allows for situations such as in ✎ 95.

A r-dimensional submanifold $S \subseteq M$ is called an *integral submanifold* of the rank r distribution $\mathscr{E}$ if all vector fields from $\mathscr{E}$ are tangent to S, or equivalently

$$T_pS = \{X_p \mid X \in \mathscr{E}\}$$

for all $p \in S$. The distribution $\mathscr{E}$ is called *integrable* if there exists an integral submanifold through every $p \in M$.

99 (answer on page ??). Let $\Phi : M \to N$ be a submersion. Show that

$$\mathscr{E} = \{X \in \mathfrak{X}(M) \,|\, X \sim_{\Phi} 0\}$$

is a rank r distribution, where $r = \dim M - \dim N$. Show that this distribution is integrable.

Using this terminology, we have the following version of Frobenius theorem:

Theorem 6.50 (Frobenius). *A rank r distribution $\mathscr{E}$ is integrable if and only if $\mathscr{E}$ is a Lie subalgebra of $\mathfrak{X}(M)$: That is, $X, Y \in \mathscr{E} \Rightarrow [X, Y] \in \mathscr{E}$.*

100 (answer on page ??). Explain how this version of Frobenius' theorem follows from the earlier version, Theorem 6.49.

7

Differential forms

In multivariable calculus, differential forms appear as a useful computational and organizational tool – unifying, for example, the various div, grad, and curl operations, and providing an elegant reformulation of the classical integration formulas of Green, Kelvin-Stokes, and Gauss. The full power of differential forms appears in their coordinate-free formulation on manifolds, which is the topic of this chapter.

7.1 Review: Differential forms on $\mathbb{R}^m$

A differential k-form on an open subset $U \subseteq \mathbb{R}^m$ is an expression of the form

$$\omega = \sum_{i_1 \cdots i_k} \omega_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

where $\omega_{i_1 \ldots i_k} \in C^\infty(U)$ are functions, and the indices are numbers

$$1 \le i_1 < \cdots < i_k \le m.$$

Let $\Omega^k(U)$ be the vector space consisting of such expressions, with the obvious addition and scalar multiplication. It is convenient to introduce a shorthand notation $I = \{i_1, \ldots, i_k\}$ for the index set, and write $\omega = \sum_I \omega_I \mathrm{d}x^I$ with

$$\omega_I = \omega_{i_1 \ldots i_k}, \quad \mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Since a k-form is determined by these functions ω_I, and since there are $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ ways of picking k-element subsets from $\{1, \ldots, m\}$, the space $\Omega^k(U)$ can be identified with vector-valued smooth functions,

$$\Omega^k(U) = C^\infty(U,\, \mathbb{R}^{\frac{m!}{k!(m-k)!}}).$$

The $\mathrm{d}x^I$ are just formal expressions; at this stage they do not have any particular significance or meaning. The motivation for writing the differentials is to suggest an associative product operation

$$\Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$

by the 'rule of computation'

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$$

for all i, j; in particular $\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$. In turn, using the product structure we may define the *exterior differential*

$$\mathrm{d}:\ \Omega^k(U) \to \Omega^{k+1}(U),\ \ \mathrm{d}\Big(\sum_I \omega_I \mathrm{d}x^I\Big) = \sum_{i=1}^m \sum_I \frac{\partial \omega_I}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I. \tag{7.1}$$

The key property of the exterior differential is the following fact:

Proposition 7.1. *The exterior differential satisfies*

$$\mathrm{d} \circ \mathrm{d} = 0,$$

i.e. $\mathrm{d}\mathrm{d}\omega = 0$ *for all* ω.

Proof. By definition,

$$\mathrm{d}\mathrm{d}\omega = \sum_{j=1}^m \sum_{i=1}^m \sum_I \frac{\partial^2 \omega_I}{\partial x^j \partial x^i} \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I,$$

which vanishes by equality of mixed partials $\frac{\partial \omega_I}{\partial x^i \partial x^j} = \frac{\partial \omega_I}{\partial x^j \partial x^i}$. (We have $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$, but the coefficients in front of $\mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\mathrm{d}x^j \wedge \mathrm{d}x^i$ are the same.) □

101 (answer on page ??). Let $U \subseteq \mathbb{R}^m$ be an open subset.
a) Show that $\Omega^k(U) = 0$ for $k > m$.
b) Write an expression for a general m-form $\omega \in \Omega^m(U)$. What is $\mathrm{d}\omega$?

102 (answer on page ??). Find the exterior differential of each of the following forms on $\mathbb{R}^3$ (with coordinates (x, y, z)).
a) $\alpha = y^2 e^x \mathrm{d}y + 2ye^x \mathrm{d}x$.
b) $\beta = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y$.
c) $\rho = e^{x^2 y} \sin z \mathrm{d}x \wedge \mathrm{d}y + 2\cos(z^3 y) \mathrm{d}x \wedge \mathrm{d}z$.
d) $\omega = \dfrac{\sin e^{xy} - \cos \sin z^3 x}{1 + (x+y+z)^4 + (7xy)^6} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

The exterior differential on forms on $\mathbb{R}^3$ is closely related to the operators div, grad, and curl from multi-variable calculus; see Problem **??** at the end of this chapter.
We will proceed to define differential forms on manifolds, beginning with 1-forms. In local charts (U, φ), 1-forms on U are identified with $\mathbb{R}^m$-valued functions, just as for vector fields. However, 1-forms on manifolds are quite different from vector fields, since their transformation properties under coordinate changes are different; in some sense they are 'dual' objects. We will therefore begin with a review of dual spaces in general.

7.2 Dual spaces

For any real vector space E, we denote by $E^* = L(E, \mathbb{R})$ its dual space, consisting of all linear maps $\alpha : E \to \mathbb{R}$. We will assume that E is finite-dimensional. Then the dual space is also finite-dimensional, and $\dim E^* = \dim E$.
It is common to write the value of $\alpha \in E^*$ on $v \in E$ as a *pairing*, using the bracket $\langle \cdot, \cdot \rangle$ notation:

$$\langle \alpha, v \rangle := \alpha(v).$$

This pairing notation emphasizes the duality between α and v. In the notation $\alpha(v)$ we think of α as a function acting on elements of E, and in particular on v. However, one may just as well think of v as acting on elements of E^* by evaluation: $v(\alpha) = \alpha(v)$ for all $\alpha \in E^*$. This symmetry manifests notationally in the pairing notation.

Let $e_1, \ldots, e_r$ be a basis of E. Any element of E^* is determined by its values on these basis vectors. For $i = 1, \ldots, r$, let $e^i \in E^*$ (with *upper* indices) be the linear functional such that

$$\langle e^i,\ e_j \rangle = \delta^i{}_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

The elements $e^1, \ldots, e^r$ are a basis of E^*; this is called the *dual basis*. The element $\alpha \in E^*$ is described in terms of the dual basis as

$$\alpha = \sum_{j=1}^{r} \alpha_j e^j, \quad \alpha_j = \langle \alpha, e_j \rangle.$$

Similarly, for vectors $v \in E$ we have

$$v = \sum_{i=1}^{r} v^i e_i, \quad v^i = \langle e^i, v \rangle.$$

Notice the placement of indices: In a given summation over $i, j, \ldots$, upper indices are always paired with lower indices.

Remark 7.2. As a special case, for $\mathbb{R}^r$ with its standard basis, we have a canonical identification $(\mathbb{R}^r)^* = \mathbb{R}^r$. For more general E with $\dim E < \infty$, there is no *canonical* isomorphism between E and E^* unless more structure is given. (For infinite-dimensional vector spaces, the dual space E^* is not in general isomorphic to E.)

103 (answer on page ??). Let V be a finite dimensional real vector space equipped with a (positive definite) inner product $(\cdot, \cdot)$. Every vector $v \in V$ determines a linear functional $A_v = (v, \cdot) \in V^*$.

a) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. Show that $\{A_{e_1}, \ldots, A_{e_n}\}$ is the corresponding dual basis of V^*.
b) Show that the linear map $V \to V^*,\ v \mapsto A_v$ is an isomorphism.

We see that the choice of an inner product determines a canonical isomorphism between V and V^*.

Given a linear map $R:\ E \to F$ between vector spaces, one defines the *dual map*

$$R^*:\ F^* \to E^*$$

(note the direction), by setting

$$\langle R^*\beta,\ v\rangle = \langle \beta, R(v)\rangle$$

for $\beta \in F^*$ and $v \in E$. This satisfies $(R^*)^* = R$, and under the composition of linear maps,

$$(R_1 \circ R_2)^* = R_2^* \circ R_1^*.$$

In terms of basis $e_1, \ldots, e_r$ of E and $f_1, \ldots, f_s$ of F, and the corresponding dual bases (with upper indices), a linear map $R:\ E \to F$ is given by the matrix with entries

$$R_i{}^j = \langle f^j,\ R(e_i)\rangle,$$

while R^* is described by the *transpose* of this matrix (the roles of i and j are reversed). Namely,

$$R(e_i) = \sum_{j=1}^{s} R_i{}^j f_j, \qquad R^*(f^j) = \sum_{i=1}^{r} R_i{}^j\ e^i.$$

Thus,

$$(R^*)^j{}_i = R_i{}^j.$$

Remark 7.3. In the physics literature, it is common and convenient to use Dirac's *'bra-ket'* notation. Elements of E are the 'kets' $v = |v\rangle$ (signified with the $|\cdot\rangle$ notation), while elements of E^* are the 'bras' $\alpha = \langle\alpha|$ (signified with the $\langle\cdot|$ notation). The pairing between elements of E^* and E is then written as bra-kets (where we write $\langle\cdot|\cdot\rangle$ instead of $\langle\cdot||\cdot\rangle$)

$$\langle\alpha|\ v\rangle = \alpha(v).$$

Concatenating in the other direction $|\cdot\rangle\,\langle\cdot|$, a ket-bra $|w\rangle\langle\alpha|$ with $|w\rangle \in F$ and $\langle\alpha| \in E^*$ signifies the linear map

$$|w\rangle\langle\alpha|:\ E \to F,\ \ |v\rangle \mapsto |w\rangle\langle\alpha|v\rangle.$$

Note that if $e_1, \ldots, e_n$ is a basis of E, with dual basis $e^1, \ldots, e^n$, then the identity operator of E may be written

$$I_E = \sum_i |e_i\rangle\langle e^i|.$$

The coefficients of a general linear map $R:\ E \to F$ with respect to bases $e_1, \ldots, e_n$ of E and $f_1, \ldots, f_m$ of F are $R_i{}^j = \langle f^j\,|R\,|e_i\rangle$, and one has the suggestive formula

$$|Re_i\rangle = R|e_i\rangle = \sum_j |f_j\rangle\langle f^j\,|R\,|e_i\rangle.$$

Denote by $\langle\alpha|^* = |\alpha\rangle$ the elements of E^*, but now playing the role of the 'given' vector space, and $|v\rangle^* = \langle v|$ the elements of E but now viewed as elements of $(E^*)^* \cong$

E (assuming $\dim E < \infty$). Then $\langle v|\alpha\rangle = \langle \alpha|v\rangle$, and the definition of dual map reads as

$$\langle \alpha|R|v\rangle = \langle v|R^*|\alpha\rangle.$$

(Note: Here we only considered real vector spaces. In quantum mechanics, one mainly deals with *complex* vector spaces, and declares $\langle v|\alpha\rangle = \langle \alpha|v\rangle^*$ (where the star denotes complex conjugate). Accordingly, the *conjugate transpose* is defined by $\langle \alpha|R|v\rangle = \langle v|R^*|\alpha\rangle^*$.)

7.3 Cotangent spaces

After this general discussion of dual spaces, we now consider the duals of tangent spaces.

Definition 7.4. *The dual of the tangent space T_pM of a manifold M is called the* cotangent space *at p, denoted*

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called cotangent vectors, or simply covectors*. Given a smooth map $F \in C^\infty(M,N)$, and any $p \in M$ we have the* cotangent map

$$T_p^*F = (T_pF)^* : \; T_{F(p)}^*N \to T_p^*M$$

defined as the dual to the tangent map.

Thus, a covector at p is a linear functional on the tangent space, assigning to each tangent vector a number. The very definition of the tangent space suggests one such functional: Every function $f \in C^\infty(M)$ defines a linear map, $T_pM \to \mathbb{R}$ taking the tangent vector v to $v(f)$. This linear functional is denoted $(\mathrm{d}f)_p \in T_p^*M$.

Definition 7.5. *Let $f \in C^\infty(M)$ and $p \in M$. The covector*

$$(\mathrm{d}f)_p \in T_p^*M, \quad \langle (\mathrm{d}f)_p, v\rangle = v(f).$$

is called the differential of f at *p.*

104 (answer on page ??). Show that under the identification of tangent spaces $T_a\mathbb{R} \cong \mathbb{R}$ for $a \in \mathbb{R}$, the differential of f at p is the same as the tangent map

$$T_pf : \; T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}.$$

Lemma 7.6. *For $F \in C^\infty(M,N)$ and $g \in C^\infty(N)$,*

$$\mathrm{d}(F^*g)_p = T_p^*F((\mathrm{d}g)_{F(p)}).$$

Proof. Every element of the dual space is completely determined by its action on vectors; so it suffices to show that the pairing with any $v \in T_pM$ is the same. This is done by unpacking the definitions:

$$\begin{aligned}
\langle d(F^*g)_p, v\rangle &= v(F^*g) \quad \text{by definition of the differential}\\
&= v(g \circ F) \quad \text{by definition of the pullback of functions}\\
&= (T_pF(v))(g) \quad \text{by definition of the tangent map}\\
&= \langle (dg)_{F(p)}, T_pF(v)\rangle \quad \text{by definition of the differential}\\
&= \langle T_p^*F((dg)_{F(p)}), v\rangle \quad \text{by definition of the dual map.}\square
\end{aligned}$$

Consider an open subset $U \subseteq \mathbb{R}^m$, with coordinates $x^1, \ldots, x^m$. Here $T_pU \cong \mathbb{R}^m$, with basis

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p \in T_pU \tag{7.2}$$

The basis of the dual space T_p^*U, dual to the basis (7.2), is given by the differentials of the coordinate functions:

$$(\mathrm{d}x^1)_p, \ \ldots, \ (\mathrm{d}x^m)_p \in T_p^*U.$$

Indeed,

$$\Big\langle (\mathrm{d}x^i)_p, \ \frac{\partial}{\partial x^j}\Big|_p \Big\rangle = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \delta^i{}_j$$

as required. For $f \in C^\infty(M)$, the coefficients of $(\mathrm{d}f)_p = \sum_i \langle (\mathrm{d}f)_p, \ e_i\rangle e^i$ are determined as

$$\Big\langle (\mathrm{d}f)_p, \ \frac{\partial}{\partial x^i}\Big|_p \Big\rangle = \frac{\partial}{\partial x^i}\Big|_p (f) = \frac{\partial f}{\partial x^i}\Big|_p.$$

Thus,

$$(\mathrm{d}f)_p = \sum_{i=1}^m \frac{\partial f}{\partial x^i}\Big|_p \ (\mathrm{d}x^i)_p.$$

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, with coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$. For $F \in C^\infty(U, V)$, the tangent map is described by the Jacobian matrix, with entries

$$(D_pF)_i{}^j = \frac{\partial F^j}{\partial x^i}(p)$$

for $i = 1, \ldots, m, \ j = 1, \ldots, n$. We have:

$$(T_pF)\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_{j=1}^n (D_pF)_i{}^j \ \frac{\partial}{\partial y^j}\Big|_{F(p)},$$

hence dually

$$(T_pF)^*(\mathrm{d}y^j)_{F(p)} = \sum_{i=1}^m (D_pF)_i{}^j \ (\mathrm{d}x^i)_p. \tag{7.3}$$

We see that, as matrices in the given bases, *the coefficients of the cotangent map are the transpose of the coefficients of the tangent map*.

105 (answer on page ??). Consider $\mathbb{R}^3$ with standard coordinates denoted x, y, z, and $\mathbb{R}^2$ with standard coordinates denoted u, v. Let $F:\ \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$(x,y,z) \mapsto (x^2 y + e^z, yz - x).$$

Find $T_pF\left(\frac{\partial}{\partial x}\Big|_p\right)$, and similarly $(T_p^*F)\big((du)_{F(p)}\big)$, for $p = (1,1,1)$

7.4 1-forms

Similarly to the definition of vector fields, one can define *covector fields*, more commonly known as *1-forms*: Collections of covectors $\alpha_p \in T_p^*M$ depending smoothly on the base point. One approach of making precise the smooth dependence on the base point is to observe that in local coordinates, 1-forms are given by expressions $\sum_i f_i \mathrm{d}x^i$, and smoothness should mean that the coefficient functions are smooth.
We will use the following (equivalent) approach. (Compare to the definition of a vector field and to Proposition 6.4.)

Definition 7.7. *A 1-form on M is a linear map*

$$\alpha:\ \mathfrak{X}(M) \to C^\infty(M), \quad X \mapsto \alpha(X) = \langle \alpha, X \rangle,$$

which is $C^\infty(M)$-linear *in the sense that*

$$\begin{aligned} \alpha(X+Y) &= \alpha(X) + \alpha(Y), \\ \alpha(fX) &= f\alpha(X) \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ *and* $f \in C^\infty(M)$. *The vector space of 1-forms is denoted* $\Omega^1(M)$.

A 1-form can be regarded as a collection of covectors:

Lemma 7.8. *Let* $\alpha \in \Omega^1(M)$ *be a 1-form, and* $p \in M$. *Then there is a unique covector* $\alpha_p \in T_p^*M$ *such that*

$$\alpha(X)_p = \alpha_p(X_p)$$

for all $X \in \mathfrak{X}(M)$. *In particular,* $\alpha(X)_p = 0$ *when* $X_p = 0$.

(We indicate the value of the function $\alpha(X)$ at p by a subscript, just like we did for vector fields.)

Proof. We have to show that $\alpha(X)_p$ depends only on the value of X at p. By considering the difference of vector fields having the same value at p, it is enough to show that if $X_p = 0$, then $\alpha(X)_p = 0$. But any vector field vanishing at p can be written as a finite sum $X = \sum_i f_i Y_i$ where $f_i \in C^\infty(M)$ vanish at p. (For example, using local coordinates, we can take the Y_i to correspond to $\frac{\partial}{\partial x^i}$ near p, and the f_i to the coefficient functions.) By C^∞-linearity, this implies that

$$\alpha(X) = \alpha(\sum_i f_i Y_i) = \sum_i f_i \alpha(Y_i)$$

vanishes at p. □

The first example of a 1-form is described in the following definition.

Definition 7.9. *The* exterior differential *of a function $f \in C^\infty(M)$ is the 1-form*

$$\mathrm{d}f \in \Omega^1(M),$$

defined in terms of its pairings with vector fields $X \in \mathfrak{X}(M)$ as $\langle \mathrm{d}f,\, X\rangle = X(f)$.

(The reader should verify that this definition conforms to Definition 7.7.) Clearly, $\mathrm{d}f$ is the 1-form defined by the family of covectors $(\mathrm{d}f)_p$, as in Definition 7.5. Note that critical points of f may be described as the *zero set* of this 1-form: $p \in M$ is a critical point of f if and only if $(\mathrm{d}f)_p = 0$.
Similarly to vector fields, 1-forms can be multiplied by functions. (This makes $\Omega^1(M)$ into a module over $C^\infty(M)$.) Hence one has more general examples of 1-forms as finite sums,

$$\alpha = \sum_i f_i \,\mathrm{d}g_i$$

where $f_i, g_i \in C^\infty(M)$.
Also similarly to vector fields, 1-forms can be *restricted* to open subsets $U \subseteq M$:

Lemma 7.10. *Given an open subsets $U \subseteq M$ and any $\alpha \in \Omega^1(M)$, there is a unique 1-form $\alpha|_U \in \Omega^1(U)$ such that*

$$(\alpha|_U)_p = \alpha_p$$

for all $p \in U$.

106 (answer on page ??). Prove this Lemma. (You will need to define $\alpha|_U(Y)$ for all $Y \in \mathfrak{X}(U)$; here Y need not be a restriction of a vector field on M. Use bump functions to resolve this issue.)

Let us describe the space of 1-forms on open subsets $U \subseteq \mathbb{R}^m$. Given $\alpha \in \Omega^1(U)$, we have

$$\alpha = \sum_{i=1}^m \alpha_i \,\mathrm{d}x^i$$

with coefficient functions $\alpha_i = \left\langle \alpha,\, \frac{\partial}{\partial x^i} \right\rangle \in C^\infty(U)$: the right hand side takes on the correct values at any $p \in U$, and is uniquely determined by those values. General vector fields on U may be written

$$X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}$$

(to match the notation for 1-forms, we write the coefficients as X^i rather than a^i, as we did in the past), where the coefficient functions are recovered as $X^i = \langle \mathrm{d}x^i,\, X\rangle$. The pairing of the 1-form α with the vector field X is then

$$\langle \alpha,\, X\rangle = \sum_{i=1}^{m} \alpha_i X^i.$$

Lemma 7.11. *Let $\alpha:\ p \mapsto \alpha_p \in T_p^*M$ be a collection of covectors. Then α defines a 1-form, with*

$$\alpha(X)_p = \alpha_p(X_p)$$

for $p \in M$, if and only if for all charts (U, φ), the coefficient functions for α in the chart are smooth.

107 (answer on page ??). Prove Lemma 7.11. (You may want to use Lemma 7.10.)

7.5 Pull-backs of function and 1-forms

Recall again that for any manifold M, the vector space $C^\infty(M)$ of smooth functions is an algebra, with product the pointwise multiplication. Any smooth map $F \in C^\infty(M,N)$ between manifolds defines an algebra homomorphism, called the *pull-back*

$$F^*:\ C^\infty(N) \to C^\infty(M),\ \ f \mapsto F^*(f) := f \circ F.$$

108 (answer on page ??). Show that the pull-back is indeed an algebra homomorphism by showing that it preserves sums and products:

$$F^*(f) + F^*(g) = F^*(f+g) \ \ ; \ \ F^*(f)F^*(g) = F^*(fg).$$

Next, show that if $F:\ M \to N$ and $G:\ N \to M$ are two smooth maps between manifolds, then

$$(G \circ F)^* = F^* \circ G^*.$$

(Note the order.)

Recall that for vector fields, there are no general 'push-forward' or 'pull-back' operations under smooth maps $F \in C^\infty(M,N)$, unless F is a diffeomorphism. For 1-forms the situation is better. Indeed, for any $p \in M$ one has the dual to the tangent map

$$T_p^*F = (T_pF)^*:\ T^*_{F(p)}N \to T_p^*M.$$

For a 1-form $\beta \in \Omega^1(N)$, we can therefore define

$$(F^*\beta)_p := (T_p^* F)(\beta_{F(p)}). \tag{7.4}$$

The following Lemma shows that this collection of covectors on M defines a 1-form.

Lemma 7.12. *There is a unique 1-form $F^*\beta \in \Omega^1(M)$ such that the covectors $(F^*\beta)_p \in T_p^*M$ are given by* (7.4)*.*

Proof. We shall use Lemma 7.11. To check smoothness near a given $p \in M$, choose coordinate charts (V, ψ) around $F(p)$ and (U, φ) around p, with $F(U) \subseteq V$. Using these charts, we may in fact assume that $M = U$ is an open subset of $\mathbb{R}^m$ (with coordinates x^i) and $N = V$ is an open subset of $\mathbb{R}^n$ (with coordinates y^j). Write

$$\beta = \sum_{j=1}^{n} \beta_j(y)\, \mathrm{d}y^j.$$

By (7.3), the pull-back of β is given by

$$F^*\beta = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \beta_j(F(x)) \frac{\partial F^j}{\partial x^i} \Big) \mathrm{d}x^i. \tag{7.5}$$

In particular, the coefficients are smooth. $\square$

Lemma 7.12 shows that we have a well-defined pull-back map

$$F^* : \ \Omega^1(N) \to \Omega^1(M), \ \ \beta \mapsto F^*\beta.$$

Under composition of two smooth maps F_1, F_2, this pullback operation on 1-forms satisfies $(F_1 \circ F_2)^* = F_2^* \circ F_1^*$. Also, if $g \in C^\infty(N)$ is a smooth function and $F^*g = g \circ F$ its pullback to M,

$$F^*(g\beta) = F^*g\, F^*\beta.$$

Another important relation with the pullback of functions is the formula

$$F^*(\mathrm{d}g) = \mathrm{d}(F^*g)$$

which follows from Lemma 7.6. (Note that on the left we are pulling-back a form, and on the right a function.)

The formula (7.5) (itself a consequence of (7.3)) shows how to compute the pullback of forms in local coordinates. In fact, suppose $F : \ U \to V$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ with coordinates x^i and y^j, respectively. Given $\beta = \sum_j \beta_j(y) \mathrm{d}y^j$, one computes $F^*\beta$ by *replacing y with $F(x)$*:

$$F^*\beta = \sum_j \beta_j(F(x))\mathrm{d}(F(x))^j = \sum_{ij} \beta_j(F(x)) \frac{\partial F^j}{\partial x^i} \mathrm{d}x^i.$$

109 (answer on page ??). Consider the map

$$F:\ \mathbb{R}^3 \to \mathbb{R}^2, \quad (x,y,z) \mapsto (x^3 e^{yz}, \sin x).$$

Let u,v be the coordinates on the target space $\mathbb{R}^2$.

a) Compute the pullback under F of the 1-forms

$$du, \quad v\cos(u)\mathrm{d}v$$

b) Let $g \in C^\infty(\mathbb{R}^2)$ be the function $g(u,v) = uv$. Verify Lemma 7.6 by computing $\mathrm{d}g$, $F^*(\mathrm{d}g)$, as well as F^*g and $\mathrm{d}(F^*g)$.

In the case of vector fields, one has neither pullback nor push-forward in general, but instead works with the notion of related vector fields, $X \sim_F Y$. This fits nicely with the pullback of 1-forms:

Proposition 7.13. *Let $F \in C^\infty(M,N)$, and let $\beta \in \Omega^1(N)$. If $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related, i.e. $X \sim_F Y$, then*

$$\langle F^*\beta, X\rangle = F^*\langle \beta, Y\rangle.$$

Proof. We verify this identity pointwise, at any $p \in M$:

$$\begin{aligned}
\langle F^*\beta, X\rangle_p &= \langle (F^*\beta)_p, X_p\rangle \\
&= \langle T_p^*F(\beta_{F(p)}), X_p\rangle \\
&= \langle \beta_{F(p)}, T_pF(X_p)\rangle \\
&= \langle \beta_{F(p)}, Y_{F(p)}\rangle \\
&= \langle \beta, Y\rangle_{F(p)} \\
&= (F^*\langle \beta, Y\rangle)_p \square
\end{aligned}$$

110 (answer on page ??). One may verify that $\gamma:\ \mathbb{R} \to \mathbb{R}^2,\ \ t \mapsto (\cos t, \sin t)$ is a solution curve of the vector field

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

That is, $\frac{\partial}{\partial t} \sim_\gamma X$. (Cf. Equation (6.7).) Let $\beta = \mathrm{d}x - \mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. Verify the conclusion of Proposition 7.13 by computing each of

$$\langle \beta, X\rangle,\ \ \gamma^*\beta,\ \ \langle \gamma^*\beta, \frac{\partial}{\partial t}\rangle,\ \ \gamma^*\langle \beta, X\rangle.$$

7.6 Integration of 1-forms

Given a curve $\gamma\colon J \to M$ in a manifold, and any 1-form $\alpha \in \Omega^1(M)$, we can consider the pull-back $\gamma^*\alpha \in \Omega^1(J)$. By the description of 1-forms on $\mathbb{R}$, this is of the form

$$\gamma^*\alpha = f(t)\mathrm{d}t \tag{7.6}$$

for a smooth function $f \in C^\infty(J)$.
To discuss integration, it is convenient to work with closed intervals rather than open intervals. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A map $\gamma\colon\ [a,b] \to M$ into a manifold will be called *smooth* if it extends to a smooth map from an open interval containing $[a,b]$. We will call such a map a smooth *path*.

Definition 7.14. *Given a smooth path $\gamma\colon\ [a,b] \to M$, we define the integral of a 1-form $\alpha \in \Omega^1(M)$ along γ as*

$$\int_\gamma \alpha = \int_a^b f(t)\mathrm{d}t$$

where f is the function defined by $\gamma^*\alpha = f(t)\mathrm{d}t$.

The fundamental theorem of calculus has the following consequence for manifolds. It is a special case of *Stokes' theorem* (Theorem 8.7).

Proposition 7.15. *Let $\gamma\colon\ [a,b] \to M$ be a smooth path, with end points $\gamma(a) = p,\ \gamma(b) = q$. For any $f \in C^\infty(M)$, we have*

$$\int_\gamma \mathrm{d}f = f(q) - f(p).$$

In particular, the integral of $\mathrm{d}f$ depends only on the end points, rather than the path itself.

Proof. We have

$$\gamma^*\mathrm{d}f = \mathrm{d}\gamma^*f = \mathrm{d}(f\circ\gamma) = \frac{\partial(f\circ\gamma)}{\partial t}\mathrm{d}t.$$

Integrating from a to b, we obtain, by the fundamental theorem of calculus, $f(\gamma(b)) - f(\gamma(a))$. $\square$

111 (answer on page ??). Let $\gamma\colon\ [a,b] \to \mathbb{R}$ be a smooth path in $M = \mathbb{R}$, and let $f \in C^\infty(\mathbb{R})$ be a smooth function, so that $f\mathrm{d}x$ is a 1-form. Let $F \in C^\infty(\mathbb{R})$ be a primitive, i.e. $F'(x) = f(x)$.

a) Verify that

$$\int_\gamma f\mathrm{d}x = \int_{\gamma(a)}^{\gamma(b)} f(s)\mathrm{d}s$$

where the right hand side is a Riemann integral. (Hint: $f\mathrm{d}x = \mathrm{d}F$.)

b) Use part a to prove the "integration by substitution" formula:

$$\int_{\gamma(a)}^{\gamma(b)} f(x)\mathrm{d}x = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t.$$

where both sides are Riemann integrals.

Let M be a manifold, and $\gamma\colon\ [a,b] \to M$ a smooth path. A *reparametrization* of the path is a path $\gamma \circ \kappa\colon\ [c,d] \to M$, where $\kappa\colon\ [c,d] \to [a,b]$ is a diffeomorphism (in the sense that it extends to a diffeomorphism on slightly larger open intervals). The reparametrization is called *orientation preserving* if $\kappa(c) = a,\ \kappa(d) = b$, orientation reversing if $\kappa(c) = b,\ \kappa(d) = a$.

Proposition 7.16 (Reparametrization invariance of the integral). *Given a reparametrization $\gamma \circ \kappa$ of the path γ as above, and any $\alpha \in \Omega^1(M)$,*

$$\int_\gamma \alpha = \pm \int_{\gamma\circ\kappa} \alpha,$$

with the plus sign if κ preserves orientation and minus sign of it reverses orientation.

Proof. Since $(\gamma \circ \kappa)^* = \kappa^* \circ \gamma^*$ we have

$$\int_{\gamma\circ\kappa} \alpha = \int_c^d (\gamma\circ\kappa)^*\alpha = \int_c^d \kappa^*(\gamma^*\alpha) = \int_{\kappa(c)}^{\kappa(d)} \gamma^*\alpha$$

where the last equality follows from integration by substitution, as in 111. If κ is orientation preserving, we therefore have

$$\int_{\gamma\circ\kappa} \alpha = \int_{\kappa(c)}^{\kappa(d)} \gamma^*\alpha = \int_a^b \gamma^*\alpha = \int_\gamma \alpha.$$

If κ is orientation reversing, we have

$$\int_{\gamma\circ\kappa} \alpha = \int_{\kappa(c)}^{\kappa(d)} \gamma^*\alpha = \int_b^a \gamma^*\alpha = -\int_a^b \gamma^*\alpha = -\int_\gamma \alpha.$$

Remark 7.17. We did not fully use the fact that κ is a diffeomorphism. The conclusion holds for any smooth path $\kappa\colon\ [c,d] \to \mathbb{R}$ taking values in $[a,b]$, so that $\gamma \circ \kappa$ is defined.

112 (answer on page ??). Consider the 1-form on $\mathbb{R}^2$

$$\alpha = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y.$$

Find the integral of α along the path

$$\gamma\colon\ [0,1] \to M,\ \ t \mapsto (\sin(\pi t/2), t^3).$$

A 1-form $\alpha \in \Omega^1(M)$ such that $\alpha = \mathrm{d}f$ for some function $f \in C^\infty(M)$ is called *exact*. Proposition 7.15 gives a necessary condition for exactness: The integral of α along paths should depend only on the end points.

Remark 7.18. This condition is also sufficient: define f on the connected components of M, by fixing a base point p_0 on each such component, and putting $f(p) = \int_\gamma \alpha$ for any path from p_0 to p. With a little work (using charts), one verifies that f defined in this way is smooth.

If M is an open subset $U \subseteq \mathbb{R}^m$, so that $\alpha = \sum_i \alpha_i \mathrm{d}x^i$, then $\alpha = \mathrm{d}f$ means that $\alpha_i = \frac{\partial f}{\partial x^i}$. A necessary condition is the equality of the mixed partial derivatives,

$$\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i}, \tag{7.7}$$

In multivariable calculus one learns that this condition is also sufficient, provided U is simply connected (e.g., convex). Using the exterior differential of forms in $\Omega^1(U)$, this condition becomes $\mathrm{d}\alpha = 0$. Since α is a 1-form, $\mathrm{d}\alpha$ is a 2-form. Thus, to obtain a coordinate-free version of the condition, we need higher order forms.

7.7 k-forms

To get a feeling for higher degree forms, and constructions with higher forms, we first discuss 2-forms.

7.7.1 2-forms.

Skew-symmetry

Definition 7.19. *A 2-form on M is a $C^\infty(M)$-bilinear skew-symmetric map*

$$\alpha: \ \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M), \ \ (X,Y) \mapsto \alpha(X,Y).$$

The space of 2-forms is denoted $\Omega^2(M)$.

Here skew-symmetry means that $\alpha(X,Y) = -\alpha(Y,X)$ for all vector fields X,Y, while $C^\infty(M)$-bilinearity means that for any fixed Y, the map $X \mapsto \alpha(X,Y)$ is $C^\infty(M)$-linear, and for any fixed X, the map $Y \mapsto \alpha(X,Y)$ is $C^\infty(M)$-linear. (Actually, by skew-symmetry it suffices to require $C^\infty(M)$-linearity in the first argument.)
By the same argument as for 1-forms, the value $\alpha(X,Y)_p$ depends only on the values X_p, Y_p. Also, if α is a 2-form then so is $f\alpha$ for any smooth function f.

Our first examples of 2-forms are obtained from 1-forms: Let $\alpha, \beta \in \Omega^1(M)$. Then we define a *wedge product* $\alpha \wedge \beta \in \Omega^2(M)$, as follows:

$$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{7.8}$$

113 (answer on page ??). Show that Equation (7.8) indeed defines a 2-form.

Another example of a 2-form is the exterior differential of a 1-form. We will soon give a general definition of the differential of any k-form; the following will be a special case:

114 (answer on page ??). Show that for any 1-form $\alpha \in \Omega^1(M)$, the following formula defines a 2-form $\mathrm{d}\alpha \in \Omega^2(M)$:

$$(\mathrm{d}\alpha)(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]).$$

Show furthermore that if $\alpha = \mathrm{d}f$ for a function $f \in C^\infty(M)$, then $\mathrm{d}\alpha = 0$.

For an open subset $U \subseteq \mathbb{R}^m$, a 2-form $\omega \in \Omega^2(U)$ is by $C^\infty(U)$-bilinearity, uniquely determined by its values on coordinate vector fields. By skew-symmetry the functions

$$\omega_{ij} = \omega\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)$$

satisfy $\omega_{ij} = -\omega_{ji}$; hence it suffices to know these functions for $i < j$. As a consequence, we see that the most general 2-form on U is

$$\omega = \sum_{i<j} \omega_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

115 (answer on page ??). Using 114 as the definition of exterior differential of a 2-form, show that the differential of $\alpha = \sum_i \alpha_i \mathrm{d}x^i \in \Omega^1(U)$ is

$$\mathrm{d}\alpha = \sum_{i<j} \Big(\frac{\partial \alpha^j}{\partial x^i} - \frac{\partial \alpha^i}{\partial x^j}\Big) \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Review the discussion around Equation (7.7) from this perspective.

7.7.2 *k*-forms

We now generalize to forms of arbitrary degree.

Definition 7.20. *Let k be a non-negative integer. A* k-form *on M is a* $C^\infty(M)$-*multilinear, skew-symmetric map*

$$\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ times}} \to C^\infty(M).$$

The space of k-forms is denoted $\Omega^k(M)$*; in particular* $\Omega^0(M) = C^\infty(M)$.

Here, *skew-symmetry* means that $\alpha(X_1,\ldots,X_k)$ changes sign under exchange of any two of its arguments. For example,

$$\alpha(X_1,X_2,X_3,\ldots) = -\alpha(X_2,X_1,X_3,\ldots).$$

More generally, denoting by S_k the group of permutations of $\{1,\ldots,k\}$, and by $\mathrm{sign}(s)=\pm1$ the sign of a permutation $s\in S_k$ ($+1$ for an even permutation, -1 for an odd permutation) then

$$\alpha(X_{s(1)},\ldots,X_{s(k)}) = \mathrm{sign}(s)\alpha(X_1,\ldots,X_k). \tag{7.9}$$

(See Appendix A.2 for more on permutations.) The $C^\infty(M)$*-multilinearity* means that for any index i, and any given vector fields $X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_k\in\mathfrak{X}(M)$, the map

$$\mathfrak{X}(M)\to C^\infty(M),\;\; X\mapsto \alpha(X_1,\ldots,X_{i-1},X,X_{i+1},\ldots,X_k)$$

is $C^\infty(M)$-linear, that is a 1-form. (Given the skew-symmetry, it suffices to check $C^\infty(M)$-linearity in any one of the arguments, for instance for $i=1$.)

The $C^\infty(M)$-multilinearity implies, in particular, that α is *local* in the sense that the value of $\alpha(X_1,\ldots,X_k)$ at any given $p\in M$ depends only on the values $X_1|_p,\ldots,X_k|_p\in T_pM$. (This is an application of Lemma 7.8 to any of the arguments.) One thus obtains a skew-symmetric multilinear form

$$\alpha_p:\; T_pM\times\cdots\times T_pM\to\mathbb{R},$$

for all $p\in M$. For any open subset $U\subseteq M$, one has a *restriction map*

$$\Omega^k(M)\to\Omega^k(U),\;\; \alpha\mapsto\alpha|_U$$

such that $(\alpha|_U)_p=\alpha_p$ for all $p\in M$. The argument for this is essentially the same as for 1-forms, see Lemma 7.10.

Given a $C^\infty(M)$-multilinear map $\eta:\;\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to C^\infty(M)$ (with k arguments), not necessarily skew-symmetric, one obtains a k-form $\alpha=\mathrm{Sk}\eta\in\Omega^k(M)$ through the process of *skew-symmetrization*:

$$(\mathrm{Sk}\eta)(X_1\ldots,X_k)=\sum_{s\in S_k}\mathrm{sign}(s)\,\eta(X_{s(1)},\ldots,X_{s(k)}).$$

116 (answer on page ??). Confirm that $\mathrm{Sk}\eta$ does indeed define a k-form. Also show that if $\alpha\in\Omega^k(M)$ (so that α is already skew-symmetric) then $\mathrm{Sk}\alpha=k!\,\alpha$.

This may be applied, for example, to define the wedge product of 1-forms $\alpha_1,\ldots,\alpha_k\in\Omega^1(M)$, as the skew-symmetrization of the multilinear form

$$(X_1,\ldots,X_k)\mapsto\langle\alpha_1,X_1\rangle\cdots\langle\alpha_k,X_k\rangle.$$

That is, $\alpha_1 \wedge \cdots \wedge \alpha_k \in \Omega^k(M)$ is given by the formula

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(X_1, \ldots, X_k) = \sum_{s \in S_k} \operatorname{sign}(s) \alpha_1(X_{s(1)}) \cdots \alpha_k(X_{s(k)})$$

(More general wedge products will be discussed below.)
Let us describe the space of k-forms on open subsets $U \subseteq \mathbb{R}^m$. Using C^∞-multilinearity, a k-form $\alpha \in \Omega^k(U)$ is uniquely determined by its values on coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$, i.e. by the functions

$$\alpha_{i_1 \ldots i_k} = \alpha\Big(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\Big).$$

By skew-symmetry we only need to consider *ordered* index sets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$, that is, $i_1 < \cdots < i_k$. Using the wedge product notation, we obtain

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}. \tag{7.10}$$

For k-forms ω on general manifolds M, this gives a description of $\omega|_U$ in coordinate charts (U, φ). Let us also note the following useful consequence:

Lemma 7.21. *Every differential form $\alpha \in \Omega^k(M)$ is locally, near a given point $p \in M$, a linear combination of k-forms of the type*

$$f_0 \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in \Omega^k(M), \tag{7.11}$$

where $f_0, \ldots, f_k \in C^\infty(M)$.

Proof. Given $p \in M$, choose a coordinate chart (U, φ) around p. In these coordinates, $\alpha|_U$ has the form (7.10). This is not quite the desired form since the coordinate functions and coefficient functions are only defined on U. But this is easily fixed: Choose $h_1, \ldots, h_m \in C^\infty(M)$ so that $h_i \circ \varphi^{-1}$ agrees with the coordinate function x^i near $\varphi(p)$, and choose $g_{i_1, \ldots, i_k} \in C^\infty(M)$ such that $g_{i_1, \ldots, i_k} \circ \varphi^{-1}$ agrees with $\alpha_{i_1 \ldots i_k}$ near p. Then

$$\sum_{i_1 < \cdots < i_k} g_{i_1, \ldots, i_k} \mathrm{d}h_{i_1} \wedge \cdots \wedge \mathrm{d}h_{i_k}$$

agrees with α near p.

7.7.3 Wedge product

We next turn to the definition of a *wedge product* of forms of arbitrary degree.

Definition 7.22. *The wedge product of $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ is the element*

$$\alpha \wedge \beta \in \Omega^{k+l}(M)$$

given by the formula

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+l}) = \frac{1}{k!\,l!} \sum_{s \in S_{k+l}} \operatorname{sign}(s) \alpha(X_{s(1)}, \ldots, X_{s(k)})\, \beta(X_{s(k+1)}, \ldots, X_{s(k+l)}) \tag{7.12}$$

117 (answer on page ??). Show that Definition 7.22 is consistent with our previous definition of the wedge product of two 1-forms (Equation (7.8)).

Thus, up to a factor $1/k!\,l!$, the wedge product of α and β is the skew-symmetrization of the map

$$(X_1,\ldots,X_{k+l}) \mapsto \alpha(X_1,\ldots,X_k)\beta(X_{k+1},\ldots,X_{k+l});$$

in particular, it is indeed a $k+l$-form. Note that many of the $(k+l)!$ terms in the sum over S_{k+l} coincide, since α is skew-symmetric in its arguments to begin with, and likewise for β. Indeed, one can get a simpler expression involving only permutations where

$$s(1) < \cdots < s(k), \quad s(k+1) < \cdots < s(k+l).$$

A permutation $s \in S_{k+l}$ having this property is called a (k,l)-*shuffle*. Denote by $S_{k,l}$ the set of (k,l)-shuffles. Every (k,l)-shuffle is uniquely determined by a k-element subset of $\{1,\ldots,k+l\}$ (by taking this subset to be $s(1),\ldots,s(k)$). In particular, there are

$$\binom{k+l}{k} = \frac{(k+l)!}{k!l!}$$

different (k,l)-shuffles.

Example 7.23. The permutation

$$(1\ 4\ 2\ 3\ 5) \in S_5$$

(meaning: $s(1)=1,\ s(2)=4,\ s(3)=2,\ s(4)=3,\ s(5)=5$) is a $(3,2)$-shuffle, since the first three elements are in order, and likewise the last two elements.

118 (answer on page ??). List all $(3,2)$-shuffles in S_5. How many elements does $S_{3,2}$ have?

Using the notion of k,l-shuffle, the wedge product is also given by the formula

$$(\alpha\wedge\beta)(X_1,\ldots,X_{k+l}) = \sum_{s\in S_{k,l}} \operatorname{sign}(s)\alpha(X_{s(1)},\ldots,X_{s(k)})\ \beta(X_{s(k+1)},\ldots,X_{s(k+l)}) \tag{7.13}$$

We tend to prefer this version since it has fewer terms.

119 (answer on page ??). For $\alpha,\beta \in \Omega^2(M)$, and $X_1,X_2,X_3,X_4 \in \mathfrak{X}(M)$, write all the terms of

$$(\alpha\wedge\beta)(X_1,X_2,X_3,X_4).$$

It will be better to use Formula (7.13) with only 6 terms, rather than (7.12) with $4! = 24$ terms (which coincide in groups of 4).

120 (answer on page ??). Identify $S_k \times S_l$ as the subgroup of S_{k+l} preserving $\{1,\dots,k\}$ and (hence) also $\{k+1,\dots,k+l\}$. Show that every $s \in S_{k+l}$ is uniquely a product

$$s = s's'',$$

where $s' \in S_{k,l}$ and $s'' \in S_k \times S_l$. Use this to prove the second formula (7.13) for the wedge product, as a sum over k,l-shuffles.

It is clear that the wedge product $\alpha \wedge \beta$ is $C^\infty(M)$-bilinear in α and in β: that is, for fixed β, the map

$$\Omega^k(M) \to \Omega^{k+l}(M),\ \alpha \mapsto \alpha \wedge \beta$$

is $C^\infty(M)$-linear, and for fixed α the map $\beta \mapsto \alpha \wedge \beta$ is $C^\infty(M)$-linear. In addition, the wedge product has the following properties:

Proposition 7.24. *a) The wedge product is* graded commutative*: If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ then*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

b) The wedge product is associative: Given $\alpha_i \in \Omega^{k_i}(M)$ we have

$$(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3).$$

The associativity allows us to drop parentheses when writing wedge products.

Proof. a) There is a canonical bijection between (k,l)-shuffles and (l,k)-shuffles, obtained by switching the interchanging the first k components and last l components. For example,

$$(1\ 2\ 4\ 3\ 5) \in S_{3,2} \quad \leftrightarrow \quad (3\ 5\ 1\ 2\ 4) \in S_{2,3}.$$

In more detail, let $\sigma \in S_{k+l}$ be the permutation

$$\sigma(1) = k+1, \dots,\ \sigma(k) = k+l,\ \sigma(k+1) = 1, \dots, \sigma(k+l) = l.$$

This has sign

$$\mathrm{sign}(\sigma) = (-1)^{kl}. \tag{7.14}$$

and we have that

$$s \in S_{k,l} \Leftrightarrow s' = s \circ \sigma \in S_{l,k}. \tag{7.15}$$

Therefore, for any $X_1, \dots, X_{k+l} \in \mathfrak{X}(M)$,

$$\begin{aligned}
&(\beta \wedge \alpha)(X_1, \dots, X_{k+l}) \\
&= \sum_{s' \in S_{l,k}} \mathrm{sign}(s') \beta(X_{s'(1)}, \dots, X_{s'(l)}) \alpha(X_{s'(l+1)}, \dots, X_{s'(k+l)}) \\
&= \sum_{s \in S_{k,l}} \mathrm{sign}(s \circ \sigma) \beta(X_{(s\circ\sigma)(1)}, \dots, X_{(s\circ\sigma)(l)}) \alpha(X_{(s\circ\sigma)(l+1)}, \dots, X_{(s\circ\sigma)(k+l)}) \\
&= \mathrm{sign}(\sigma) \sum_{s \in S_{k,l}} \mathrm{sign}(s) \beta(X_{s(k+1)}, \dots, X_{s(k+l)}) \alpha(X_{s(1)}, \dots, X_{s(k)}) \\
&= (-1)^{kl} (\alpha \wedge \beta)(X_{\sigma(1)}, \dots, X_{\sigma(k+l)})
\end{aligned}$$

b) Define a (k,l,m)-shuffle to be a permutation s in S_{k+l+m} such that

$$s(1) < \cdots < s(k), \;\; s(k+1) < \cdots < s(k+l), \;\; s(k+l+1) < \cdots s(k+l+m).$$

By careful bookkeeping, one finds that both

$$((\alpha_1 \wedge \alpha_2) \wedge \alpha_3)(X_1, \ldots, X_{k_1+k_2+k_3}), \quad (\alpha_1 \wedge (\alpha_2 \wedge \alpha_3))(X_1, \ldots, X_{k_1+k_2+k_3})$$

are given by

$$\sum_{s \in S_{k_1,k_2,k_3}} \operatorname{sign}(s) \alpha_1(X_{s(1)}, \ldots, X_{s(k_1)}) \alpha_2(X_{s(k_1+1)}, \ldots, X_{s(k_1+k_2)}) \alpha_3(X_{s(k_1+k_2+1)}, \ldots, X_{s(k_1+k_2+k_3)}).$$

□

121 (answer on page ??). Give some details of the 'careful bookkeeping' in the proof of part (b).

7.7.4 Exterior differential

Recall that we have defined the exterior differential on functions by the formula

$$(\mathrm{d}f)(X) = X(f). \tag{7.16}$$

(In 114, we also indicated a possible definition of d on 1-forms.) We will now extend this definition to all forms.

Theorem 7.25. *There is a unique collection of linear maps* $\mathrm{d}: \; \Omega^k(M) \to \Omega^{k+1}(M)$*, extending the map (7.16) for* $k = 0$*, such that* $\mathrm{d}(\mathrm{d}f) = 0$ *for* $f \in C^\infty(M)$*, and such that the graded product rule*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta \tag{7.17}$$

holds, for $\alpha \in \Omega^k(M)$ *and* $\beta \in \Omega^l(M)$*. This* exterior differential *has the property*

$$\mathrm{d}(\mathrm{d}\alpha) = 0$$

for all $\alpha \in \Omega^k(M)$.

Proof. Let us first *assume* that such an exterior differential exists. We will establish uniqueness, and along the way give a formula.
Observe first that d is necessarily *local*, in the sense that for any open subset $U \subseteq M$ the restriction $(\mathrm{d}\alpha)|_U$ depends only on $\alpha|_U$. Equivalently, $\alpha|_U = 0 \Rightarrow (\mathrm{d}\alpha)|_U = 0$. Indeed, if $\alpha|_U = 0$, and given any $p \in U$, we may choose $f \in C^\infty(M) = \Omega^0(M)$ such that $\operatorname{supp}(f) \subseteq U$ and $f|_p = 1$. Then $f\alpha = 0$, hence the product rule (7.17) gives

$$0 = \mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha.$$

Evaluating at p we obtain $(\mathrm{d}\alpha)_p = 0$, as claimed.
But locally, a k-form is a linear combination of expressions $f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ (see Lemma 7.21). The graded product rule and the property $\mathrm{dd}f = 0$ force us to define

$$\mathrm{d}(f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k;$$

this specifies d uniquely. We also see that $\mathrm{d} \circ \mathrm{d} = 0$ on k-forms of any degree, since this holds true on $f_0\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$.
For open subsets of $U \subseteq \mathbb{R}^m$, we are forced to define the differential of

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \in \Omega^k(U).$$

as

$$\mathrm{d}\alpha = \sum_{i_1 < \cdots < i_k} \mathrm{d}\alpha_{i_1 \cdots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \in \Omega^{k+1}(U).$$

Conversely, we may use this explicit formula to *define* $(\mathrm{d}\alpha)|_U = \mathrm{d}(\alpha|_U)$ for a coordinate chart domain U; by uniqueness the local definitions agree on overlaps of any two coordinate chart domains. □

Definition 7.26. *A k-form* $\omega \in \Omega^k(M)$ *is called* exact *if* $\omega = \mathrm{d}\alpha$ *for some* $\alpha \in \Omega^{k-1}(M)$. *It is called* closed *if* $\mathrm{d}\omega = 0$.

Since $\mathrm{d} \circ \mathrm{d} = 0$, the exact k-forms are a subspace of the space of closed k-forms. For the case of 1-forms, we have seen that the integral $\int_\gamma \alpha$ of an exact 1-form $\alpha = \mathrm{d}f$ along a smooth path $\gamma\colon [a,b] \to M$ is given by the difference of the values at the end points $p = \gamma(a)$ and $q = \gamma(b)$; in particular, for an exact 1-form the integral does not depend on the choice of path from p to q. In particular, if γ is a *loop* (that is, $p = q$) the integral is zero. A *necessary* condition for α to be exact is that it is closed. An example of a 1-form that is closed but not exact is

$$\alpha = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 \backslash \{\mathbf{0}\}).$$

122 (answer on page ??). Prove that the 1-form α above is closed but not exact.

The quotient space (closed k-forms modulo exact k-forms) is a vector space called the k-th (de Rham) *cohomology*

$$H^k(M) = \frac{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is closed }\}}{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is exact }\}}. \tag{7.18}$$

(See Appendix A.4 for generalities about quotient spaces.) Note that when M is connected, then $H^0(M) = \mathbb{R}$: A closed 0-form is a function f that is locally constant; hence for connected M it is globally constant; on the other hand there is no nonzero exact 0-form.

123 (answer on page ??). Explain why there are no exact 0-forms other than 0.

It turns out that whenever M is compact (and often also if M is non-compact), $H^k(M)$ is a finite-dimensional vector space. The dimension of this vector space

$$b_k(M) = \dim H^k(M)$$

is called the *k-th Betti number* of M; these numbers are important invariants of M which one can use to distinguish non-diffeomorphic manifolds. For example, if $M = \mathbb{C}\mathrm{P}^n$ one can show that

$$b_k(\mathbb{C}\mathrm{P}^n) = 1 \quad \text{for} \quad k = 0, 2, \ldots, 2n$$

and $b_k(\mathbb{C}\mathrm{P}^n) = 0$ otherwise. For $M = S^n$ the Betti numbers are (see Problem **??** at the end of this chapter, and Problem **??** in the next chapter)

$$b_k(S^n) = 1 \quad \text{for} \quad k = 0, n$$

while $b_k(S^n) = 0$ for all other k. Hence $\mathbb{C}\mathrm{P}^n$ cannot be diffeomorphic to S^{2n} unless $n = 1$. For an example when M is not compact we have

$$b_k(\mathbb{R}^n) = 1 \quad \text{for} \quad k = 0,$$

while $b_k(\mathbb{R}^n) = 0$ for all other k. Put differently, for $k > 0$, every closed k-form on $\mathbb{R}^n$ is exact. This fact is known as the *Poincaré Lemma*, and you will prove it in Problems **??** and **??** at the end of this chapter.

7.8 Lie derivatives and contractions

A given vector field $X \in \mathfrak{X}(M)$ determines a $C^\infty(M)$-linear map

$$\iota_X : \ \Omega^k(M) \to \Omega^{k-1}(M)$$

called *contraction by* X. Thinking of $\alpha \in \Omega^k(M)$ as a $C^\infty(M)$-multilinear form, one simply puts X into the first slot:

$$(\iota_X \alpha)(X_1, \ldots, X_{k-1}) = \alpha(X, X_1, \ldots, X_{k-1}).$$

(If $k = 0$ so that α is a function, one puts $\iota_X \alpha = 0$.)
Given two vector fields X, Y, we have that

$$\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0 \tag{7.19}$$

as operators on forms, due to $\alpha(X,Y,X_1,\ldots,X_{k-2}) = -\alpha(Y,X,X_1,\ldots,X_{k-2})$. Contractions have the following compatibility with the wedge product, similar to that for the exterior differential:

$$\iota_X(\alpha\wedge\beta) = \iota_X\alpha\wedge\beta + (-1)^k\alpha\wedge\iota_X\beta, \tag{7.20}$$

for all $\alpha\in\Omega^k(M)$, and $\beta\in\Omega^l(M)$.

124 (answer on page ??). Prove Equation (7.20).

125 (answer on page ??). Compute $\iota_Z\alpha$ for

$$\alpha = \sin(x)\mathrm{d}x\wedge\mathrm{d}y,\;\; Z = e^x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

The exterior differential d and the contraction operators ι_X are both examples of *odd superderivations*, where ‘super’ refers to the signs appearing in the product rule. In the ‘super’ world, a sign change appears whenever two odd objects move past each other: For example, in (7.20) there is a minus in the second term whenever α is odd, since the ‘odd’ operator ι_X appears to the right of α in the second term.

More formally, a collection of linear maps

$$D:\;\Omega^k(M)\to\Omega^{k+r}(M)$$

is called a *degree r superderivation* if it has the property

$$D(\alpha\wedge\beta) = D(\alpha)\wedge\beta + (-1)^{rk}\alpha\wedge D\beta$$

for $\alpha\in\Omega^k(M),\;\beta\in\Omega^l(M)$. Thus, d is a superderivation of degree 1, while ι_X is a superderivation of degree -1.

126 (answer on page ??). Show that if D_1, D_2 are degree r_1, r_2 superderivations on differential forms, then their *supercommutator* (using the $[\cdot,\cdot]$ notation)

$$[D_1,D_2] = D_1\circ D_2 - (-1)^{r_1 r_2} D_2\circ D_1$$

is a degree r_1+r_2 superderivation.

Note that the identity $\mathrm{d}\circ\mathrm{d}=0$ may be written using supercommutators, as $[\mathrm{d},\mathrm{d}]=0$, while the identity $\iota_X\circ\iota_Y+\iota_Y\circ\iota_X=0$ now reads as $[\iota_X,\iota_Y]=0$. From the contractions and differential, we obtain a superderivation $[\mathrm{d},\iota_X]$ of degree $1+(-1)=0$. We shall temporarily take this to be the definition of the *Lie derivative* $L_X:\ \Omega^k(M)\to\Omega^k(M)$,

$$L_X=\mathrm{d}\circ\iota_X+\iota_X\circ\mathrm{d}. \tag{7.21}$$

Thus, $L_X=[\mathrm{d},\iota_X]$ in supercommutator notation. A 'better' definition will be given in the next section, where the simple formula (7.21) will be realized as the result of a theorem. Note that (7.21) is consistent with the earlier notion of the Lie derivative of functions $f\in C^\infty(M)=\Omega^0(M)$:

$$L_X(f)=(\mathrm{d}\circ\iota_X)(f)+(\iota_X\circ\mathrm{d})(f)=\iota_X(\mathrm{d}f)=X(f).$$

By the general result from 126, L_X satisfies the product rule

$$L_X(\alpha\wedge\beta)=L_X\alpha\wedge\beta+\alpha\wedge L_X\beta \tag{7.22}$$

for $\alpha\in\Omega^k(M)$ and $\beta\in\Omega^l(M)$, as a consequence of the product rules for ι_X and d. The product rule is frequently used for computations:

127 (answer on page ??). For each of the following vector-fields $X\in\mathfrak{X}(\mathbb{R}^3)$ and differential forms $\alpha\in\Omega^k(\mathbb{R}^3)$ on $\mathbb{R}^3$, compute the Lie derivative $L_X\alpha$:

a) $X=y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$ and $\alpha=-y\mathrm{d}x-x\mathrm{d}y-z\mathrm{d}z$.
b) $X=\sin x\frac{\partial}{\partial y}-y^2\frac{\partial}{\partial x}$ and $\alpha=x^2-\sin(y)$.
c) $X=\sin x\frac{\partial}{\partial y}-y^2\frac{\partial}{\partial x}$ and $\alpha=(x^2+y^2)\mathrm{d}x\wedge\mathrm{d}z$.

To summarize, we have introduced three operators

$$\mathrm{d}:\ \Omega^k(M)\to\Omega^{k+1}(M),\ \ L_X:\ \Omega^k(M)\to\Omega^k(M),\ \ \iota_X:\ \Omega^k(M)\to\Omega^{k-1}(M)$$

respectively called the exterior differential, the Lie derivative with respect to $X\in\mathfrak{X}(M)$, and the contraction by $X\in\mathfrak{X}(M)$. These are superderivations of degree $1,0,-1$ respectively. One might expect getting other superderivations by taking further supercommutators, but nothing new is obtained:

Theorem 7.27 (Cartan calculus). *The operators exterior differentiation, Lie derivation, and contraction satisfy the supercommutator relations*

$$[\mathrm{d},\mathrm{d}]=0, \tag{7.23}$$
$$[L_X,L_Y]=L_{[X,Y]}, \tag{7.24}$$
$$[\iota_X,\iota_Y]=0, \tag{7.25}$$
$$[\mathrm{d},L_X]=0, \tag{7.26}$$
$$[L_X,\iota_Y]=\iota_{[X,Y]}, \tag{7.27}$$
$$[\mathrm{d},\iota_X]=L_X. \tag{7.28}$$

This collection of identities is referred to as the *Cartan calculus*, after Élie Cartan (1861-1951), and in particular the last identity is called the *Cartan formula* [2]. Basic contributions to the theory of differential forms were made by his son Henri Cartan (1906-1980), who also wrote a textbook [3] on the subject.

Proof. The identities (7.23), (7.25), (7.28) have already been discussed. The identity (7.26) is proved from the definitions, and using $\mathrm{d}\circ\mathrm{d}=0$:

$$\begin{aligned}[\mathrm{d},L_X] &= \mathrm{d}\circ L_X - L_X\circ\mathrm{d}\\ &= \mathrm{d}\circ(\iota_X\circ\mathrm{d}+\mathrm{d}\circ\iota_X)-(\iota_X\circ\mathrm{d}+\mathrm{d}\circ\iota_X)\circ\mathrm{d}\\ &= 0.\end{aligned}$$

Consider the identity (7.27). Both $D=[L_X,\iota_Y]$ and $D'=\iota_{[X,Y]}$ are superderivations of degree -1. The identity $D=D'$ is true for functions $f\in C^\infty(M)=\Omega^0(M)$ since both sides act as zero on functions for degree reasons. The identity also holds for differentials of functions, since

$$\begin{aligned}[L_X,\iota_Y]\mathrm{d}f &= (L_X\circ\iota_Y)\mathrm{d}f-(\iota_Y\circ L_X)\mathrm{d}f\\ &= L_XL_Yf-\iota_Y\mathrm{d}L_Xf \quad \text{using (7.28), (7.26)}\\ &= L_XL_Yf-L_YL_Xf \quad \text{using (7.28)}\\ &= L_{[X,Y]}f\end{aligned}$$

Here the last line uses $[L_X,L_Y]=L_{[X,Y]}$ *on functions*, by definition of the Lie bracket. By 128 below, the equality of D,D' on functions and differentials of functions shows $D=D'$. The remaining identity (7.24) is left to the reader (see 129). □

128 (answer on page ??).
a) Show that if D is a degree r superderivation on differential forms, and $U\subseteq M$ is open, then $(D\alpha)|_U$ depends only on $\alpha|_U$.
b) Show that if two degree r superderivations D,D', satisfy $Df=D'f$ and $D(\mathrm{d}f)=D'(\mathrm{d}f)$ for all functions $f\in C^\infty(M)$, then $D=D'$.

129 (answer on page ??). Complete the proof of Theorem 7.27 by proving (7.24).

130 (answer on page ??). Show $\iota_X\circ\iota_X=0$ as a consequence of the Cartan calculus.

131 (answer on page ??).

a) Use the Cartan calculus to prove the following formula for the exterior differential of a 1-form $\alpha \in \Omega^1(M)$ given in 114

$$(\mathrm{d}\alpha)(X,Y) = L_X\big(\alpha(Y)\big) - L_Y\big(\alpha(X)\big) - \alpha([X,Y]).$$

b) Prove a similar formula for the exterior differential of a 2-form.

The formulas for the differentials of 1-forms and 2-forms generalize to arbitrary k-forms $\alpha \in \Omega^k(M)$. For $X_1,\ldots,X_{k+1} \in \mathfrak{X}(M)$ we have that

$$\begin{aligned}(\mathrm{d}\alpha)(X_1,\ldots,X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}L_{X_i}(\alpha(X_1,\ldots,\widehat{X_i},\ldots X_{k+1}) \\ &+\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{k+1}),\end{aligned}$$

where the hat notation $\widehat{X_i}$ indicates that the entry X_i is absent. We leave this general formula as a homework problem (see Problem **??** at the end of this chapter).

7.9 Pull-backs

Let $F \in C^\infty(M,N)$ be a smooth map between manifolds. Similarly to the pull-back of functions (0-forms) and 1-forms, we have a pull-back operation for k-forms.

Proposition 7.28. *There is a well-defined linear map*

$$F^*:\ \Omega^k(N) \to \Omega^k(M)$$

such that for all $\beta \in \Omega^k(N)$ and all $p \in M$,

$$(F^*\beta)_p(v_1,\ldots,v_k) = \beta_{F(p)}\big(T_pF(v_1),\ldots,T_pF(v_k)\big). \tag{7.29}$$

The pullback operation satisfies

$$F^*(\beta_1 \wedge \beta_2) = F^*\beta_1 \wedge F^*\beta_2 \tag{7.30}$$

as well as

$$F^* \circ \mathrm{d} = \mathrm{d} \circ F^*. \tag{7.31}$$

Proof. We have to check that the collection of multilinear forms on T_pM given by (7.29) does indeed define a *smooth* k-form $F^*\beta \in \Omega^k(M)$. Observe that the pullback operation is *local*: If $U \subseteq M,\ V \subseteq N$ are open subsets with $F(U) \subseteq V$, then $(F^*\beta)_p$ for $p \in U$ depends only on $\beta|_V$. Note also that for wedge products, $F^*(\beta_1 \wedge \beta_2)_p = (F^*\beta_1)_p \wedge (F^*\beta_2)_p$. Locally, near a given point $F(p) \in N$, every β is a linear combination of forms

$$g_0\,\mathrm{d}g_1\wedge\cdots\wedge\mathrm{d}g_k\in\Omega^k(N) \tag{7.32}$$

with $g_i\in C^\infty(N)$. (See Lemma 7.21.) Hence, it is enough to show that $F^*\beta$ is smooth when β is a function g or the differential of such a function. But $F^*g=g\circ F$ is the usual pullback of functions, hence is smooth. We claim $F^*(\mathrm{d}g)=\mathrm{d}F^*g$. which shows that $F^*(\mathrm{d}g)$ is smooth as well. Given $p\in M$ and $v\in T_pM$ we calculate

$$(F^*(\mathrm{d}g))_p(v)=(\mathrm{d}g)_{F(p)}(T_pF(v))=(T_pF(v))(g)=v(F^*g)=\mathrm{d}(F^*g)_p(v),$$

proving the claim. This shows that the pullback operation is well-defined. Equation (7.30) follows from the pointwise property, and (7.31) is proved by applying both sides to expressions (7.32), using that (7.31) holds true on function g and differentials of functions $\mathrm{d}g$. $\square$

Equation (7.31) shows how F^* interacts with the differential. As for contractions and Lie derivatives with respect to vector fields, we have the following statement: Let $X\in\mathfrak{X}(M),\ Y\in\mathfrak{X}(N)$, with $X\sim_F Y$. Then

$$\iota_X\circ F^*=F^*\circ\iota_Y,\quad L_X\circ F^*=F^*\circ L_Y.$$

We leave the proof as Problem **??** at the end of this chapter.
In local coordinates, if $F:\ U\to V$ is a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, with coordinates $x^1,\ldots,x^m$ and $y^1,\ldots,y^n$, the pull-back just amounts to 'putting $y=F(x)$'.

132 (answer on page ??). Denote the coordinates on $\mathbb{R}^3$ by x,y,z and those on $\mathbb{R}^2$ by u,v. Let $F:\ \mathbb{R}^3\to\mathbb{R}^2,\ (x,y,z)\mapsto(y^2z,\ x)$. Compute

$$F^*(\mathrm{d}u\wedge\mathrm{d}v).$$

The following proposition is the 'key fact' toward the definition of an integral of differential forms.

Proposition 7.29. *Let $U\subseteq\mathbb{R}^m$ with coordinates x^i, and $V\subseteq\mathbb{R}^n$ with coordinates y^j, and let $F\in C^\infty(U,V)$. Suppose $m=n$. Then*

$$F^*(\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^n)=J\ \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$$

where J is the Jacobian determinant,

$$J(\mathbf{x})=\det\Big(\frac{\partial F^i}{\partial x^j}\Big)_{i,j=1}^n.$$

Proof.

$$\begin{aligned}
F^*\beta &= \mathrm{d}F^1 \wedge \cdots \wedge \mathrm{d}F^n \\
&= \sum_{i_1 \dots i_n} \frac{\partial F^1}{\partial x^{i_1}} \cdots \frac{\partial F^n}{\partial x^{i_n}} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\
&= \sum_{s \in S_n} \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^{s(1)} \wedge \cdots \wedge \mathrm{d}x^{s(n)} \\
&= \sum_{s \in S_n} \mathrm{sign}(s) \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\
&= J\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.
\end{aligned}$$

(Where we have used the characterization of the determinant in terms of the group of permutations.) In this calculation, we noted that the wedge product $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$ is zero unless the indices $i_1, \dots, i_n$ are obtained from $1, \dots, n$ by a permutation s, in which case it is given by $\mathrm{sign}(s)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. $\square$

One may regard this result as giving a new (and in some sense, better) definition of the Jacobian determinant.
Having the notion of pullback of forms, we may reconsider the Lie derivative $L_X \alpha$ of a differential form with respect to a vector field X. Let us assume for simplicity that X is complete, so that the flow Φ_t is globally defined. The following formula shows that L_X measures (infinitesimally) the extent to which α is invariant under the flow of X.

Theorem 7.30. *For X a complete vector field, and $\alpha \in \Omega^k(M)$,*

$$L_X \alpha = \frac{d}{dt}\Big|_{t=0} \Phi_t^* \alpha. \tag{7.33}$$

Here the derivative on the right hand side is to be understood pointwise, at any $p \in M$, as the derivative of $(\Phi_t^*\alpha)_p$ (a function of t with values in the finite-dimensional vector space of multilinear forms on T_pM). The theorem also holds for incomplete vector fields; indeed, to define the right hand side at a given $p \in M$ one only needs Φ_t *near* p, and for *small* $|t|$.

Proof. To prove this identity, it suffices to check that the right hand side satisfies a product rule with respect to the wedge product of forms, and that it has the correct values on functions and on differentials of functions. In detail, let $D\alpha := \frac{d}{dt}\big|_{t=0} \Phi_t^*\alpha$. Then

$$\begin{aligned}
D(\alpha_1 \wedge \alpha_2) &= \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\alpha_1 \wedge \Phi_t^*\alpha_2) \\
&= \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\alpha_1) \wedge \alpha_2 + \alpha_1 \wedge \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\alpha_2) \\
&= D\alpha_1 \wedge \alpha_2 + \alpha_1 \wedge D\alpha_2
\end{aligned}$$

(pointwise, at any $p \in M$). By the usual argument, this implies that $D\alpha|_U$ depends only on $\alpha|_U$. Next, on functions $f \in C^\infty(M)$ we find

$$Df = \frac{d}{dt}\Big|_{t=0} \Phi_t^* f = L_X f$$

by definition of the flow of a vector field, and of differentials of functions we have

$$D(\mathrm{d}f) = \frac{d}{dt}\Big|_{t=0} \Phi_t^* \mathrm{d}f = \mathrm{d}\frac{d}{dt}\Big|_{t=0} \Phi_t^* f = \mathrm{d}L_X f = L_X \mathrm{d}f.$$

This verifies $L_X = D$ on functions and on differential of functions, and since any form is locally a sum of expressions $f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ it shows $L_X = D$ on k-forms. □

8

Integration

Differential forms of top degree and of compact support can be integrated over *oriented* manifolds, and more generally over domains (with boundary) in such manifolds. A key result concerning integration is *Stokes' theorem*, a far-reaching generalization of the fundamental theorem of calculus. The Stokes theorem has numerous important applications, such as to winding numbers and linking numbers, mapping degrees, de Rham cohomology, and many more. We begin with a quick review of integration on open subsets of Euclidean spaces.

8.1 Integration of differential forms

Suppose $U \subseteq \mathbb{R}^m$ is open, and $\omega \in \Omega^m(U)$ is a form of top degree $k = m$. Such a differential form is an expression

$$\omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f \in C^\infty(U)$ (see 101). If $\mathrm{supp}(f)$ is compact, one defines the integral of ω to be the usual Riemann integral:

$$\int_U \omega = \int_{\mathbb{R}^m} f(x^1, \ldots, x^m) \, \mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{8.1}$$

Note that we can regard f as a function on all of $\mathbb{R}^m$, due to the compact support condition. Let us now generalize this to manifolds.

8.1.1 Integration of differential forms on manifolds

The *support* of a differential form $\omega \in \Omega^k(M)$ is the smallest closed subset $\mathrm{supp}(\omega) \subseteq M$ with the property that ω is zero outside of $\mathrm{supp}(\omega)$ (cf. Definition 3.3). Let M be an oriented manifold of dimension m, and $\omega \in \Omega^m(M)$. If $\mathrm{supp}(\omega)$ is contained in an oriented coordinate chart (U, φ), then one defines

$$\int_M \omega = \int_{\mathbb{R}^m} f(\mathbf{x})\,\mathrm{d}x^1 \cdots \mathrm{d}x^m$$

where $f \in C^\infty(\mathbb{R}^m)$ is the function, with $\mathrm{supp}(f) \subseteq \varphi(U)$, determined from

$$(\varphi^{-1})^*\omega = f\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

This definition does not depend on the choice of oriented coordinate chart. Indeed, suppose (V, ψ) is another oriented chart with $\mathrm{supp}(\omega) \subseteq V$, and write

$$(\psi^{-1})^*\omega = g\ \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m,$$

where $y^1, \ldots, y^m$ are the coordinates on V. Letting $F = \psi \circ \varphi^{-1}$ be the change of coordinates $\mathbf{y} = F(\mathbf{x})$, Proposition 7.29 shows

$$F^*(g\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m) = (g \circ F)\,J\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

where $J(\mathbf{x}) = \det(DF(\mathbf{x}))$ is the determinant of the Jacobian matrix of F at $\mathbf{x}$. Hence, $f(\mathbf{x}) = g(F(\mathbf{x}))J(\mathbf{x})$, and we obtain

$$\int_{\psi(U)} g(\mathbf{y})\mathrm{d}y^1 \cdots y^m = \int_{\varphi(U)} g(F(\mathbf{x}))J(\mathbf{x})\mathrm{d}x^1 \cdots \mathrm{d}x^m = \int_{\varphi(U)} f(\mathbf{x})\mathrm{d}x^1 \cdots \mathrm{d}x^m,$$

as required.

Remark 8.1. Here we used the *change-of-variables formula* from multivariable calculus. It was important to work with *oriented* charts, guaranteeing that $J > 0$ everywhere. Indeed, for general changes of variables, the change-of-variables formula involves $|J|$ rather than J itself.

More generally, if the support of ω is compact but not necessarily contained in a single oriented chart, we proceed as follows. Let $(U_i, \varphi_i),\quad i = 1, \ldots, r$ be a finite collection of oriented charts covering $\mathrm{supp}(\omega)$. Together with $U_0 = M\backslash\mathrm{supp}(\omega)$ this is an open cover of M. For any such open cover, there exists a *partition of unity subordinate to the cover*, i.e., functions $\chi_i \in C^\infty(M)$ with

$$\mathrm{supp}(\chi_i) \subseteq U_i, \qquad \sum_{i=0}^r \chi_i = 1.$$

A proof for the existence of such partitions of unity (for *any* open cover, not only finite ones) is given in in Appendix B. The partition of unity allows us to write ω as a sum

$$\omega = (\sum_{i=0}^r \chi_i)\,\omega = \sum_{i=0}^r (\chi_i \omega)$$

where each $\chi_i\omega$ has compact support in a coordinate chart. (We may drop the term for $i = 0$, since $\chi_0\omega = 0$.) Accordingly, we define

$$\int_M \omega = \sum_{i=0}^r \int_M \chi_i \omega$$

We have to check that this is well-defined, independent of the various choices we have made. To that end, let (V_j, ψ_j) for $j = 0, \ldots, s$ be another collection of oriented coordinate charts covering $\mathrm{supp}(\omega)$, put $V_0 = M - \mathrm{supp}(\omega)$, and let $\sigma_0, \ldots, \sigma_s$ be a corresponding partition of unity subordinate to the cover by the V_i's.
Then $\{U_i \cap V_j : \; i = 0, \ldots, r, j = 0, \ldots, s\}$ is an open cover, with the collection of products $\chi_i \sigma_j$ a partition of unity subordinate to this cover. We obtain

$$\sum_{j=0}^{s} \int_M \sigma_j \omega = \sum_{j=0}^{s} \int_M (\sum_{i=0}^{r} \chi_i) \, \sigma_j \omega = \sum_{j=0}^{s} \sum_{i=0}^{r} \int_M \sigma_j \chi_i \omega = \sum_{i=0}^{r} \sum_{j=0}^{s} \int_M \sigma_j \chi_i \omega = \sum_{i=0}^{r} \int_M \chi_i \omega.$$

8.1.2 Integration over oriented submanifolds

Let M be a manifold (not necessarily oriented), and S a k-dimensional oriented submanifold, with inclusion $i : \; S \to M$. We define the integral over S, of any k-form $\omega \in \Omega^k(M)$ such that $S \cap \mathrm{supp}(\omega)$ is compact, as follows:

$$\int_S \omega = \int_S i^* \omega.$$

Of course, this definition works equally well for *any* smooth map from S into M, it does not have to be an embedding as a submanifold. For example, the integral of compactly supported 1-forms along arbitrary curves $\gamma \colon \; \mathbb{R} \to M$ is defined. (Compare with the definition of integrals of 1-forms along paths in Section 7.6; where γ was defined on *closed* intervals.)

8.2 Stokes' theorem

Let M be an m-dimensional oriented manifold.

Definition 8.2. *A* region with (smooth) boundary *in M is a closed subset $D \subseteq M$ of the form*

$$D = \{p \in M | \; f(p) \leq 0\},$$

where $f \in C^\infty(M, \mathbb{R})$ is a smooth function having 0 *as a regular value.*

We do not consider f itself as part of the definition of D, only the existence of f is required. The interior of a region with boundary, given as the largest open subset contained in D, is

$$\mathrm{int}(D) = \{p \in M | \; f(p) < 0\},$$

and the boundary is

$$\partial D = \{p \in M | \; f(p) = 0\},$$

a codimension 1 submanifold (i.e., hypersurface) in M.

Example 8.3. The unit disk

$$D = \{(x, y) \in \mathbb{R}^2 | \; x^2 + y^2 \leq 1\}$$

is a region with boundary, with defining function $f(x, y) = x^2 + y^2 - 1$.

Example 8.4. Recall that for $0 < r < R$, the function $f \in C^\infty(\mathbb{R}^3)$ given by

$$f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - R)^2 - r^2$$

has zero as a regular value, with $f^{-1}(0)$ a 2-torus. The corresponding region with boundary $D \subseteq \mathbb{R}^3$ is the *solid torus*.

Recall that we are considering D inside an *oriented* manifold M. The boundary ∂D may be covered by oriented submanifold charts (U, φ), in such a way that ∂D is given in the chart by the condition $x^1 = 0$, and D by the condition $x^1 \leq 0$:

$$\varphi(U \cap D) = \varphi(U) \cap \{x \in \mathbb{R}^m |\, x^1 \leq 0\}.$$

(Indeed, given an oriented submanifold chart for which D lies on the side where $x_1 \geq 0$, one obtains a suitable chart by composing with the orientation-preserving coordinate change $(x^1, \ldots, x^m) \mapsto (-x^1, -x^2, x^3 \ldots, x^m)$.) We shall call oriented submanifold charts of this kind 'region charts' (this is not a standard name).

Remark 8.5. We originally defined submanifold charts in such a way that the last $m-k$ coordinates are zero on S, here we require that the first coordinate be zero. It doesn't matter, since one can simply reorder coordinates, but works better for our description of the 'induced orientation'.

Lemma 8.6. *The restriction of the region charts to ∂D form an oriented atlas for ∂D.*

Proof. Let (U, φ) and (V, ψ) be two region charts, defining coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^m$, and let $F = \psi \circ \varphi^{-1} :\ \varphi(U \cap V) \to \psi(U \cap V),\ \mathbf{x} \mapsto \mathbf{y} = F(\mathbf{x})$. It restricts to a map

$$F' :\ \{\mathbf{x} \in \varphi(U \cap V) |\, x^1 = 0\} \to \{\mathbf{y} \in \psi(U \cap V) |\, y^1 = 0\}.$$

Since $y^1 > 0$ if and only if $x^1 > 0$, the change of coordinates satisfies

$$\frac{\partial y^1}{\partial x^1}\Big|_{x^1=0} > 0, \quad \frac{\partial y^1}{\partial x^j}\Big|_{x^1=0} = 0, \quad \text{for } j > 0.$$

Hence, the Jacobian matrix $DF(\mathbf{x})|_{x^1=0}$ has a positive $(1,1)$ entry, and all other entries in the first row equal to zero. Using expansion of the determinant across the first row, it follows that

$$\det\big(DF(0, x^2, \ldots, x^m)\big) = \frac{\partial y^1}{\partial x^1}\Big|_{x^1=0} \det\big(DF'(x^2, \ldots, x^m)\big).$$

which shows that $\det(DF') > 0$. $\square$

In particular, ∂D is again an oriented manifold. To repeat: If $x^1, \ldots, x^m$ are local coordinates near $p \in \partial D$, compatible with the orientation and such that D lies on the side $x^1 \leq 0$, then $x^2, \ldots, x^m$ are local coordinates on ∂D. This convention of 'induced orientation' is arranged in such a way that Stokes' theorem holds without an extra sign.

For a top degree form $\omega \in \Omega^m(M)$ such that $\mathrm{supp}(\omega) \cap D$ is compact, the integral

$$\int_D \omega$$

is defined similarly to the case of $D = M$: One covers $D \cap \mathrm{supp}(\omega)$ by finitely many submanifold charts (U_i, φ_i) with respect to ∂D (this includes charts that are entirely in the interior of D), and puts

$$\int_D \omega = \sum \int_{D \cap U_i} \chi_i \omega$$

where the χ_i are supported in U_i and satisfy $\sum_i \chi_i = 1$ over $D \cap \mathrm{supp}(\omega)$. By the same argument as for $D = M$, this definition of the integral is independent of the choices made.

Theorem 8.7 (Stokes' theorem). *Let M be an oriented manifold of dimension m, and $D \subseteq M$ a region with smooth boundary ∂D. Let $\alpha \in \Omega^{m-1}(M)$ be a form of degree $m-1$, such that* $\mathrm{supp}(\alpha) \cap D$ *is compact. Then*

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha.$$

As in Section 8.1.2, the right hand side means $\int_{\partial D} i^* \alpha$, where $i: \ \partial D \hookrightarrow M$ is the inclusion map.

Proof. We shall see that Stokes' theorem is just a coordinate-free version of the fundamental theorem of calculus. Let (U_i, φ_i) for $i = 1, \ldots, r$ be a finite collection of region charts covering $\mathrm{supp}(\alpha) \cap D$. Let $\chi_1, \ldots, \chi_r \in C^\infty(M)$ be functions with $\chi_i \geq 0$, $\mathrm{supp}(\chi_i) \subseteq U_i$, and such that $\chi_1 + \cdots + \chi_r$ is equal to 1 on $\mathrm{supp}(\alpha) \cap D$. (For instance, we may take $U_1, \ldots, U_r$ together with $U_0 = M \backslash \mathrm{supp}(\alpha)$ as an open covering, and take the $\chi_0, \ldots, \chi_r \in C^\infty(M)$ to be a partition of unity subordinate to this cover.) Since

$$\int_D \mathrm{d}\alpha = \sum_{i=1}^r \int_D \mathrm{d}(\chi_i \alpha), \qquad \int_{\partial D} \alpha = \sum_{i=1}^r \int_{\partial D} \chi_i \alpha,$$

it suffices to consider the case that α is supported in a region chart.

Using the corresponding coordinates, it hence suffices to prove Stokes' theorem for the case that $\alpha \in \Omega^{m-1}(\mathbb{R}^m)$ is a compactly supported form in $\mathbb{R}^m$:

$$D = \{x \in \mathbb{R}^m | \ x^1 \leq 0\}.$$

That is, α has the form

$$\alpha = \sum_{i=1}^m f_i \, \mathrm{d}x^1 \wedge \cdots \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m,$$

with compactly supported $f_i \in C^\infty(\mathbb{R}^m)$, where the hat means that the corresponding factor is to be omitted. Only the $i = 1$ term contributes to the integral over $\partial D = \mathbb{R}^{m-1}$, and

$$\int_{\mathbb{R}^{m-1}} \alpha = \int f_1(0, x^2, \ldots, x^m)\, \mathrm{d}x^2 \cdots \mathrm{d}x^m.$$

On the other hand,

$$\mathrm{d}\alpha = \Big(\sum_{i=1}^m (-1)^{i+1} \frac{\partial f_i}{\partial x^i} \Big)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

Let us integrate each summand over the region D given by $x^1 \le 0$. For $i > 1$, we have

$$\int_\infty^\infty \cdots \int_{-\infty}^\infty \int_{-\infty}^0 \frac{\partial f_i}{\partial x_i}(x^1, \ldots, x^m)\, \mathrm{d}x^1 \cdots \mathrm{d}x^m = 0$$

where we used Fubini's theorem to carry out the x^i-integration first, and applied the fundamental theorem of calculus to the x^i-integration (keeping the other variables fixed). Since the integrand is the derivative of a compactly supported function, the x^i-integral is zero. It remains to consider the case $i = 1$. Here we have, again by applying the fundamental theorem of calculus,

$$\begin{aligned} \int_D \mathrm{d}\alpha &= \int_\infty^\infty \cdots \int_{-\infty}^\infty \int_{-\infty}^0 \frac{\partial f_1}{\partial x_1}(x^1, \ldots, x^m) \mathrm{d}x^1 \cdots \mathrm{d}x^m \\ &= \int_\infty^\infty \cdots \int_{-\infty}^\infty f_1(0, x^2, \ldots, x^m) \mathrm{d}x^2 \cdots \mathrm{d}x^m \\ &= \int_{\partial D} \alpha. \end{aligned}$$ □

As a special case (where $D = M$ with $\partial D = \emptyset$) we have:

Corollary 8.8. *Let $\alpha \in \Omega^{m-1}(M)$ be a compactly supported form on the oriented manifold M. Then*

$$\int_M \mathrm{d}\alpha = 0.$$

Note that it does not suffice that $\mathrm{d}\alpha$ has compact support. For example, if $f(t)$ is a function with $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for $t > 1$, then $\mathrm{d}f$ has compact support, but $\int_{\mathbb{R}\setminus\{0\}} \mathrm{d}f = 1$.

A typical application of Stokes' theorem shows that for a closed form $\omega \in \Omega^k(M)$, the integral of ω over an oriented compact submanifold does not change with smooth deformations of the submanifold.

Theorem 8.9. *Let $\omega \in \Omega^k(M)$ be a closed form on a manifold M, and S a compact, oriented manifold of dimension k. Let $F \in C^\infty(\mathbb{R} \times S, M)$ be a smooth map, thought of as a smooth family of maps*

$$F_t = F(t, \cdot) : \ S \to M.$$

Then the integrals

$$\int_S F_t^* \omega$$

do not depend on t.

Proof. Let $a < b$, and consider the domain $D = [a,b] \times S \subseteq \mathbb{R} \times S$. The boundary ∂D has two components, both diffeomorphic to S. At $t = b$ the orientation is the given orientation on S, while at $t = a$ we get the opposite orientation. Hence,

$$0 = \int_D F^* \mathrm{d}\omega = \int_D \mathrm{d}F^* \omega = \int_{\partial D} F^* \omega = \int_S F_b^* \omega - \int_S F_a^* \omega.$$

Hence $\int_S F_b^* \omega = \int_S F_a^* \omega$. □

Remark 8.10. If the F_t's are embeddings, then $\int_S F_t^* \omega$ can be regarded as the integrals of ω over the time dependent family of submanifolds $F_t(S) \subseteq M$.

Remark 8.11. Suppose that one member of this family of maps, say the map F_1, takes values in a $k-1$-dimensional submanifold. Then $F_1^* \omega = 0$. (Indeed, the assumption means that $F_1 = j \circ F_1'$, where j is the inclusion of a $k-1$-submanifold and F_1' takes values in that submanifold. But $j^* \omega = 0$ for degree reasons.) It then follows that $\int_S F_t^* \omega = 0$ for all t. A special case of this situation is when one member of the family is the constant map to a point.

Given a smooth map $\varphi: S \to M$, one refers to a smooth map $F: \mathbb{R} \times S \to M$ with $F_0 = \varphi$ as an 'smooth deformation' (or 'isotopy') of φ. We say that φ can be smoothly deformed into φ' if there exists a smooth isotopy F with $\varphi = F_0$ and $\varphi' = F_1$. The theorem shows that if S is oriented, and if there is a closed form $\omega \in \Omega^k(M)$ with

$$\int_S \varphi^* \omega \neq \int_S (\varphi')^* \omega$$

then φ cannot be smoothly deformed into φ'. This observation has many applications; here are some of them.

Example 8.12. Suppose $\varphi: S \to M$ is a smooth map, where S is an oriented manifold of dimension k. Suppose $\omega \in \Omega^k(M)$ is closed, $\mathrm{d}\omega = 0$. If $\int_S \varphi^* \omega \neq 0$, then φ cannot be 'deformed' into a map taking values in a lower-dimensional submanifold; in particular it cannot be deformed into a constant map. Indeed, Remark 8.11 shows that if such a deformation existed, the integral would have to be zero.

Example 8.13 (Winding number). Let $\omega \in \Omega^2(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ be the 1-form

$$\omega = \frac{1}{2\pi(x^2+y^2)}(x\mathrm{d}y - y\mathrm{d}x)$$

In polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, one has that $\omega = \frac{1}{2\pi}\mathrm{d}\theta$. Using this fact one sees that ω is closed (but not exact, since θ is not a globally defined function on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.) Hence, if

$$\gamma\colon\; S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

is any smooth map (a 'loop'), then the integral

$$w(\gamma) = \int_{S^1} \gamma^* \omega$$

does not change under deformations (isotopies) of the loop. In particular, γ cannot be deformed into a constant map, unless the integral is zero. The number $w(\gamma)$ is the *winding number* of γ. (One can show that this is always an integer, and that two loops can be deformed into each other if and only if they have the same winding number.)

133 (answer on page ??). Consider S^1 as the quotient $\mathbb{R}/\sim$ under the equivalence relation $t \sim t' \Leftrightarrow t' - t \in \mathbb{Z}$. Find the winding number of the loop

$$\gamma\colon\; S^1 \to \mathbb{R}^2 \setminus \{\mathbf{0}\},\;\; \gamma([t]) = \big(\cos(2\pi n t), \sin(2\pi n t)\big).$$

Let M be a compact, connected, oriented manifold. There exists a top degree differential form $\omega \in \Omega^m(M)$ such that $\int_M \omega = 1$. The form ω is closed for degree reasons (the space of $m+1$-forms on an m-dimensional manifold is trivial).

134 (answer on page ??). Show that ω is *not* exact.

In terms of de Rham cohomology, the theorem tells us that for a compact, oriented manifold M of dimension $m = \dim(M)$, the m-th cohomology group $H^m(M)$ must be nontrivial. In fact, more is true:

Theorem 8.14. *Let M be a compact, connected, and oriented manifold of dimension m. Then the integration map* $\int_M\colon\; \Omega^m(M) \to \mathbb{R}$ *induces an isomorphism in cohomology,*

$$H^m(M) \cong \mathbb{R}. \tag{8.2}$$

That is, an m-form on M is exact if and only if its integral vanishes.

Thus, any top degree form ω with $\int_M \omega = 1$ gives the basis element $[\omega]$ corresponding to 1 under this isomorphism. The proof of this theorem will be left as a homework problem at the end of this chapter.

Example 8.15. Let M, N be two compact, connected, oriented manifolds of the same dimension $m = n$, and let $\omega \in \Omega^m(N)$ with $\int_N \omega = 1$. Given a smooth map $F \in C^\infty(M, N)$, we can define the *degree of F*

$$\deg(F) = \int_M F^* \omega.$$

By Theorem 8.9, the degree is invariant under smooth deformations of F. It is also independent of the choice of ω: adding an exact form $\mathrm{d}\alpha$ changes $F^*\omega$ by an exact form $F^*\mathrm{d}\alpha = \mathrm{d}F^*\alpha$, which does not affect the integral. It turns out that $\deg(F)$ is always an integer. We can prove this under the assumption that F has at least one regular value q. (It is a non-trivial result from differential topology that regular values *always* exist). The pre-image $F^{-1}(q) \subseteq M$ is a finite set (by compactness), say $\{p_1, \ldots, p_r\}$, and for all p_i the map F restricts to a local diffeomorphism $U_i \to F(U_i)$. Letting $V = \bigcap_{i=1}^r F(U_i)$, we may take ω, with $\int_N \omega = 1$, to be supported in V. Then $F^*\omega$ is supported in the disjoint union of the U_i's. We have that $\int_{U_i} F^*\omega = \varepsilon_{p_i} \int_V \omega = \varepsilon_{p_i}$, where the sign $\varepsilon_{p_i} = \pm 1$ comes from a possible change of orientation. Consequently,

$$\deg(F) = \int_M F^*\omega = \sum_{i=1}^r \int_{U_i} F^*\omega = \sum_{i=1}^r \varepsilon_{p_i} = \deg_q(F) \in \mathbb{Z},$$

the local mapping degree of F defined in 4.23. Note that this argument finally confirms the independence of $\deg_q(F)$ of the choice of regular value q.

135 (answer on page ??). The winding number of a path in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ can also be regarded as a degree of a map. Explain how.

Example 8.16 (Linking number). Let $f, g : S^1 \to \mathbb{R}^3$ be two smooth maps whose images don't intersect, that is, with $f(z) \neq g(w)$ for all $z, w \in S^1$ (we regard S^1 as the unit circle in $\mathbb{C}$). Define a new map

$$F : S^1 \times S^1 \to S^2, \ \ (z,w) \mapsto \frac{f(z) - g(w)}{||f(z) - g(w)||}.$$

On S^2, we have a 2-form ω of total integral 1. It is the pullback of

$$\frac{1}{4\pi}\big(x\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y\big) \in \Omega^2(\mathbb{R}^3)$$

to the 2-sphere. The integral

$$L(f,g) = \int_{S^1 \times S^1} F^*\omega$$

is called the *linking number* of f and g. Note that it is the degree of the map $S^1 \times S^1 \to S^2$ obtained from f, g. Note that if it is possible to deform one of the loops, say f, into a constant loop through loops that are always disjoint from g, then the linking number is zero. In his case, we consider f, g as 'unlinked'.

8.3 Volume forms

A top degree differential form $\Gamma \in \Omega^m(M)$ is called a *volume form* if it is non-vanishing everywhere: $\Gamma_p \neq 0$ for all $p \in M$. In a local coordinate chart (U, φ), this means that

$$(\varphi^{-1})^* \Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f(x) \neq 0$ for all $x \in \varphi(U)$.

Suppose $S \subseteq M$ is a submanifold of codimension 1 (a hypersurface), and $X \in \mathfrak{X}(M)$ a vector field that is *nowhere tangent* to S. Let $i: \; S \to M$ be the inclusion. Given a volume form on M, the form

$$i^* \left(\iota_X \Gamma\right) \in \Omega^{m-1}(S)$$

is a volume form on S.

136 (answer on page ??). Verify the claim that $i^* \left(\iota_X \Gamma\right)$ is a volume form on S.

Example 8.17. The Euclidean space $\mathbb{R}^m$ has a *standard volume form*

$$\Gamma = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

If S is a hypersurface given as a level set $f^{-1}(0)$, where 0 is a regular value of f, then the gradient vector field

$$X = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

has the property that X is nowhere tangent to S. By the above, it follows that S inherits a volume form $i^*(\iota_X \Gamma)$.

Example 8.18. As a special case, let $i: \; S^n \to \mathbb{R}^{n+1}$ be the inclusion of the standard n-sphere. Let $X = \sum_{i=0}^n x^i \frac{\partial}{\partial x^i}$. Then

$$\iota_X (\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n) = \sum_{i=0}^{n} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n$$

pulls back to a volume form on S^n.

Proposition 8.19. *A volume form $\Gamma \in \Omega^m(M)$ determines an orientation on M, by taking as the oriented charts those charts (U, φ) such that*

$$(\varphi^{-1})^* \Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

with $f > 0$ everywhere on $\Phi(U)$.

Proof. It is clear that we can cover M with such charts (as f locally has the same sign, and we may multiply φ by -1 if necessary). We have to check that the condition is consistent. Suppose (U, φ) and (V, ψ) are two charts, where $(\varphi^{-1})^* \Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ and $(\psi^{-1})^* \Gamma = g \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m$ with $f > 0$ and $g > 0$. If $U \cap V$ is non-empty, let $F = \psi \circ \varphi^{-1}: \; \varphi(U) \to \psi(V)$ be the transition function. Then

$$F^*(\psi^{-1})^*\Gamma|_{U\cap V} = (\varphi^{-1})^*\Gamma|_{U\cap V},$$

hence

$$g(F(\mathbf{x}))\, J(\mathbf{x})\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m = f(\mathbf{x})\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

where J is the Jacobian determinant of the transition map $F = \psi \circ \varphi^{-1}$. Hence $f = J\,(g \circ F)$ on $\varphi(U \cap V)$. Since $f > 0$ and $g > 0$, it follows that $J > 0$. Hence the two charts are oriented compatible. □

Theorem 8.20. *A manifold M is orientable if and only if it admits a volume form. In this case, any two volume forms compatible with the orientation differ by an everywhere positive smooth function:*

$$\Gamma' = f\Gamma, \quad f > 0.$$

Proof. As we saw above, any volume form determines an orientation. Conversely, if M is an oriented manifold, there exists a volume form compatible with the orientation: Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas on M. Then each

$$\Gamma_\alpha = \varphi_\alpha^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m) \in \Omega^m(U_\alpha)$$

is a volume form on U_α; on overlaps $U_\alpha \cap U_\beta$ these are related by the Jacobian determinants of the transition functions, which are *strictly positive* functions. Let $\{\chi_\alpha\}$ be a locally finite partition of unity subordinate to the cover $\{U_\alpha\}$, see Appendix B.4. The forms $\chi_\alpha \Gamma_\alpha$ have compact support in U_α, hence they extend by zero to global forms on M (somewhat imprecisely, we use the same notation for this extension). The sum

$$\Gamma = \sum_\alpha \chi_\alpha \Gamma_\alpha \in \Omega^m(M)$$

is a well-defined volume form. Indeed, near any point p at least one of the summands is non-zero; and if other summands in this sum are non-zero, they differ by a positive function. □

For a compact manifold M with a given volume form $\Gamma \in \Omega^m(M)$, one can define the volume of M,

$$\mathrm{vol}(M) = \int_M \Gamma.$$

Here the orientation used in the definition of the integral is taken to be the orientation given by Γ. Thus $\mathrm{vol}(M) > 0$. By the discussion around Theorem 8.14, this means that Γ cannot be exact, and so represents a non-trivial cohomology class. The compactness of M is essential here: For instance, $\mathrm{d}x$ is an exact volume form on the real line $\mathbb{R}$.

A

Background material

A.1 Notions from set theory

A.1.1 Countability

A set X is *countable* if it is either finite (possibly empty), or there exists a bijective map $f: \mathbb{N} \to X$. We list some basic facts about countable sets:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable, $\mathbb{R}$ is not countable.
- If X_1, X_2 are countable, then the cartesian product $X_1 \times X_2$ is countable.
- If X is countable, then any subset of X is countable.
- If X is countable, and $f: X \to Y$ is surjective, then Y is countable.
- If $(X_i)_{i \in I}$ are countable sets, indexed by a countable set I, then the (disjoint) union $\bigsqcup_{i \in I} X_i$ is countable.

A.1.2 Equivalence relations

A *relation* from a set X to a set Y is simply a subset

$$R \subseteq Y \times X.$$

We write $x \sim_R y$ if and only if $(y,x) \in R$. When R is understood, we write $x \sim y$. If $Y = X$ we speak of a *relation on* X.

Example A.1. Any map $f: X \to Y$ defines a relation, given by its *graph*

$$\mathrm{graph}(f) = \{(f(x),x) | x \in X\}.$$

In this sense relations are generalizations of maps; for example, they are often used to describe 'multi-valued' maps.

Remark A.2. Given another relation $S \subseteq Z \times Y$, one defines a composition $S \circ R \subseteq Z \times X$, where

$$S \circ R = \{(z,x) |\ \exists y \in Y:\ (z,y) \in S,\ (y,x) \in R\}.$$

Our conventions are set up in such a way that if $f: X \to Y$ and $g: Y \to Z$ are two maps, then $\mathrm{graph}(g \circ f) = \mathrm{graph}(g) \circ \mathrm{graph}(f)$.

Example A.3. On the set $X = \mathbb{R}$ we have relations $\geq, >, <, \leq, =$. But there is also the relation defined by the condition $x \sim x' \Leftrightarrow x' - x \in \mathbb{Z}$, and many others.

A relation $\sim$ on a set X is called an *equivalence relation* if it has the following properties,

1. Reflexivity: $x \sim x$ for all $x \in X$,
2. Symmetry: $x \sim y \Rightarrow y \sim x$,
3. Transitivity: $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Given an equivalence relation, we define the *equivalence class* of $x \in X$ (using the notation $[\cdot]$) to be the subset

$$[x] = \{y \in X \,|\, x \sim y\}.$$

Note that X is a disjoint union of its equivalence classes. We denote by $X/\sim$ the set of equivalence classes. That is, all the elements of a given equivalence class are lumped together and represent a single element of $X/\sim$. One defines the *quotient map*

$$q:\ X \to X/\sim, \quad x \mapsto [x].$$

By definition, the quotient map is surjective.

Remark A.4. There are two other useful ways to think of equivalence relations:

- An equivalence relation R on X amounts to a decomposition $X = \bigsqcup_{i \in I} X_i$ as a disjoint union of subsets. Given R, one takes X_i to be the equivalence classes; given the decomposition, one defines $R = \{(y,x) \in X \times X | \exists i \in I:\ x, y \in X_i\}$.
- An equivalence relation amounts to a surjective map $q:\ X \to Y$. Indeed, given R one takes $Y := X/\sim$ with q the quotient map; conversely, given q one defines $R = \{(y,x) \in X \times X |\ q(x) = q(y)\}$.

Remark A.5. Often, we will not write out the entire equivalence relation. For example, if we say "*the equivalence relation on S^2 given by $x \sim -x$*", then it is understood that we also have $x \sim x$, since reflexivity holds for any equivalence relation. Similarly, when we say "*the equivalence relation on $\mathbb{R}$ generated by $x \sim x+1$*", it is understood that we also have $x \sim x+2$ (by transitivity: $x \sim x+1 \sim x+2$) as well as $x \sim x-1$ (by symmetry), hence $x \sim x+k$ for all $k \in \mathbb{Z}$. (Any relation $R_0 \subseteq X \times X$ extends to a unique smallest equivalence relation R; one says that R is the equivalence relation generated by R_0.)

Example A.6. Consider the equivalence relation on S^2 given by

$$(x,y,z) \sim (-x,-y,-z).$$

The equivalence classes are pairs of antipodal points; they are in 1-1 correspondence with lines in $\mathbb{R}^3$. That is, the quotient space $S^2/\sim$ is naturally identified with $\mathbb{R}\mathrm{P}^2$.

Example A.7. The quotient space $\mathbb{R}/\sim$ for the equivalence relation $x \sim x+1$ on $\mathbb{R}$ is naturally identified with S^1. If we think of S^1 as a subset of $\mathbb{R}$, the quotient map is given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

Example A.8. Similarly, the quotient space for the equivalence relation on $\mathbb{R}^2$ given by $(x,y) \sim (x+k, y+l)$ for $k,l \in \mathbb{Z}$ is the 2-torus T^2.

Example A.9. Let E be a k-dimensional real vector space. Given two ordered bases $(e_1,\ldots,e_k)$ and $(e'_1,\ldots,e'_k)$, there is a unique invertible linear transformation $A\colon E \to E$ with $A(e_i) = e'_i$. The two ordered bases are called *equivalent* if $\det(A) > 0$. One checks that equivalence of bases is an equivalence relation. There are exactly two equivalence classes; the choice of an equivalence class is called an *orientation* on E. For example, $\mathbb{R}^n$ has a standard orientation defined by the standard basis $(e_1,\ldots,e_n)$. The opposite orientation is defined, for example, by $(-e_1,e_2,\ldots,e_n)$. A permutation of the standard basis vectors defines the standard orientation if and only if the permutation is even (see below).

A.2 Permutations

Let X be a set with $n < \infty$ elements. A *permutation* of X is the same as an invertible map $s\colon X \to X$. The set of permutations form a group, with product the composition of permutations, and with identity element the trivial permutation. A permutation interchanging two elements of X, while fixing all other elements, is called a *transposition*.

By choosing an enumeration of the elements of X, we may assume

$$X = \{1,\ldots,n\};$$

the corresponding group is denoted S_n. For $i < j$, we denote by t_{ij} the transposition of the indices i,j (leaving all others fixed).
It is standard practice to denote a permutation $s(1) = i_1, \ldots, s(n) = i_n$ by a symbol*

$$(i_1,\ i_2,\ \cdots, i_n).$$

Alternatively, one can simply list where the elements map to, e.g.

$$1 \to i_1,\ 2 \to i_2, \ldots,\ n \to i_n.$$

Example A.10. The notation

$$(2,4,1,3) \to (3,2,4,1)$$

* Overloading the parenthesis $(\cdot,\cdot)$ notation. One uses context to distinguish between a permutation, an ordered tuple, and an open interval.

signifies the permutation $s(2)=3,\ s(4)=2,\ s(3)=1,\ s(1)=4$. After listing the elements in the proper order $s(1)=4,\ s(2)=3,\ s(3)=1,\ s(4)=2$, it is thus described by the symbol

$$(4,3,1,2).$$

By induction, one may prove that every permutation is a product of transpositions. In fact, it is enough to consider transpositions of *adjacent* elements, i.e., those of the form $t_{i\ i+1}$.

Example A.11. For $s=(4,3,1,2)$, use the following transpositions to put s back to the original position:

$$(4,3,1,2) \to (4,1,3,2) \to (1,4,3,2) \to (1,4,2,3) \to (1,2,4,3) \to (1,2,3,4).$$

Reversing arrows, this shows how to write s as a product of five transpositions of adjacent elements: $s = t_{13}\, t_{14}\, t_{23}\, t_{24}\, t_{34}$.

A permutation $s \in S_n$ of $\{1,\dots,n\}$ is called *even* if the number of pairs (i,j) such that $i<j$ but $s(i)>s(j)$ is even, and is called *odd* if the number of such 'wrong order' pairs is odd. In particular, every transposition is odd.

Example A.12. Let $s \in S_4$ be the permutation $s=(4,3,1,2)$ has five pairs of indices in the wrong order,

$$(4,3),(4,1),(4,2),(3,1),(3,2).$$

Hence, s is odd.

Of course, computing the sign by listing all pairs in the wrong order can be cumbersome. Fortunately, there are much simpler ways of funding the parity. Define a map

$$\mathrm{sign}:\ S_n \to \{1,-1\}$$

by setting $\mathrm{sign}(s)=1$ if the permutation is even, $\mathrm{sign}(s)=-1$ if the permutation is odd. View $\{1,-1\}$ as a group, with product the multiplication.

Theorem A.13. *The map* $\mathrm{sign}:\ S_n \to \{1,-1\}$ *is a group homomorphism. That is,* $\mathrm{sign}(s's)=\mathrm{sign}(s')\mathrm{sign}(s')$ *for all* $s,s' \in S_n$.

Proof (Sketch). This may be proved by examining the effect of precomposing a given permutation s with a transposition $t_{i,i+1}$ of two adjacent elements. If $i,i+1$ were a 'right order' pair for s, then they will be a 'wrong order' pair for $\tilde{s}$; the relative order for all other pairs remains unchanged. Consequently, the signs of $\tilde{s}$ and s are opposite. It follows by induction that if s can be written as a product of N transpositions of adjacent elements, then $\mathrm{sign}(s)=(-1)^N$. A similar reasoning applies to s', so that $\mathrm{sign}(s')=(-1)^{N'}$. The expressions for s,s' as products of transpositions of adjacent elements gives another such expression for $s's$, involving $N+N'$ transpositions.

A simple consequence is that $\mathrm{sign}(s)=(-1)^N$ whenever s is a product of N transpositions (not necessarily adjacent ones).

Example A.14. We saw that $s = (4,3,1,2)$ is a product of five transpositions of adjacent elements, hence $\operatorname{sign}(s) = (-1)^5 = -1$. But if we use general transpositions, we only need three steps to put $(4,3,1,2)$ in the initial position:

$$(4,3,1,2) \to (1,3,4,2) \to (1,2,4,3) \to (1,2,3,4)$$

We once again see that $\operatorname{sign}(s) = (-1)^3 = -1$.

The permutation group, and the sign function, appear in the formula for the determinant of an $n \times n$-matrix A with entries A_{ij}:

$$\det(A) = \sum_{s \in S_n} \operatorname{sign}(s) A_{s(1)1} \cdots A_{s(n)n}$$

A.3 Algebras

A.3.1 Definition and examples

An *algebra* (over the field $\mathbb{R}$ of real numbers) is a vector space $\mathscr{A}$, together with a *multiplication* (product) $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$, $(a,b) \mapsto ab$ such that

1. The multiplication is associative: That is, for all $a,b,c \in \mathscr{A}$
$$(ab)c = a(bc).$$
2. The multiplication map is linear in both arguments: That is,
$$(\lambda_1 a_1 + \lambda_2 a_2)b = \lambda_1(a_1 b) + \lambda_2(a_2 b),$$
$$a(\mu_1 b_1 + \mu_2 b_2) = \mu_1(ab_1) + \mu_2(ab_2),$$
for all $a, a_1, a_2, b, b_1, b_2 \in \mathscr{A}$ and all scalars $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

The algebra is called *commutative* if $ab = ba$ for all $a,b \in \mathscr{A}$. A *unital algebra* is an algebra $\mathscr{A}$ with a distinguished element $1_{\mathscr{A}} \in \mathscr{A}$ (called the *unit*), with

$$1_{\mathscr{A}} a = a = a 1_{\mathscr{A}}$$

for all $a \in \mathscr{A}$.

Remark A.15. One can also consider non-associative product operations on vector spaces, most importantly one has the class of *Lie algebras*. If there is risk of confusion with these or other concepts, we may refer to *associative algebras*.

Remark A.16. One can also consider algebras over other fields.

Example A.17. The space $\mathbb{C}$ of complex numbers (regarded as a real vector space $\mathbb{R}^2$) is a unital, commutative algebra, containing $\mathbb{R} \subseteq \mathbb{C}$ as a subalgebra.

Example A.18. A more sophisticated example is the algebra $\mathbb{H} \cong \mathbb{R}^4$ of quaternions, which is a unital non-commutative algebra. Elements of $\mathbb{H}$ are written as

$$x = a + ib + jc + kd$$

with $a, b, c, d \in \mathbb{R}$; here i, j, k are just formal symbols. The multiplication of quaternions is specified by the rules

$$i^2 = j^2 = k^2 = -1, \;\; ij = k = -ji, \;\; jk = i = -kj, \;\; ki = j = -ik.$$

Elements of the form $z = a + ib$ form a subalgebra of $\mathbb{H}$ isomorphic to $\mathbb{C}$. The norm of a quaternion is defined by $|x| = \sqrt{a^2 + b^2 + c^2 + d^2}$; it has the properties $|x_1 + x_2| \leq |x_1 + x_2|$ and $|x_1 x_2| = |x_1||x_2|$. The algebra of quaternions may also be described as an algebra of complex 2×2-matrices of the form

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

with $z, w \in \mathbb{C}$, with the algebra multiplication given by the multiplication of matrices.

Example A.19. For any n, the space $\mathrm{Mat}_{\mathbb{R}}(n)$ of $n \times n$ matrices, with product the matrix multiplication, is a non-commutative unital algebra. One can also consider matrices with coefficients in $\mathbb{C}$, denoted $\mathrm{Mat}_{\mathbb{C}}(n)$, or in fact with coefficients in any given algebra.

Example A.20. For any set X, the space of functions $f : \; X \to \mathbb{R}$ is a unital commutative algebra, where the product is given by pointwise multiplication. Given a topological space X, one has the unital algebra $C(X)$ of *continuous* $\mathbb{R}$-valued functions. IF X is non-compact, this has a (non-unital) subalgebra $C_0(X)$ of continuous functions vanishing outside a compact set.

A.3.2 Homomorphisms of algebras

A *homomorphism of algebras* $\Phi : \; \mathscr{A} \to \mathscr{A}'$ is a linear map preserving products:

$$\Phi(ab) = \Phi(a)\Phi(b).$$

(For a homomorphism of unital algebras, one asks in addition that $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{A}'}$.) It is called an *isomorphism of algebras* if Φ is invertible. For the special case $\mathscr{A}' = \mathscr{A}$, these are also called *algebra automorphisms* of $\mathscr{A}$. Note that the algebra automorphisms form a group under composition.

Example A.21. Consider $\mathbb{R}^2$ as an algebra, with product coming from the identification $\mathbb{R}^2 = \mathbb{C}$. The complex conjugation $z \mapsto \overline{z}$ defines an automorphism $\Phi : \; \mathbb{R}^2 \to \mathbb{R}^2$ of this algebra.

Example A.22. The algebra $\mathbb{H}$ of quaternions has an automorphism given by cyclic permutation of the three imaginary units:

$$\Phi(x + iu + jv + kw) = x + ju + kv + iw.$$

Example A.23. Let $\mathscr{A} = \mathrm{Mat}_{\mathbb{R}}(n)$ the algebra of $n \times n$-matrices. If $U \in \mathscr{A}$ is invertible, then $X \mapsto \Phi(X) = UXU^{-1}$ is an algebra automorphism.

Example A.24. Suppose $\mathscr{A}$ is a unital algebra. Let $\mathscr{A}^{\times}$ be the set of invertible elements, that is, elements $u \in \mathscr{A}$ for which there exists $v \in \mathscr{A}$ with $uv = vu = 1_{\mathscr{A}}$. Given u, such v is necessarily unique (write $v = u^{-1}$), and the map $\mathscr{A} \to \mathscr{A},\ a \mapsto uau^{-1}$ is an algebra automorphism. Such automorphisms are called *inner*.

A.3.3 Derivations of algebras

Definition A.25. *A* derivation *of an algebra $\mathscr{A}$ is a linear map $D:\ \mathscr{A} \to \mathscr{A}$ satisfying the product rule*

$$D(a_1a_2) = D(a_1)a_2 + a_1D(a_2).$$

If $\dim \mathscr{A} < \infty$, then a derivation may be regarded as an *infinitesimal automorphism* of an algebra. Indeed, let $U:\ \mathbb{R} \to \mathrm{End}(\mathscr{A}),\ t \mapsto U_t$ be a smooth curve with $U_0 = I$, such that each U_t is an algebra automorphism. Consider the Taylor expansion,

$$U_t = I + tD + \ldots$$

here

$$D = \frac{d}{dt}\Big|_{t=0} U_t$$

is the velocity vector at $t = 0$. By taking the derivative of the condition

$$U_t(a_1a_2) = U_t(a_1)U_t(a_2)$$

at $t = 0$, we get the derivation property for D. Conversely, if D is a derivation, then

$$U_t = \exp(tD) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n$$

(using the exponential of a matrix) is a well-defined curve of algebra automorphisms. We leave it as an exercise to check the automorphism property; it involves proving the property

$$D^n(a_1a_2) = \sum_k \binom{n}{k} D^k(a_1)\, D^{n-k}(a_2)$$

for all $a_1, a_2 \in \mathscr{A}$.
If $\mathscr{A}$ has infinite dimensions, one may still want to think of derivations D as infinitesimal automorphisms, even though the discussion will run into technical problems. (For instance, the exponential map of infinite rank endomorphisms is not well-defined in general.)
A collection of facts about derivations of algebras $\mathscr{A}$:

1. Any given $x \in \mathscr{A}$ defines a derivation

$$D(a) = [x,a] := xa - ax.$$

(Exercise: Verify that this is a derivation.) These are called *inner derivations*. If $\mathscr{A}$ is commutative (for example $\mathscr{A} = C^\infty(M)$) the inner derivations are all trivial. At the other extreme, for the matrix algebra $\mathscr{A} = \mathrm{Mat}_{\mathbb{R}}(n)$, one may show that every derivation is inner.
2. If $\mathscr{A}$ is a unital algebra, with unit $1_{\mathscr{A}}$, then $D(1_{\mathscr{A}}) = 0$ for all derivations D. (This follows by applying the defining property of derivations to $1_{\mathscr{A}} = 1_{\mathscr{A}} 1_{\mathscr{A}}$.)
3. Given two derivations D_1, D_2 of an algebra $\mathscr{A}$, their commutator (using the $[\cdot,\cdot]$ notation)

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

is again a derivation. Indeed, if $a, b \in \mathscr{A}$ then

$$\begin{aligned} D_1 D_2(ab) &= D_1\big(D_2(a)b + aD_2(b)\big) \\ &= (D_1D_2)(a)b + a(D_1D_2)(b) + D_1(a)D_2(b) + D_2(a)D_1(b). \end{aligned}$$

Subtracting a similar expression with indices 1 and 2 interchanged, one obtains the derivation property of $[D_1, D_2]$.

A.3.4 Modules over algebras

Definition A.26. *A (left)* module *over an algebra $\mathscr{A}$ is a vector space $\mathscr{E}$ together with a map (module action) $\mathscr{A} \times \mathscr{E} \to \mathscr{E}$, $(a,x) \mapsto ax$ such that*

1. For $a, b \in \mathscr{A}$ and $x \in \mathscr{E}$,

$$(ab)x = a(bx).$$

2. The module action is linear in both arguments: That is,

$$(\lambda_1 a_1 + \lambda_2 a_2)x = \lambda_1(a_1 x) + \lambda_2(a_2 x),$$

$$a(\mu_1 x_1 + \mu_2 x_2) = \mu_1(ax_1) + \mu_2(ax_2),$$

for all $a, a_1, a_2 \in \mathscr{A}$, $x, x_1, x_2 \in \mathscr{E}$, and all scalars $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

1. Every algebra $\mathscr{A}$ is a module over itself, with the module action given by algebra multplication from the left.
2. If the algebra $\mathscr{A}$ is commutative, then the space of derivations is a module over $\mathscr{A}$: if D is a derivation and $x \in \mathscr{A}$ then $a \mapsto (xD)(a) := xD(a)$ is again a derivation:

$$(xD)(ab) = x(D(ab)) = x(D(a)b + a(D(b)) = (xD)(a)b + a(xD)(b),$$

where we used $xa = ax$.

A.4 Dual spaces and quotient spaces

Let E be a vector space over a field $\mathbb{F}$. (We mostly have in mind the cases $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.) The *dual space* is the space of linear functionals $\varphi : E \to \mathbb{F}$,

$$E^* = L(E, \mathbb{F}),$$

it is again a vector space over $\mathbb{F}$. If $n = \dim E < \infty$, and $v_1, \ldots, v_n$ is a basis of E, then the dual space E^* has a basis $\varphi^1, \ldots, \varphi^n$ such that $\varphi^i(v_j) = \delta^i_j$ for all i, j. In particular, $\dim E^* = \dim E$, and in particular E and E^* are isomorphic, for example by the map taking v_i to φ^i. Note however that there is no *canonical* isomorphism: the isomorphism just described depends on the choice of a basis, and for a general E it is not possible to describe an isomorphism $E \to E^*$ without making extra choices. Furthermore, if $\dim E = \infty$ is then E and E^* are in fact *not* isomorphic; intuitively, E^* is 'more' infinite-dimensional than E. For instance, if E has a countable (infinite) basis then E^* does not admit a countable basis.

For any vector space E, any $v \in E$ defines a linear functional on the dual space E^*, given by evaluation:

$$\mathrm{ev}_v : E^* \to \mathbb{F}, \;\; \varphi \mapsto \varphi(v).$$

This defines a canonical linear map

$$E \to E^{**} = (E^*)^*, \quad v \mapsto \mathrm{ev}_v.$$

This map is injective; hence, if $\dim E < \infty$ it is an isomorphism.

Suppose that $E' \subseteq E$ is a subspace. Define an equivalence relation on E, where $v_1 \sim v_2 \Leftrightarrow v_1 - v_2 \in E'$. The set of equivalence classes is called the *quotient space*, and is denoted by

$$E/E' = \{[v] \in E\}.$$

It has a unique vector space structure in such a way that the quotient map $E \to E/E'$ is linear; specifically, $[v_1] + [v_2] = [v_1 + v_2]$ for $v_1, v_2 \in E$ and $\lambda[v] = [\lambda v]$ for $v \in E$ and $\lambda \in \mathbb{F}$. Note that the quotient map $E \to E/E'$ is surjective, with kernel (null space) equal to E'; conversely, if $E \to E''$ is any surjective linear map with kernel E', then E'' is canonically isomorphic to E/E'.

The subspace $E' \subseteq E$ also determines a subspace of the dual space, namely its *annihilator*

$$\mathrm{ann}(E') = \{\varphi \in E^* | \; \varphi(v) = 0 \text{ for all } v \in E'\}.$$

(Also common are notations such as $(E')^0$, or also $\mathrm{ann}_{E^*}(E')$ to indicate the ambient space.) For $\varphi \in \mathrm{ann}(E')$, one obtains a linear functional on E/E', by

$$E/E' \to \mathbb{F}, \;\; [v] \mapsto \varphi(v).$$

This is well-defined exactly because φ vanishes on E'. Conversely, given a linear functional on E/E', its composition with the quotient map $E \to E/E'$ is a linear functional on E vanishing on E'. This defines an isomorphism

$$(E/E')^* \xrightarrow{\cong} \mathrm{ann}(E').$$

Note that E, E' may be infinite-dimensional here; if they are finite-dimensional then we obtain, in particular,

$$\dim \mathrm{ann}(E') = \dim(E/E') = \dim E - \dim E'.$$

Furthermore, in this case, $\mathrm{ann}(\mathrm{ann}(E')) = E'$ under the identification of E^{**} with E.

Consider next the situation that E comes equipped with a symmetric bilinear form $\beta:\ E\times E\to\mathbb{F}$. That is, β is linear in each argument, leaving the other fixed, and $\beta(v_1,v_2)=\beta(v_2,v_1)$ for all v_1,v_2. Let us assume for the remainder of this subsection that $\dim E<\infty$. Then β is called *non-degenerate* if it has the property that for all $v\in V$, one has $\beta(v,w)=0$ for all $w\in E$ only if $v=\mathbf{0}$. Equivalently, the linear map $\beta^\flat:\ E\to E^*,\ v\mapsto\beta(v,\cdot)$ is an isomorphism. Hence, a non-degenerate bilinear form gives a concrete isomorphism between E and E^*. For a subspace E' we define the orthogonal subspace as

$$(E')^\perp = \{v\in E|\ \beta(v,w)=0 \text{ for all } w\in E'\}.$$

The isomorphism $\beta^\flat:\ E\to E^*$ restricts to an isomorphism $(E')^\perp\to\mathrm{ann}(E')$; in particular, $\dim(E')^\perp=\dim E-\dim E'$ and $(E')^{\perp\perp}=E'$.
If $\mathbb{F}=\mathbb{R}$ and if the bilinear form β is positive definite (i.e. $\beta(v,v)>0$ for $v\neq\mathbf{0}$) then $(E')^\perp\cap E'=\{\mathbf{0}\}$, and one often refers to $(E')^\perp$ as the *orthogonal complement*. However, in more general situations the space $(E')^\perp$ need not be a complement to E'.
As a final remark, note that a similar discussion goes through for nondegenerate *skew-symmetric* bilinear forms ω, i.e. such that $\omega(v_1,v_2)=-\omega(v_2,v_1)$ and the map $\omega^\flat:\ E\to E^*$ is an isomorphism. In particular, for $E'\subseteq E$ one can define the ω-orthogonal space $(E')^\omega$, and one has $(E')^{\omega\omega}=E'$.

B

Topology of manifolds

B.1 Topological notions

A *topological space* is a set X together with a collection of subsets $U \subseteq X$ called *open subsets* , with the following properties:

- $\emptyset, X$ are open.
- If U, U' are open then $U \cap U'$ is open.
- For any collection $\{U_\alpha\}$ of open subsets, the union $\bigcup_\alpha U_\alpha$ is open.

The collection of open subsets is called the *topology* of X. In the third condition, the index set need not be finite, or even countable.
The space $\mathbb{R}^n$ has a standard topology given by the usual open subsets. Likewise, the open subsets of a manifold M define a topology on M. For any set X, one has the *trivial topology* where the only open subsets are $\emptyset$ and X, and the *discrete topology* where every subset is considered open. An *open neighborhood* of a point p is an open subset containing it. A topological space is called *Hausdorff* of any two distinct points have disjoint open neighborhoods.
Let X be a topological space. Then any subset $A \subseteq X$ has a *subspace topology*, with open sets the collection of all intersections $U \cap A$ such that $U \subseteq X$ is open. Given a surjective map $q: X \to Y$, the space Y inherits a *quotient topology*, whose open sets are all $V \subseteq Y$ such that the pre-image $q^{-1}(V) = \{x \in X \mid q(x) \in V\}$ is open.
A subset A is *closed* if its complement $X \backslash A$ is open. Dual to the statements for open sets, one has

- $\emptyset, X$ are closed.
- If A, A' are closed then $A \cup A'$ is closed.
- For any collection $\{A_\alpha\}$ of closed subsets, the intersection $\bigcap_\alpha A_\alpha$ is closed.

For any subset A, denote by $\overline{A}$ its *closure*, given as the smallest closed subset containing A.

B.2 Manifolds are second countable

A *basis* for the topology on X is a collection $\mathscr{B} = \{U_\alpha\}$ of open subsets of X such that every U is a union from sets from $\mathscr{B}$. (Equivalently, for all open U and all $p \in U$ there exists α such that $p \in U_\alpha \subseteq U$.) In contrast to the notion of basis of a vector space, this collection does not have to be minimal in any sense; for instance, the collection of *all* open subsets of a topological space is a basis.

Example B.1. Let $X = \mathbb{R}^n$. Then the collection of all open balls $B_\varepsilon(x)$, with $\varepsilon > 0$ and $x \in \mathbb{R}^n$, is a basis for the topology on $\mathbb{R}^n$.

A topological space is said to be *second countable* if its topology has a countable basis.

Proposition B.2. $\mathbb{R}^n$ *is second countable.*

Proof. A countable basis is given by the collection of all *rational balls* (cf. ✎ 47), by which we mean ε-balls $B_\varepsilon(x)$ such that $x \in \mathbb{Q}^m$ and $\varepsilon \in \mathbb{Q}_{>0}$. To check that it is a basis, let $U \subseteq \mathbb{R}^m$ be open, and $p \in U$. Choose $\varepsilon \in \mathbb{Q}_{>0}$ such that $B_{2\varepsilon}(p) \subseteq U$. There exists a rational point $x \in \mathbb{Q}^n$ with $||x - p|| < \varepsilon$. This then satisfies $p \in B_\varepsilon(x) \subseteq U$. Since p was arbitrary, this proves the claim.

The same reasoning shows that for any open subset $U \subseteq \mathbb{R}^m$, the rational ε-balls that are contained in U form a basis of the topology of U.

Proposition B.3. *Manifolds are second countable.*

Proof. Given a manifold M, let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a countable atlas. Then the set of all $\varphi_\alpha^{-1}(B_\varepsilon(x))$, where $B_\varepsilon(x)$ is a rational ball contained in $\varphi_\alpha(U_\alpha)$, is a countable basis for the topology of M. Indeed, any open subset U is a countable union over all $U \cap U_\alpha$, and each of these intersections is a countable union over all $\varphi_\alpha^{-1}(B_\varepsilon(x))$ such that $B_\varepsilon(x)$ is a rational ε-ball contained in $U \cap U_\alpha$. $\square$

B.3 Manifolds are paracompact

A collection $\{U_\alpha\}$ of open subsets of X is called an *open covering* of $A \subseteq X$ if $A \subseteq \bigcup_\alpha U_\alpha$. Consider the case $A = X$. A *refinement* of an open cover $\{U_\alpha\}$ of X is an open cover $\{V_\beta\}$ of X such that each V_β is contained in some U_α. It is a *subcover* if each V_β's is equal to some U_α.

A topological space X is called *compact* if every open cover of X has a finite subcover. A topological space is called *paracompact* if every open cover $\{U_\alpha\}$ has a locally finite refinement $\{V_\beta\}$: that is, every point has an open neighborhood meeting only finitely many V_β's.

Proposition B.4. *Manifolds are paracompact.*

We will need the following auxiliary result.

Lemma B.5. *For any manifold M, there exists a sequence of open subsets $W_1, W_2, \ldots$ of M such that*

$$\bigcup W_i = M,$$

and such that each W_i has compact closure with $\overline{W_i} \subseteq W_{i+1}$.

Proof. Start with a a countable open cover $O_1, O_2, \ldots$ of M such that each O_i has compact closure $\overline{O}_i$. (We saw in the proof of Proposition B.3 how to construct such a cover, by taking pre-images of ε-balls in coordinate charts.) Replacing O_i with $O_1 \cup \cdots \cup O_i$ we may assume $O_1 \subseteq O_2 \subseteq \cdots$. For each i, the covering of the compact set $\overline{O}_i$ by the collection of all O_j's admits a finite subcover. Since the sequence of O_j's is nested, this just means $\overline{O}_i$ is contained in O_j for j sufficiently large. We can thus define $W_1, W_2, \ldots$ as a subsequence $W_i = O_{j(i)}$, starting with $W_1 = O_1$, and inductively letting $j(i)$ for $i > 1$ be the smallest index $j(i)$ such that $\overline{W}_{i-1} \subseteq O_{j(i)}$.

Proof (of Proposition B.4). Let $\{U_\alpha\}$ be an open cover of M. Let W_i be a sequence of open sets as in Lemma B.5. For every i, the compact subset $\overline{W}_{i+1} \backslash W_i$ is contained in the open set $W_{i+2} \backslash \overline{W}_{i-1}$, hence it is covered by the collection of open sets

$$(W_{i+2} \backslash \overline{W}_{i-1}) \cap U_\alpha. \tag{B.1}$$

By compactness, $\overline{W}_{i+1} \backslash W_i$ is already covered by finitely many of the subsets (B.1). Let $\mathscr{V}^{(i)}$ be this finite collection, and $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}^{(i)}$ the union. Then

$$\mathscr{V} = \{V_\beta\}$$

is the desired countable open cover of M. Indeed, if $V_\beta \in \mathscr{V}^{(i)}$, then $V_\beta \cap W_{i-1} = \emptyset$. That is, a given W_i meets only V_β's from $\mathscr{V}^{(k)}$ with $k \leq i$. Since these are finitely many V_β's, it follows that the cover $\mathscr{V} = \{V_\beta\}$ is locally finite. $\square$

Remark B.6. (See Lang [11], page 35.) One can strengthen the result a bit, as follows: Given a cover $\{U_\alpha\}$, we can find a refinement to a cover $\{V_\beta\}$ such that each V_β is the domain of a coordinate chart (V_β, ψ_β), with the following extra properties, for some $0 < r < R$:

(i) $\psi_\beta(V_\beta) = B_R(0)$, and
(ii) M is already covered by the smaller subsets $V'_\beta = \psi_\beta^{-1}(B_r(0))$.

To prove this, we modify the second half of the proof as follows: For each $p \in \overline{W}_{i+1} \backslash W_i$ choose a coordinate chart (V_p, ψ_p) such that $\psi_p(p) = 0$, $\psi_p(V_p) = B_R(0)$, and $V_p \subseteq (W_{i+2} \backslash \overline{W}_{i-1}) \cap U_\alpha$. Let $V'_p \subseteq V_p$ be the pre-image of $B_r(0)$. The V'_p cover $\overline{W}_{i+1} \backslash W_i$; let $\mathscr{V}^{(i)}$ be a finite subcover and proceed as before. This remark is useful for the construction of partitions of unity.

B.4 Partitions of unity

Let M be a manifold. The *support* $\mathrm{supp}(f)$ of a function $f: M \to \mathbb{R}$ is the smallest closed subset such that f vanishes on $M \backslash \mathrm{supp}(f)$. Equivalently, $p \in M \backslash \mathrm{supp}(f)$ if and only if f vanishes on some open neighborhood of p.

Definition B.7. *A* partition of unity subordinate to an open cover $\{U_\alpha\}$ *of a manifold M is a collection of smooth functions $\chi_\alpha \in C^\infty(M)$, with $0 \le \chi_\alpha \le 1$, such that* $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$*, and*

$$\sum_\alpha \chi_\alpha = 1.$$

Proposition B.9 below states that every open cover admits a partition of unity. To prove it, we will need the following result from multivariable calculus.

Lemma B.8 (Bump functions). *For all $0 < r < R$, there exists a function $f \in C^\infty(\mathbb{R}^m)$, with* $\mathrm{supp}(f) \subseteq B_R(0)$*, such that $f(\mathbf{x}) = 1$ for $||\mathbf{x}|| \le r$.*

[PICTURE]

Proof. Recall that the function

$$h(t) = \begin{cases} 0 & \text{if } t \le 0, \\ \exp(-1/t) & \text{if } t > 0 \end{cases}$$

is smooth even at $t = 0$. Choose $R_1 \in \mathbb{R}$ with $r < R_1 < R$. We claim that

$$f(\mathbf{x}) = 1 - \frac{h(||\mathbf{x}|| - r)}{h(||\mathbf{x}|| - r) + h(R_1 - ||\mathbf{x}||)}$$

is well-defined, and has the desired properties. Indeed, for $||\mathbf{x}|| \le r$ we have that $h(R_1 - ||\mathbf{x}||) > 0$ while $h(||\mathbf{x}|| - r) = 0$, hence $f(\mathbf{x}) = 1$. For $||\mathbf{x}|| > r$ we have that $h(||\mathbf{x}|| - r) > 0$, hence the denominator is > 0 and the expression is well-defined. Finally, if $||\mathbf{x}|| \ge R_1$ we have that $h(R_1 - ||\mathbf{x}||) = 0$, hence the enumerator becomes equal to the denominator and hence $f(\mathbf{x}) = 0$. □

Proposition B.9. *For any open cover $\{U_\alpha\}$ of a manifold, there exists a partition of unity $\{\chi_\alpha\}$ subordinate to that cover. One can take this partition of unity to be locally finite: That is, for any $p \in M$ there is an open neighborhood U meeting the support of only finitely many χ_α's.*

Proof. Let V_β be a locally finite refinement of the cover U_α, given as domains of coordinate charts (V_β, ψ_β) of the kind described in Remark B.6, and let $V'_\beta \subseteq V_\beta$ be as described there. Since the images of $V'_\beta \subseteq V_\beta$ are $B_r(0) \subseteq B_R(0)$, we can use Lemma B.8 to define a function $f_\beta \in C^\infty(M)$ with $\mathrm{supp}(f_\beta) \subseteq V_\beta$, and equal to 1 on the closure $\overline{V'_\beta}$. Since the collection of sets V_β is a locally finite cover, the sum $\sum_\beta f_\beta$ is well-defined (near any given point, only finitely many terms are non-zero). Since

already the smaller sets V'_β are a cover of M, and the f_β are > 0 on these sets, the sum is strictly positive everywhere.
For each index β, pick an index α such that $V_\beta \subseteq U_\alpha$. This defines a map $d:\ \beta \mapsto d(\beta)$ between the indexing sets. The functions

$$\chi_\alpha = \frac{\sum_{\beta\in d^{-1}(\alpha)} f_\beta}{\sum_\gamma f_\gamma}.$$

give the desired partition of unity: The support is in U_α (since each f_β in the enumerator is supported in U_α), and the sum over all χ_α's is equal to 1. Furthermore, the partition of unity is locally finite, since near any given point p only finitely many f_β's are nonzero. $\square$

An important application of partitions of unity is the following result, a weak version of the *Whitney embedding theorem*.

Theorem B.10. *Let M be a manifold admitting a finite atlas with r charts. Then there is an embedding of M as a submanifold of $\mathbb{R}^{r(m+1)}$.*

Proof. Let $\{(U_i,\varphi_i),\ i=1,\dots,r\}$ be a finite atlas for M, and $\chi_1,\dots,\chi_r$ a partition of unity subordinate to the cover by coordinate charts. Then the products $\chi_i\varphi_i:\ U_i \to \mathbb{R}^m$ extend by zero to smooth functions $\psi_i:\ M \to \mathbb{R}^m$. The map

$$F:\ M \to \mathbb{R}^{r(m+1)},\ p \mapsto (\psi_1(p),\dots,\psi_r(p),\chi_1(p)\dots,\chi_r(p))$$

is the desired embedding. Indeed, F is injective: if $F(p)=F(q)$, choose i with $\chi_i(p)>0$. Then $\chi_i(q)=\chi_i(p)>0$, hence both $p,q\in U_i$, and the condition $\psi_i(p)=\psi_i(q)$ gives $\varphi_i(p)=\varphi_i(q)$, hence $p=q$. Similarly T_pF is injective: For $v\in T_pM$ in the kernel of T_pF, choose i such that $\chi_i(p)>0$, thus $v\in T_pU_i$. Then v being in the kernel of $T_p\psi_i$ and of $T_p\chi_i$ implies that it is in the kernel of $T_p\varphi_i$, hence $v=0$ since φ_i is a diffeomorphism. This shows that we get an injective immersion, we leave it as an exercise to verify that the image is a submanifold (e.g., by constructing submanifold charts).

The theorem applies in particular to all compact manifolds. Actually, one can show that *all* manifolds admit a finite atlas; for a proof see, e.g., the book [8]. Hence, every manifold can be realized as a submanifold of Euclidean space.

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List of Symbols