MAT 367: Differential Geometry
Assignment #6
Due on Friday August 6, 2021 by 11:59 pm

Note: This assignment covers material from the lectures Frobenius theorem and differential 1-forms.

Problem #1

In this problem, you will fill in the details for the proof of Frobenius theorem.

Let Δ be a C^∞ distribution of rank k on an n dimensional manifold M.
Suppose Δ is involutive and let $p \in M$. Let $X_1, ..., X_k \in \mathfrak{X}(U)$ be a local frame of Δ on a neighbourhood U of p.

(a) Show that Δ is involutive if and only if $\Gamma(\Delta)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Remark: Recall that a subset of a Lie algebra is a Lie subalgebra if it's a Lie algebra with respect to the same bracket.

(b) * Show that there exists another local frame $Y_1, ..., Y_k \in \mathfrak{X}(\tilde{U})$ of Δ on a neighbourhood $\tilde{U} \subseteq U$ of p such that $[Y_i, Y_j] = 0$ for $i, j = 1, ..., k$.

Suppose for simplicity that $k = 2$. To prove that Δ is completely integrable (i.e. there exists a flat chart for Δ near p), it suffices to show that there exists a chart such that the first 2 coordinate vector fields are Y_1 and Y_2. It was argued in lectures that such a chart will indeed be a flat chart of Δ near p.

Upon possibly shrinking $\tilde{U}$, we fix a chart $(\tilde{U}, \phi = (x^1, ..., x^n))$ with the property that $\phi(p) = 0$, and $\frac{\partial}{\partial x^i}\big|_p = Y_{ip}$ for $i = 1, 2$. We define the map $\psi : V \to \tilde{U}$ by

$$\psi(y^1, ..., y^n) = F_{y^1} \circ G_{y^2}\Big(\phi^{-1}((0, 0, y^3, ..., y^n))\Big)$$

where $V \subset \mathbb{R}^n$ is small enough neighbourhood of 0 so that ψ makes sense, and F and G are the flows of Y_1 and Y_2. I defined ψ in lectures incorrectly.

(c) * Show that ψ^{-1} defines a chart near p satisfying the desired properties. i.e. show that the coordinate vector fields with respect to ψ^{-1} satisfy $\frac{\partial}{\partial y^i} = Y_i$ for $i = 1, 2$.

Remark: We have then proven that if Δ is involutive, then there exists a flat chart of Δ near every point, and so Δ is completely integrable.

(d) **(*bonus*)** Where do the coordinate vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ agree?

We define a k dimensional foliation on M to be a collection $\mathcal{F}$ of disjoint connected k dimensional submanifolds of M (called the leaves of the foliation), whose union is M, and such that there exists a flat chart for $\mathcal{F}$ near each point. For example, spheres in $\mathbb{R}^3$ with radius $r > 0$ make a $n-1$ dimensional foliation on $\mathbb{R}^3 \setminus \{0\}$

(e) Let $\mathcal{F}$ be a k dimensional foliation. Show that there exists an involutive smooth distribution Δ such that the leaves of the foliation are the integral submanifolds of Δ.

(f) Let Δ be an involutive smooth distribution of rank k. Show that for each $p \in M$, there exists a coordinate open set U and a k dimensional foliation on U with the leaves being integral submanifolds of Δ.

Remark: This means that any involutive distribution admits a local foliation near every point. In fact, if we allow the leaves of a foliation to be immersed submanifolds, then Δ admits a global foliation with the leaves being the maximal integral submanifolds of Δ. However, this is not trivial. It must be shown first that for each point p, there exists a maximal integral immersed submanifold passing through p.

Problem # 2

In this problem, you will see some examples of Frobenius theorem.

(a) * Consider the radial vector field $X = \sum x^i \frac{\partial}{\partial x^i}$ on $\mathbb{R}^n \setminus 0$. Let Δ_p be the orthogonal complement of the subspace spanned by X_p and define the distribution $\Delta := \cup_p \Delta_p$. Show that Δ is a smooth involutive distribution of rank $n-1$. What are the integral submanifolds?

(b) Consider the distribution Δ spanned by the vector fields

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z},$$

$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$

Show that Δ is involutive, and find a flat chart in a neighbourhood of the origin.

(c) * Let Δ be the distribution on $\mathbb{R}^3$ spanned by

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y}$$

Is Δ involutive? Find an integral submanifold of Δ passing through the origin.

(d) * Consider on $\mathbb{R}^3$ the following distribution Δ determined by the vector fields

$$X = \frac{\partial}{\partial x} + \frac{2xz}{1+x^2+y^2}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{2yz}{1+x^2+y^2}\frac{\partial}{\partial z}$$

Show that Δ is involutive and compute the flows of X and Y. Then find the integral surfaces of Δ.

Problem # 3

In this question, you will study how Frobenius theorem can be applied to existence theorems for certain types of PDEs.
We start with a very simple example. Let U be an open subset of $\mathbb{R}^2$ containing the origin. Suppose we are looking for a function $f : U' \to \mathbb{R}$ satisfying

$$\frac{\partial f}{\partial x} = g_1, \qquad \frac{\partial f}{\partial y} = g_2, \qquad f(0,0) = c$$

where $g_1, g_2 \in C^\infty(U)$ are given, $U' \subseteq U$ is a neighbourhood of the origin in $\mathbb{R}^2$, and $c \in \mathbb{R}$. This is an overdetermined system of first order linear PDEs. We wish to find necessary and sufficient conditions on the givens g_1 and g_2 for the existence and uniqueness of a solution for every choice of $c \in \mathbb{R}$.

(a) Find the obvious necessary condition on g_1 and g_2 such that there exists a solution for every choice of $c \in \mathbb{R}$. Show the solution is unique if it exists.

Remark: Uniqueness here means that if f_1 and f_2 are solutions defined on U_1' and U_2' satisfying $f_1(0,0) = f_2(0,0) = c$, then $f_1 = f_2$ on $U_1' \cap U_2'$.

Now suppose there is a solution on U for each choice of $c \in \mathbb{R}$. If f is a solution satisfying $f(0,0) = c$, then $\Gamma_{f_c} := \{(x, y, f(x,y)) : (x,y) \in U\} \subseteq \mathbb{R}^3$ is submanifold of $\mathbb{R}^3$ passing through $(0,0,c)$. These submanifolds form a 2 dimensional foliation on $U \times \mathbb{R}$, which is associated to an involutive smooth distribution Δ as shown by problem 3d. In fact, Δ will only depend on the givens g_1 and g_2!

(b) Use Frobenius theorem to show that the condition you found in (b) is also sufficient for the existence of a solution for every choice of $c \in \mathbb{R}$.

This can be proven without using Frobenius theorem. We can first define $f(x,0)$ so that $\frac{\partial f}{\partial x}(x,0) = g_1(x,0)$ and $f(0,0) = c$:

$$f(x,0) = c + \int_0^x g_1(t,0)dt$$

Then we define $f(x, y)$ so that $\frac{\partial f}{\partial y}(x, y) = g_2(x, y)$:

$$f(x,y) = f(x,0) + \int_0^y g_2(x,t)dt = c + \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt$$

Note that this function does not necessarily solve the PDE since $\frac{\partial f}{\partial x}(x, y) = g_1(x, y)$ might not be satisfied for $y \neq 0$. One can show that f is a solution if and only if the condition you found in (a) is satisfied. This gives an alternative simpler proof of the existence theorem of first order linear PDEs. However, Frobenius theorem can be applied to a much more general family of PDEs including non-linear ones!

Let U and V be open subsets of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ such that U contains the origin. Let $g_i : U \times V \to \mathbb{R}^{n-k}$ be given smooth functions. Suppose we are looking for a function $f : U' \to V$ on a neighbourhood of the origin satisfying

$$\frac{\partial f}{\partial x^i}(x) = g_i(x, f(x)), \qquad f(0) = c$$

for $i = 1, ..., k$ and $c \in V$. This is also an overdetermined system of first order (possibly non-linear) PDEs.

(c) Find a necessary and sufficient condition on g_i for the existence and uniqueness of a solution f for each $c \in V$.

(d) Let Δ be a k dimensional distribution on $\mathbb{R}^n$. Formulate the result of Frobenius theorem in the language of PDEs.

Hint: Recall that a submanifold of $\mathbb{R}^n$ is locally the graph of a function. We expect that the problem of the integrability of a distribution locally boils down to the above system of PDEs, and that the condition you found in (c) is equivalent to the involutivity of the distribution.

(e) Consider the following system of PDEs on the function $z(x, y)$:

$$\frac{\partial z}{\partial x} = e^{xz}, \qquad \frac{\partial z}{\partial y} = xe^{yz}$$

Determine if there exists a solution in a neighbourhood of the origin satisfying $z(0, 0) = z_0$ for every choice of $z_0 \in \mathbb{R}$. Interpret the results in the language of differential geometry.

Problem # 4

Let M be a smooth manifold. We defined in lectures the operator $d : C^\infty(M) \to \Omega^1(M)$ taking a function $f \in C^\infty(M)$ to its differential $df \in \Omega^1(M)$. This operator is called the exterior derivative. In this problem, you will prove some of the properties of this operator.

(a) Prove the following properties of d:

- $d : C^\infty(M) \to \Omega^1(M)$ is $\mathbb{R}$-linear (linear with respect to the vector space structure of $C^\infty(M)$ and $\Omega^1(M)$).
- For any $f, g \in C^\infty(M)$, $d(fg) = fdg + gdf$.
- For any $f, g \in C^\infty(M)$ such that g is non-vanishing, $d\left(\frac{f}{g}\right) = \frac{1}{g^2}(gdf - fdg)$.
- For any $f \in C^\infty(M)$ and $h \in C^\infty(\mathbb{R})$, $d(h \circ f) = (h' \circ f)df$.
- $df = 0$ if and only if f is constant on connected components.

By the last property, we can easily see that d is not injective. But is it surjective? i.e. is it true that every smooth 1-form the differential of a smooth function? We say that $\omega \in \Omega^1(M)$ is exact if $\omega = df$ for some $f \in C^\infty(M)$.

If ω is exact, then we notice that on a coordinate chart, we have that $a_i dx^i = \frac{\partial f}{\partial x^i} dx^i$ implying that $a_i = \frac{\partial f}{\partial x^i}$. In particular, by taking the derivatives of both sides with respect to $\frac{\partial}{\partial x^j}$ and using Clairaut's theorem , we get that the coefficients of ω must satisfy

$$\frac{\partial a^i}{\partial x^j} = \frac{\partial a^j}{\partial x^i} \tag{1}$$

This gives us a necessary condition for a smooth 1-form to be exact: If ω is exact, then on any chart, the coefficients must satisfy equation 1. It is clear that not any 1-form satisfies this since we can pick the coefficients to be anything. This particularly implies that d is not surjective.

Let us study this condition a little more deeply. We wish to write that condition in a coordinate independent way. We can rewrite equation 1 in the following way:

$$\frac{\partial}{\partial x^i}\left(\omega\left(\frac{\partial}{\partial x^j}\right)\right) - \frac{\partial}{\partial x^j}\left(\omega\left(\frac{\partial}{\partial x^i}\right)\right) = 0$$

It is then tempting to say that ω satisfies that necessary condition if and only if for all $X, Y \in \mathfrak{X}(M)$,

$$X(\omega(Y)) - Y(\omega(X)) = 0$$

However, by choosing X and Y to be non commuting vector fields, one can easily see that this is not the correct coordinate independent formulation of that necessary condition. The next question corrects this.

(b) * Show that $\omega \in \Omega^1(M)$ satisfies equation 1 on every chart if and only if for all $X, Y \in \mathfrak{X}(M)$,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X,Y])$$

We sat that $\omega \in \Omega^1(M)$ is closed if it satisfies the above condition. We have then proved that if ω is exact, then it's closed.
But is that condition sufficient? i.e. if ω is closed, does that mean it's exact?

(c) * Consider the following smooth 1-form on $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

Is ω closed? Is ω exact?

(d) * Suppose that $\omega \in \Omega^1(M)$ is non-vanishing and closed. Show that the distribution Δ defined by $\Delta_p := \mathrm{Ker}(\omega_p)$ is a smooth rank $n-1$ involutive distribution.

(e) * Since Δ is completely integrable by Frobenius theorem, we can fix a flat chart (U, ϕ) for Δ near p. Show that ω is exact on U.

We have then used Frobenius theorem to show that any closed non-vanishing 1-form is locally exact. This means that for any $p \in M$, there exists a neighbourhood U of p and a function $f \in C^\infty(U)$ such that $\omega = df$ on U.
In fact, the non-vanishing condition can be dropped.

(f) **(*bonus*)** Use the results of problem # 3 to show that if $\omega \in \Omega^1(M)$ is closed, then it's locally exact.

It happens that if M satisfies some condition, then every smooth closed 1-form is (globally) exact. What's absolutely surprising is that this condition is completely topological.

Problem # 5

In this problem, you will study some properties of differential forms and will practice computations.

(a) Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$. Let $(U, \phi = (x^1, ..., x^n))$ and $(U, \psi = (y^1, ..., y^n))$ be two charts on M. Then $X = a^i \frac{\partial}{\partial x^i}, \omega = b_i dx^i$ on U and $X = \tilde{a}^i \frac{\partial}{\partial y^i}, \omega = \tilde{b}^i dy^i$ on U. Derive a relation between: a^i and $\tilde{a}^i$, $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$, b^i and $\tilde{b}^i$, dx^i and dy^i.

Remark: Recall that $X|_U(x^i) = a^i$ and $\omega|_U(\frac{\partial}{\partial x^i}) = b_i$ on U, where $X|_U$ and $\omega|_U$ is the vector field X and the 1-form ω restricted to U. We will usually drop the "$|_U$" if it's understood from the context. So for any open set U, we can think of $X : C^\infty(U) \to C^\infty(U)$ as a derivation on $C^\infty(U)$ and $\omega : \mathfrak{X}(U) \to C^\infty(U)$ as a C^∞-linear map on $\mathfrak{X}(U)$.

(b) * Consider the following vector fields on $\mathbb{R}^3$.

$$X_1 = (2+y^2)e^z \frac{\partial}{\partial x}, \quad X_2 = 2xy\frac{\partial}{\partial x} + (2+y^2)\frac{\partial}{\partial y}, \quad X_3 = -2xy^2\frac{\partial}{\partial x} - y(2+y^2)\frac{\partial}{\partial y} + (2+y^2)\frac{\partial}{\partial z}$$

Show that these vector fields are a basis for the module $\mathfrak{X}(\mathbb{R}^3)$ and express the dual basis ω^i in terms of dx, dy, dz.

(c) Let $\omega \in \Omega^1(\mathbb{R}^4)$ be defined by $\omega = xdy - ydx + zdt - tdz$. Show that the restriction of ω to S^3 is no-where vanishing.

(d) * Solve problem 19.11. In this problem, you will construct a no-where vanishing top form on any regular level set of a function on Euclidean space.

Remark: It then follows that any regular level set in Euclidean spaces is orientable. It also follows that non orientable spaces like the Klein bottle, $\mathbb{R}P^2$ and the mobius strip cannot be realized as the regular level set of a C^∞ function.

(e) * In this problem, we want to generalize the idea of Lagrange multipliers to manifolds. Let $S \subseteq M$ be a submanifold of M of codimension k and let $f \in C^\infty(M)$. Suppose $p \in S$ is a point at which f attains a local maximum or minimum value among points in S. Let $F : U \to \mathbb{R}^k$ be a smooth function on a neighbourhood U of p such that $U \cap S$ is the regular zero set of F (0 is a regular value and $F^{-1}(0) = U \cap S$). Show that there exists real numbers $\lambda_1, ..., \lambda_k$ called Lagrange multipliers such that

$$df_p = \sum_{i=1}^{k} \lambda_i \, dF^i\big|_p$$