MAT 367: Differential Geometry
Assignment #4
Due on Sunday July 11, 2021 by 11:59 pm

Note: This assignment covers material from lectures and sections 12, 13, 14.

Problem # 1

Fix a point p on a smooth manifold M. We will give a different definition of a point derivation at p. We say $D : C^\infty(M) \to \mathbb{R}$ is a point derivation at p if it's $\mathbb{R}$-linear (linear with respect to the vector space structure of $C^\infty(M)$), and satisfies

$$D(fg) = f(p)D(g) + g(p)D(f)$$

for all $f, g \in C^\infty(M)$. Denote by $\mathcal{D}_p^*$ the space of all point derivations at p. We want to show that $\mathcal{D}_p^*$ is an equivalent definition of the tangent space T_pM .

(a) Show that for any $D \in \mathcal{D}_p^*$ and any $f \in C^\infty M$, $D(f)$ only depends on the behaviour of f near p.

(b) Let $\Phi : T_pM \to \mathcal{D}_p^*$ be the map defined by $\Phi(v)(f) = v([f])$ for $v \in T_pM$ and $f \in C^\infty$. Show that Φ is an isomorphism.

(c) Let $X \in \mathfrak{X}(M)$ be a smooth vector field on M, and define $D : C^\infty(M) \to \mathbb{R}$ by $D(f) = X(f)|_p$ for $f \in C^\infty(M)$. Show that $D \in \mathcal{D}_p^*$ and $\Phi(X_p) = D$.

Using this new definition of the tangent space, we can redefine the tangent bundle $TM^* := \cup_{p\in M}\mathcal{D}_p^*$. This gives rise to a new definition of a vector field $X^* \in \Gamma^*(M)$ as a smooth section of TM^*. As a consequence of (c), there is a natural isomorphism between $\Gamma^*(M)$ and the space of derivations on the algebra $C^\infty(M)$ denoted by $Der(C^\infty(M))$, which in turn is naturally isomorphic to $\Gamma(M)$. We identify the three definitions of vector fields and we continue to denote the space of vector fields by $\mathfrak{X}(M)$.

Problem #2

Read section 14.5-14.6.

(a) Solve questions 14.13 and 14.14.

(b) Let $F : N \to M$ be an injective smooth map, and let $Y \in \mathfrak{X}(M)$. Note there doesn't necessarily exist a unique vector field $X \in \mathfrak{X}(N)$ that is F-related to Y. Show that

- If F is an immersion, then uniqueness holds.
- If F is a submersion, then existence holds.

Remark: How does this fail when you remove the injectivity condition on F? What if F is surjective? We say that X, if it exists, is a lift of Y. It follows that if F is a diffeomorphism, then every Y has a unique lift.

(c) Let S be a submanifold of M. Recall that we say that $X \in \mathfrak{X}(M)$ is tangent to S if for all $p \in S$, $X_p \in i_{*,p}(T_pS)$. Show that if X is tangent to S, there exists $Y \in \mathfrak{X}(S)$ that is i-related to X. (We will sometime identify Y with $X|_S$.)

(d) Show that X is tangent to a submanifold S if and only if $X(f) = 0$ on S for all $f \in C^\infty(M)$ that is constant on S.

(e) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f(x, y, z) = x^2 + y^2 - 1$. This defines a submanifold $S := f^{-1}(0)$ of $\mathbb{R}^3$. Consider the vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ defined by

$$X = (x^2 - 1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, \qquad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}$$

Are they tangent to S?

(f) Express the Lie bracket introduced in assignment 3 in local coordinates. (Solve question 14.12).

(g) Show that if $X, Y \in \mathfrak{X}(M)$ are tangent to a submanifold S, then so is $[X, Y]$. Conclude that if $X_1, ..., X_k \in \mathfrak{X}(M)$ is a linearly independent k-tuple of smooth vector fields that span TS, then $[X_i, X_j]$ on S is a linear combination of the k-tuple with coefficients in $C^\infty(S)$.

Remark: The vector fields span TS means that $X_{1p}, ..., X_{kp}$ span $i_{*,p}(T_pS)$ for every point $p \in S$.

Problem #3

We will now study some applications of partition of unity.

(a) Let $f : A \to \mathbb{R}$ be a C^∞ function on a set $A \subseteq M$ as defined in lecture. Show that there exists $\tilde{f} \in C^\infty(U)$ such that $\tilde{f}\big|_A = f$ and U is a neighbourhood of A in M. If A is closed, show that U can be chosen to be all M.

(b) Let $X \in \mathfrak{X}(S)$ where S is a submanifold of M. Show that there exists $Y \in \mathfrak{X}(U)$ such that $Y|_S = X$ and U is a nieghbourhood of S in M. If S is closed, show that U can be chosen to be all M.

(c) Show that any compact smooth manifold can be embedded in $\mathbb{R}^N$ for large enough N. (The proof was outlined in lectures).

(d) Show that there exists a symmetric map $\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ that is C^∞-bilinear and satisfies $\alpha(X, X)(p) = 0 \implies X_p = 0$.

Hint: On a coordinate chart, define $\beta : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U)$ by $\beta(\sum a^i \frac{\partial}{\partial x^i}, \sum b^i \frac{\partial}{\partial x^i}) = \sum a^i b^i$.

(e) Let A and B be disjoint closed sets in M. Show that there exists a function $f \in C^\infty(M)$ that vanishes on A and is identically 1 on B.

Hint: $\{A^c, B^c\}$ is an open cover for M.

We will prove the existence of a compact exhaustion. A compact exhaustion of a manifold M is a sequence of compact sets K_i satisfying $K_i \subseteq \operatorname{int}(K_{i+1})$ and $\cup_{i=1}^\infty K_i = M$. Let $\{U_i\}$ be an open cover such that $\overline{U}_i$ is compact (you should be able to show the existence of such an open cover). Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$.

(f) Define the function $f : M \to \mathbb{R}$ by $f(p) = \sum_{i=1}^\infty i\rho_i(p)$. Show that $f^{-1}(-\infty, c]$ is compact for all $c \in \mathbb{R}$.

(g) Use f to find a compact exhaustion.

We want to prove that continuous functions on manifolds are arbitrarily close to C^∞ functions. Let $f \in C(M)$ and let $\varepsilon > 0$.

(h) Show there exists a function $g \in C^\infty(M)$ such that $\sup_{p \in M} |f(p) - g(p)| < \varepsilon$.

Hint: For any $p \in M$, there exists a neighbourhood U_p of p such that $|f(q) - f(p)| < \varepsilon$ for all $q \in U_p$ by the continuity of f.

(i) Suppose f is C^∞ on a closed set A. Show that we can choose g so such that $g = f$ on A.

Problem #4

You will do some computational examples in this problem.

(a) Show that

$$\phi_t(x,y) = e^t(\cos(t)x + \sin(t)y, \cos(t)y - \sin(t)x)$$

is a flow some vector field X on $\mathbb{R}^2$. Find X expressed in the form

$$X = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y}$$

.

(b) Define the vector fields X and Y on $\mathbb{R}^2$

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$$

Compute the flows F and G of X and Y and verify that the flows do not commute by finding explicit open intervals J and K containing 0 such that $F_t \circ G_s$ and $G_s \circ F_t$ are both defined for all $(s,t) \in J \times K$ but they are unequal for some such (s,t).

(c) Find the general expression for a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ satisfying $[\frac{\partial}{\partial x}, X] = X$ and $[\frac{\partial}{\partial y}, X] = X$

(d) Let $\phi : \mathbb{R} \times TM \to TM$ defined by $\phi(t,X) = e^t X$. Show that ϕ is a global flow. This means that ϕ satisfies:

- ϕ is C^∞.
- $\phi(s+t,X) = \phi(s,\phi(t,X))$ for all $t,s \in \mathbb{R}$ and $X \in TM$. Using the notation we used in class, this means $\phi_{s+t}(X) = \phi_s \circ \phi_t(X)$.

Then find $Y \in \mathfrak{X}(TM)$ that generates ϕ.

Remark #1: From the post-lecture practice questions, we know that any map satisfying the above is the global flow generated by some vector field.

Remark #2: Note that $\phi_t : TM \to TM$ is a diffeomorphism. Letting $\mathrm{Diff}(TM)$ be the group of diffeomorphisms on TM, we note that a global flow ϕ gives rise to a group homomorphism from $\mathbb{R}$ to $\mathrm{Diff}(TM)$ defined by $\Phi : t \mapsto \phi_t$ (it's a group homomorphism since $\Phi(s+t) = \Phi(s) \circ \Phi(t)$). We refer to this as a one parameter group of diffeomorphisms.

(e) Solve problem 14.2.

Problem #5 (*bonus*)

We will prove the fundamental theorem of flows and study some of their properties. Let $X \in \mathfrak{X}(M)$. We want to prove the existence of a maximal flow generated by X.

Let $p \in M$ and let $\gamma_1 : I \to M$ and $\gamma_2 : I \to M$ be two integral curves of X starting at p, where I is open intervals containing 0. Let $A = \{t \in I : \gamma_1(t) = \gamma_2(t)\}$.

(a) Show that $A = I$.

Hint: Show that A is a non-empty clopen subset of the connected set I.

This shows that any two integral curves starting at p are equal. Let $\mathcal{D}^{(p)}$ be the union of all open intervals containing 0 on which an integral curve starting at p is defined. This proves the existence and uniqueness of a maximal integral curve starting at p denoted by $\gamma_p(t) : \mathcal{D}^{(p)} \to M$. (How would $\gamma_p(t)$ be defined).

Define $\mathcal{D} = \{(t, p) : p \in M, t \in \mathcal{D}^{(p)}\} \subseteq \mathbb{R} \times M$. Then the maximal flow $F : \mathcal{D} \to M$ is defined by $F(t, p) = \gamma_p(t)$.

(b) Show that F satisfies the properties of a flow: F is C^∞, $F(0, p) = p$, and $F(s+t, p) = F(s, F(t, p))$ for appropriate $s, t \in \mathbb{R}$ and $p \in M$. Also, it satisfies $\mathcal{D}^{(F(s,p))} = \mathcal{D}^{(p)} - s$.

(c) Show that $\mathcal{D}$ is open and so is $M_t := \{p \in M : (t, p) \in \mathcal{D}\}$.

(d) Show that $F_t : M_t \to M_{-t}$ defined by $F_t(p) = F(t, p)$ is a diffeomorphism with inverse F_{-t}.

(e) The uniqueness of maximal integral curves uses the fact that M is Hausdorff. Explain how Hausdorfness is used in the proof. Find a non-Hausdorff "manifold" and a vector field that does not have a unique maximal integral curve.

Hint: Define a vector field on the real line with two origins using charts and find two different integral curves of it.

(f) Show that if M is compact, then any vector field is complete.

Hint: Show first that there exists an $\varepsilon > 0$ such that every integral curve is defined on $(-\varepsilon, \varepsilon)$.