MAT 367: Differential Geometry
Assignment #7
Due on Friday August 13, 2021 by 11:59 pm

Note: This assignment covers material from the lectures on Cartan Calculus, Integration on manifolds.

Problem #1

Recall that we extended the definition of closed and exact to k-forms. $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$, and is exact if there $\eta \in \Omega^k(M)$ such that $\omega = d\eta$.

We say that a smooth manifold M is smoothly contractible to a point $p_0 \in M$ if there exists a C^∞ map $H : M \times [0,1] \to M$ satisfying $H(p,1) = p$ and $H(p,0) = p_0$ for all $p \in M$. Any open star convex subset of $\mathbb{R}^n$ is an example of such a manifold. Poincaré lemma asserts that if M is smoothly contractible to a point, then every closed form is exact. [1]

(a) Let $\omega = \frac{1}{2}\sum_{i,j} \omega_{ij} dx^i \wedge dx^j \in \Omega^2(\mathbb{R}^n)$ be a 2-form where $\omega_{ij} = -\omega_{ji}$. State explicitly under what condition on ω_{ij} is ω closed. If so, show that $\omega = d\eta$ where

$$\eta = \sum_{i,j} \left(\int_0^1 t\omega_{ij}(tx^1, ..., tx^n)dt \right) x^i dx^j$$

(b) ***(bonus)*** Let $\omega \in \Omega^k(M)$ be a closed form where $k > 0$, and let $p \in M$. Show there exists a closed form $\eta \in \Omega^k(M)$ that vanishes near p such that $\omega - \eta$ is exact.

(c) * Is S^1 contractible to a point? (check the form $i^*\omega$ from 4c in assignment 6)

(d) * Show that for any 1-form $\eta \in \Omega^1(S^1)$, there exists a unique $a \in \mathbb{R}$ such that $\eta - ai^*\omega$ is exact, where ω is the 1-form defined in problem 4c in assignment 6. This shows that every 1-form on S^1 is of the form $ai^*\omega +$ (some exact 1-form).

(e) * Define a relation on $\Omega^1(S^1)$ as follows: $\eta \sim \eta'$ if $\eta - \eta'$ is exact. Show that $\int_{S^1} \eta = 0$ if and only if η is exact and conclude that $\int_{S^1} : \Omega^1(S^1)/\sim \to \mathbb{R}$ is an isomorphism of vector spaces.

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[1] Check "A Comprehensive Introduction to Differential Geometry" by Spivak, page 221- 225

(f) Show that every closed 1-form $\omega \in \Omega^1(S^2)$ is exact.

Hint: if Consider the stereographic atlas $\{U_N, U_S\}$ from assignment 2 problem 1. Recall that both U_N and U_S are diffiomorphic to $\mathbb{R}^2$.

(g) Use the 2-form defined in problem 20.10 to argue that S^2 is not contractible to a point. (Find a closed but not exact 2-form on S^2).

Problem #2

In this problem, you will build an intuition for the exterior derivative and study some of its applications and interpretations.

The exterior derivative allows for the generalizations of notions studied in vector calculus. Let $f \in C^\infty(\mathbb{R}^n)$ and $X \in \mathfrak{X}(\mathbb{R}^n)$. Denote by $\nabla f, \nabla \times X$, and $\nabla \cdot X$ the gradient of f, curl of X (if $n = 3$) and divergence of X. If $n = 3$ and $X = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} + a^3 \frac{\partial}{\partial z}$, then define the forms

$$\omega_X = a^1 dx + a^2 dy + a^3 dz, \qquad \eta_X = a^1 dy \wedge dz + a^2 dz \wedge dx + a^3 dx \wedge dy$$

(a) *Show that $df = \omega_{\nabla f}$, $d\omega_X = \eta_{\nabla \times X}$ and $d\eta_X = \nabla \cdot X dx \wedge dy \wedge dz$. Conclude that $\nabla \times \nabla f = 0$ and $\nabla \cdot (\nabla \times X) = 0$.

(b) *Let $X \in \mathfrak{X}(U)$ where U is a star convex open set in $\mathbb{R}^3$. Show that if $\nabla \times X = 0$, then $X = \nabla f$ for some $f \in C^\infty(U)$. Similarly, if $\nabla \cdot X = 0$, then $X = \nabla \times Y$ for some $Y \in \mathfrak{X}(U)$.

Remark: This will stay true when we generalize the notion of divergence and curl in Riemannian manifolds.

(c) Solve problem 19.13. You will apply this to the modern formulation of Maxwell's equations.

(d) Prove that the exterior derivative is the only collection of $\mathbb{R}$ linear maps from $\Omega^k(M)$ to $\Omega^{k+1}(M)$ satisfying

- $d : \Omega^0(M) \to \Omega^1(M)$ is the differential.
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$.
- $d \circ F^* = F^* \circ d$ for any smooth map $F : M \to N$.

(e) ***(bonus)*** Let $\omega \in \Omega^k(\mathbb{R}^n)$ and let $v_1, ..., v_{k+1} \in T_0\mathbb{R}^n$. For simplicity, suppose that $v_i \in \text{span}\{\frac{\partial}{\partial x^i} : i = 1, .., k+1\}$. For $\epsilon > 0$, let $P_\epsilon := \{\sum_{i=1}^{k+1} t_i v_i : t_1, ..., t_{k+1} \in [0, \epsilon]\}$. Show that the $k+1$ coefficient in the Taylor expansion of $F(\epsilon) := \int_{\partial P_\epsilon} \omega$ is $d\omega_0(v_1, ..., v_{k+1})$.

Problem #3

In this problem, you will prove some properties of the Lie derivative and practice using Cartan calculus. Fix $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$. Let F be the flow of X

(a) * Show that $\mathcal{L}_X\omega = 0$ if and only if for any $X_1, ..., X_k \in \mathfrak{X}(M)$ and any appropriate $t \in \mathbb{R}$, $\omega\big((F_t)_*X_1, ..., (F_t)_*X_k\big) = \omega(X_1, ..., X_k) \circ F_{-t}$ on M_{-t}.

Remark: Note that $(F_t)_*X_i$ is a vector field on M_{-t}. Recall that $M_{-t} := \{p \in M : -t \in \mathcal{D}^{(p)}\} = \{p \in M : F(-t, p) \text{ is defined}\}$.

(b) * Suppose X is nowhere vanishing. Show that for every $p \in M$, there exists a coordinate chart (U, ϕ) such that $\mathcal{L}_X\omega = \sum_I \frac{\partial \omega_I}{\partial x^1} dx^I$ on U.

Hint: Argue that there exists a coordinate chart such that $X = \frac{\partial}{\partial x^1}$ on U.

(c) * Suppose ω is a no-where vanishing n-form. Show that there exists a unique function $g \in C^\infty(M)$ such that $\mathcal{L}_X\omega = g\omega$. Find a formula for g in local coodinates. Conclude that $g = \nabla \cdot X$ if $M = \mathbb{R}^n$ and $\omega = dx^1 \wedge ... \wedge dx^n$.

Remark: For each choice of no-where vanishing n-form, the map $X \mapsto g$ gives a candidate for the generalized definition of divergence. Note that this notion of divergence is not intrisic to the manifold since it depends on the choice of a no-where vanishing n-form. In a Riemannian manifold, there exists a unique choice of no-where vanishing n-form that takes any orthonormal basis to 1; this gives rise to a well defined notion of divergence on Riemannian manifolds.

(d) Solve problem 20.7. You will derive a formula for $\mathcal{L}_{fX}\omega$.

(e) Solve problem 20.9.

(f) Solve problem 20.10. (The 2-form should be $xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$)

(g) Show that $\iota_X(F^*\omega) = F^*(\iota_{F_*X}\omega)$ for $X \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$, and a diffeomorphism $F : M \to M$.

Problem #4

We will formulate the statement of Frobenius theorem in the language of differential forms. We first start by 2 lemmas.

(a) *Show that the smooth 1-forms $\omega^1, ..., \omega^k$ on a manifold M are linearly independent at every point if and only if $\omega^1 \wedge ... \wedge \omega^k$ is no-where vanishing.

(b) Suppose $\omega^1, ..., \omega^k \in \Omega^1(M)$ are linearly independent 1-forms on a manifold such that $k < n$. Show that for every $p \in M$, there exists $\omega^{k+1}, ..., \omega^n \in \Omega^1(U)$ on an open neighbourhood U of p such that $\omega^1, ..., \omega^n$ is a smooth local coframe.

Remark: This means we can complete k linearly independent smooth 1-forms to a smooth coframe near each point. Similarly, we can complete k linearly independent smooth vector fields to a smooth frame near each point.

Let Δ be a smooth distribution of codimension k. We say that an l-form ω annihilates Δ if $\omega(X_1, ..., X_l) = 0$ whenever $X_1, ..., X_l$ are local sections of Δ. Smooth linearly independent 1-forms $\omega^1, ..., \omega^k$ are called local defining forms for Δ if they annihilate Δ. If so, then $\Delta_q = \cap_i \text{Ker } \omega^i|_q$ for all $q \in U$.

(c) Let Δ be a distribution of codimension k. Show that Δ is a smooth distribution if and only if there exists local defining forms for Δ near every point.

(d) * Let Δ be a smooth distribution. Show that Δ is involutive if and only if $d\omega$ annihilates Δ on an open set U whenever $\omega \in \Omega^1(U)$ annihilates Δ on U.

Denote by $\ell^l(\Delta) \subseteq \Omega^k(M)$ the space of all smooth l-forms that annihilate Δ. Define $\ell(\Delta) := \ell^0(\Delta) \oplus ... \oplus \ell^n(\Delta) \subseteq \Omega^*(M)$. It is easy to see that $\ell(\Delta)$ is a subalgebra as it's closed under addition, wedge product and scalar multiplication with $C^\infty(M)$. A subalgebra of $\Omega^*(M)$ is called an ideal if it is also closed under wedge products of arbitrary elements of $\Omega^*(M)$. It is easy to see that $\ell(\Delta)$ is an ideal as $\eta \wedge \omega \in \ell(\Delta)$ for any $\eta \in \Omega^*(M)$ and $\omega \in \ell(\Delta)$.

(e) Show that $\ell(\Delta)$ is indeed a sublagebra and an ideal of $\Omega^*(M)$.

(f) Let $\omega^1, ..., \omega^k$ be local defining forms for Δ on an open set U. Show that an l-form η annihilates Δ on U if and only if $\eta \wedge \omega^1 \wedge ... \wedge \omega^k = 0$.

Remark: This implies that if η annihilates Δ on U, then $\eta = \sum_{i=1}^k \omega^i \wedge \beta^i$ for some $l-1$ forms $\beta^1, ..., \beta^k$ on U.

(g) * Use part (f) to show that Δ is involutive if and only if $d(\ell(\Delta)) \subseteq \ell(\Delta)$. (This means that if $\eta \in \ell(\Delta)$, then $d\eta \in \ell(\Delta)$; if so, we say $\ell(\Delta)$ is a differential ideal.)

We now can state Frobenius theorem in the language of differential forms:
A smooth distribution Δ is completely integrable if and only if $d(\ell(\Delta)) \subseteq \ell(\Delta)$.

(h) Consider the following vector fields on $\mathbb{R}^3$.

$$X = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Let Δ be the distribution spanned by X and Y. Find a 1-form that globally defines Δ. Is $\ell(\Delta)$ a differential ideal? Is Δ involutive?

(i) * Show that the 1-form $\omega = (1 + y^2)(xdy + ydx)$, defined on $\mathbb{R}^2 \setminus \{0\}$ is associated to a smooth rank 1 distribution Δ (defined by $\Delta_p = \text{Ker}(\omega_p)$). Show that $\ell(\Delta)$ is a differential ideal and find the integral submanifolds.

(j) Let $\omega \in \Omega^1(M)$ be a 1-form on a 3 dimensional manifold such that $\eta := d\alpha \wedge \alpha \in \Omega^3(M)$ is no where vanishing. Show that the smooth rank 2 distribution Δ associated to ω is not involutive. Conclude that $\eta(X, Y, [X, Y])$ is non-vanishing for any smooth local basis X, Y of Δ.

Problem #5

(a) * Let M be a manifold that admits an atlas with only two charts (U, ϕ) and (V, ψ) such that $U \cap V$ is connected. Show that M is orientable. That particularly proves that S^n is orientable.

(b) Study the orientability of the cylinder, mobius strip, and $\mathbb{R}P^n$.

(c) ***(bonus)*** Show that $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$.

Hint: You can use without proof that every closed n dimensional submanifold of $\mathbb{R}^3$ is the regular level set of a function $f \in C^\infty(\mathbb{R}^3)$.

Let M and N be oriented compact manifolds and let $\omega \in \Omega^m(M)$ and $\eta \in \Omega^n(N)$ be top-forms. We can define an orientation on $M \times N$ by agreeing that $\{v_1, ..., v_m, w_1, ..., w_n\}$ is positively oriented whenever $\{v_1, ..., v_m\}$ and $\{w_1, ..., w_n\}$ are positively oriented on M and N respectively. Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the projection maps.

(d) * Show that $\displaystyle\int_{M\times N} \pi_M^*\omega \wedge \pi_N^*\eta = \int_M \omega \cdot \int_N \eta$

(e) Let $h \in C^\infty(M \times N)$. Show that $\displaystyle\int_{M\times N} h\pi_M^*\omega \wedge \pi_N^*\eta = \int_M g\omega,$

where $g(p) := \displaystyle\int_N h(p, \cdot)\eta.$

(f) Show that every $m + n$ form is of the form $h\pi_M^*\omega \wedge \pi_N^*\eta$ for some $h \in C^\infty(M \times N)$, $\omega \in \Omega^m(M)$, and $\eta \in \Omega^n(N)$.