MAT 367: Differential Geometry
Assignment #5
Due on Friday July 23, 2021 by 11:59 pm

Note: This assignment covers material from the lectures on Lie derivatives.

You only need to submit the questions marked with a *

Problem #1

In this problem, you will strengthen your understanding of the Lie derivative.
You are not allowed to use the fact that the Lie derivative is the Lie bracket for questions 1a,b,c.

We extend the Lie derivative to smooth functions. Let $X \in \mathfrak{X}(M)$ with flow F and define $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ by

$$\mathcal{L}_X f(p) = \lim_{t\to 0} \frac{f \circ F_t(p) - f(p)}{t}$$

for $f \in C^\infty(M)$ and $p \in M$.

(a) * Show that the limit always exists and that $\mathcal{L}_X f = X(f)$. Then conclude that $\mathcal{L}_X$ is a derivation on the algebra $C^\infty(M)$.

Remark: We can equivalently define $\mathcal{L}_X f$ as the first order term in the Taylor expansion of $f \circ F_t(p)$ around $t = 0$: $f \circ F_t(p) = f(p) + t\mathcal{L}_X f(p) + o(t)$.

(b) * Show that for any $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, $\mathcal{L}_X(fY) = (\mathcal{L}_X f)Y + f\mathcal{L}_X Y$.

Recall that the Lie derivative on vector fields is defined by

$$\mathcal{L}_X Y\big|_p = \lim_{t\to 0} \frac{1}{t}[F_{-t_{*,F_t(p)}}(Y\big|_{F_t(p)}) - Y\big|_p]$$

Fix a coordinate chart (U, ϕ) near $p \in M$. Then on U, $X = \sum a^i \frac{\partial}{\partial x_i}$ and $Y = \sum b^i \frac{\partial}{\partial x_i}$ for some $a^i, b^i \in C^\infty(U)$.

(c) * Use (b) to express $\mathcal{L}_X Y$ in local coordinates and conclude that the limit always exists and $\mathcal{L}_X Y \in \mathfrak{X}(M)$.

Remark: Compare this with the expression of the Lie bracket in local coordinates that you derived in problem 2d in assignment 4. Are they the same? (If not, you did something wrong). This is another proof that the Lie derivative is the Lie bracket.

(d) * Justify why $\mathcal{L}_X Y|_p$ depends on X and Y on a neighbourhood of p and not only on X_p and Y_p.

(e) * Suppose we know $X \in \mathfrak{X}(M)$ and we only know Y on the integral curve of X starting at p. Is this sufficient to know $\mathcal{L}_X Y|_p$?

Let $X, Y \in \mathfrak{X}(M)$ with flows F and G. Let $p \in M$ and fix a real number t_0 close enough to 0. Then the curve $s \mapsto G_s \circ F_{t_0}(p)$ is an integral curve of Y starting at $F_{t_0}(p)$. It gets mapped by the flow of X to the curve $\gamma_{t_0} : s \mapsto F_{-t_0} \circ G_s \circ F_{t_0}(p)$, which passes through p. Note that this might not necessarily coincide with the curve $s \mapsto G_s(p)$ (why not?). We can then define the map $A : (s,t) \mapsto F_{-t} \circ G_s \circ F_t(p)$.

(f) * For fixed t_0, compute $\frac{\partial A}{\partial s}\big|_{(0,t_0)}$. Show that this agrees with the velocity vector of γ_{t_0} at $s = 0$.

(g) * Note that we constructed A in this way so that $t_0 \mapsto \frac{\partial A}{\partial s}\big|_{(0,t_0)}$ is a curve on T_pM. Compute the velocity vector of this curve at $t_0 = 0$.

Remark: This is precisely $\left.\frac{\partial^2 A}{\partial t \partial s}\right|_{(0,0)}$

This means that an integral curve of Y gets mapped by the flow of X, namely F_{-t_0}, to a curve with the velocity vector you got in (f), which changes in the direction of the vector you got in (g) as you move t_0 away from 0.

We now want to do the same thing but in the other order. We start with the curve $t \mapsto F_t \circ G_{s_0}(p)$ for some fixed s_0 close enough to 0, which is an integral curve of X starting at $G_{s_0}(p)$. It gets mapped by the flow of Y to the curve $\beta_{s_0} : t \mapsto G_{-s_0} \circ F_t \circ G_{s_0}(p)$. This defines the map $B : (s,t) \mapsto G_{-s} \circ F_t \circ G_s(p)$.

(h) In the same way as in (f) and (g), compute $\beta'_{s_0}(0)$ and $\left.\frac{\partial^2 B}{\partial s \partial t}\right|_{(0,0)}$

Remark: How does $\left.\frac{\partial^2 A}{\partial t \partial s}\right|_{(0,0)}$ relate to $\left.\frac{\partial^2 B}{\partial s \partial t}\right|_{(0,0)}$? What can you say about A and B if $\mathcal{L}_X Y \equiv 0$? This problem should give some insight on the non-intuitive formula $\mathcal{L}_X Y = -\mathcal{L}_Y X$.

Problem # 2

In this problem, you will improve your intuition of the Lie derivative by considering specifically manifolds in $\mathbb{R}^n$.

Let $X, Y \in \mathfrak{X}(\mathbb{R}^n)$. Since the tangent space at every point of $\mathbb{R}^n$ is the same, we can compare vectors in different tangent spaces directly. This gives rise to the following notion of the rate of change of Y in the direction of X: define

$$\nabla_X Y|_p := \lim_{t \to 0} \frac{Y|_{F_t(p)} - Y|_p}{t}$$

where $p \in \mathbb{R}^n$ and F is the flow of X. Think about what this measures and compare it with the Lie derivative.

(a) Show that $\nabla : \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$ is C^∞-linear with respect to the first argument (X) and only $\mathbb{R}$-linear with respect to the second argument (Y).

Remark: This shows that $\nabla_X Y$ only depends on X at the point p but depends on Y on a neighbourhood of p. It can't be the same as the Lie derivative.

(b) Show that $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$ for all $X, Y \in \mathfrak{X}(\mathbb{R}^n)$. Give a necessary and sufficient condition on X such that $\mathcal{L}_X \equiv \nabla_X$ on $\mathfrak{X}(\mathbb{R}^n)$.

Remark: The operator ∇ will be generalized once we define a Riemannian manifold. One needs a geometric structure on the manifold to generalize what ∇ measures. It's called the covariant derivative and satisfies that same relation with the Lie derivative. Try to see how ∇ implicitly uses the Euclidean geometry of $\mathbb{R}^n$.

Now suppose that M is a submanifold of $\mathbb{R}^n$. Notice quickly that the definition of $\nabla_X Y$ carries over directly for $X, Y \in \mathfrak{X}(M)$ (why?). However, for $X, Y \in \mathfrak{X}(M)$, $\nabla_X Y$ might not be tangent to M. For example, take a vector field on S^2 that is tangent to a great circle. Then $\nabla_X X$ points to the centre of the sphere and so is not tangent to it. So what ∇ measures is not intrinsic and depends on the ambient space. Nevertheless, $\nabla_X Y - \nabla_Y X$ will always be tangent to M since that's just the Lie derivative, as you will show in the following problem.

(c) Repeat (b) for $X, Y \in \mathfrak{X}(M)$.

Hint: Let $\tilde{X}, \tilde{Y}$ be extensions of X and Y on a neighbourhood of p in $\mathbb{R}^n$. What can you say about $\nabla_{\tilde{X}} \tilde{Y}$ and $\nabla_X Y$? What about $\mathcal{L}_{\tilde{X}} \tilde{Y}$ and $\mathcal{L}_X Y$?

Let F and G be the flows of the vector fields $X, Y \in \mathfrak{X}(M)$ respectively. Define the map

$$A : (s,t) \mapsto G_s \circ F_t(p) - F_t \circ G_s(p)$$

for (s,t) in a neighbourhood of $(0,0)$ in $\mathbb{R}^2$. For fixed t_0 and s_0, consider the curves $s \mapsto A(s, t_0)$ and $t \mapsto A(s_0, t)$. These curves measure the extent to which the flows fail to commute. Try to understand using pictures what these two curves are. (Assume that the origin is far from the M to help you draw the picture).

(d) * Show that $\left.\frac{\partial^2 A}{\partial t \partial s}\right|_{(0,0)} = \mathcal{L}_X Y\big|_p$.

Hint: First, compute the velocity vector of this curve $s \to A(s, t_0)$ at $s = 0$ to get $\left.\frac{\partial A}{\partial s}\right|_{(0,t_0)}$. Then compute the velocity vector of this curve $t \mapsto \left.\frac{\partial A}{\partial s}\right|_{(0,t)}$ at $t = 0$.

(e) **(*bonus*)** Compute $\left.\frac{\partial^2 A}{\partial t \partial s}\right|_{(s,t)}$.

Let F and G be the flows of the vector fields $X, Y \in \mathfrak{X}(M)$ respectively. Consider the curve

$$\gamma(t) = G_t \circ F_t \circ G_{-t} \circ F_{-t}(p)$$

for $p \in M$ and for t in an open interval containing 0. Draw a picture to understand what the curve is.

(f) * Show that $\gamma'(0) = 0$ and $\gamma''(0) = 2\, \mathcal{L}_X Y\big|_p$

Let's study an example. For any $A \in \mathrm{MAT}_n(\mathbb{R})$, define the vector field

$$X_A := \sum_{i,j} A_{ij} x^i \frac{\partial}{\partial x^j}$$

on $\mathbb{R}^n$.

(g) * Compute the Lie bracket $[X_A, X_B]$ for $A, B \in \mathrm{MAT}_n(\mathbb{R})$ and express as X_C for some $C \in \mathrm{MAT}_n(\mathbb{R})$. Derive a coordinate independent relation between A, B, and C.

(h) * Compute the first nonzero term after the 0^{th} order term in the Taylor expansion of the curve $t \mapsto F_t \circ G_t \circ F_{-t} \circ G_{-t}(p)$ around $t = 0$, where F and G are the flows of X_A and X_B.