# MAT 367: Differential Geometry
# Assignment #3
# Due on Friday June 11, 2021 by 11:59 pm

**Note:** This assignment covers material from lectures and sections 8,9,11,12.

## Problem 1

We defined two definitions of a submanifold that turned out to be equivalent (check Theorem 11.13 and 11.14). We will study one more definition and will find a way to create many examples of submanifolds.

**(a)** Show that $S \subseteq M$ is a k-dim submanifold if and only if $\forall p \in S$, there exists a $C^\infty$ map $F : U \to \mathbb{R}^{n-k}$ on a neighbourhood $U$ of $p$ in $M$ such that 0 is a regular value and $U \cap S = F^{-1}(0)$.

*Remark:* This shows that submanifolds are locally the level set of a function and gives a third equivalent definition of a submanifold. Compare this with the fact that a $k$-dim in $\mathbb{R}^n$ is locally the graph of a function.

**(b)** Let $S \subseteq M$ be a submanifold that is the regular level set of a function $F : M \to \mathbb{R}^{n-k}$. Show that the tangent space of $S$ as seen from the ambient space is Kernel$(F_{*,p})$.

*Hint*: Show Kernel$(F_{*,p}) = i_{*,p}(T_pS)$, where $i : S \hookrightarrow M$ is the inclusion map.

**(c)** Let $F : N \to M$ be a $C^\infty$ map of manifolds that is an embedding. Show that there exists a unique topology and smooth structure on $F(N)$ such that $F(N)$ is a submanifold with the property that $F : N \to F(N)$ is a diffeomorphism.

*Remark:* This gives a way to create more examples of submanifolds. Also, since $N$ and $F(N)$ are diffeomorphic smooth manifolds, one can say that the manifold $F(N)$ *is* the manifold $N$ seen as a submanifold of $M$. You will study an example in problem 3c in which you will embed the projective plane $\mathbb{R}P^2$ in $\mathbb{R}^4$

## Problem 2

It is sometimes important to consider a more general definition of a submanifold. $S \subseteq M$ is called an immersed submanifold if it's a smooth manifold such that that the inclusion map is a smooth immersion. Clearly, any submanifold is an immersed submanifold. Not every immersed submanifold is a submanifold since a smooth injective immersion is not necessarily a homeomorphism onto its image, and so not an embedding. An example is the figure eight, which is described in example 11.12.

**(a)** Show that if an immersed submanifold is equipped with the subspace topology, then it's an embedded submanifold. Show in general that the topology of an immersed submanifold is finer than the subspace topology.

*Remark:* To convince yourself, find an open subset of the figure-eight in example 11.12 that is not open in the subspace topology.

Let $F : N \to M$ be a smooth injective immersion. Note that it might not be an embedding as $F$ might not be a homeomorphism onto its image.

**(b)** Show that there exists a unique topology and smooth structure on $F(N)$ such that $F(N)$ is an immersed submanifold with the property that $F : N \to F(N)$ is a diffeomorphism.

*Remark:* Compare with problem 1c.

**(c)** Show that if $N$ is compact, then $F$ is an embedding. Conclude that if $S$ is a compact immersed submanifold of $M$, then it's a submanifold.

*Remark:* The figure-eight is however compact as a subset of $\mathbb{R}^2$. Does this contradict what you just proved?

**(d)** Show that $F$ is locally an embedding.

*Remark:* In general, any immersion is locally an embedding. This implies that in some sense $F(N)$ is locally an embedded manifold. Try to describe, possibly in the context of the figure eight, what this means rigorously.

**(e)** ***bonus*** Can there exists another smooth structure on $F(N)$ with the same topology that makes it another immersed submanifold? What if you can change the topology?

## Problem 3

In this problem, you will do some computational examples.

**(a)** Solve problem 11.4 in section 11.

**(b)** Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$F(x, y) = x^2 - y^2$$

Is $F^{-1}(0)$ a submanifold? Is there a topology on $F^{-1}(0)$ and a smooth structure such that it's an immersed submanifold?

**(c)** Show that the map

$$\begin{aligned} g &: \mathbb{R}P^2 \to \mathbb{R}^4 \\ &: (x : y : z) \mapsto \frac{1}{x^2+y^2+z^2}(xy, xz, zw, 2x^2+3y^2+4z^2) \end{aligned}$$

is an embedding.

*Hint:* It suffices to show that $g$ is an injective immersion by problem 2c. To do this, first show that the map

$$\begin{aligned} f &: S^2 \to \mathbb{R}^4 \\ &: (x, y, z) \mapsto (xy, xz, zy, 2x^2+3y^2+4z^2) \end{aligned}$$

is an immersion. Then how would you write $g$ in terms of $f$?

Denote the 2-dimensional torus by $T^2 = S^1 \times S^1 \subseteq \mathbb{C}^2$. Let $\alpha$ be an irrational number, and consider the curve $\gamma : \mathbb{R} \to T^2$ defined by

$$\gamma(t) = (e^{2\pi it}, e^{2\pi\alpha it})$$

**(d)** Show that $\gamma$ is an injective smooth immersion making the image an immersed 1-dimensional submanifold on $T^2$, but not an embedding.

**(e)** ***bonus*** Show that the curve is dense on $T^2$. (This means that the closure $\overline{\gamma(\mathbb{R})} = T^2$).

Denote by $\mathrm{SYM}(n, \mathbb{R}) \subseteq \mathrm{MAT}_{n\times n}(\mathbb{R})$ the space of symmetric n by n matrices, which is a vector space itself. Denote by $O(n) := \{A \in \mathrm{MAT}_{n\times n}(\mathbb{R}) : A^TA = I\}$ the space of orthogonal matrices, and denote by $\mathrm{SL}_n(\mathbb{R}) := \{A \in \mathrm{MAT}_{n\times n}(\mathbb{R}) : \det(A)^2 = 1\}$ the space of invertible matrices with determinant $\pm 1$.

**(f)** Consider the function $f : \mathrm{MAT}_{n\times n}(\mathbb{R}) \to \mathrm{SYM}(n, \mathbb{R})$ defined by $f(A) = A^TA$. Compute the derivative $D_Af : T_A\mathrm{MAT}_{n\times n}(\mathbb{R}) \to T_{f(A)}\mathrm{SYM}(n, \mathbb{R})$.

*Remark:* If our manifold is a vector space, we can Identify the tangent space with the space itself. $D_Af$ is then the standard derivative of a function from a vector space to a vector space, which is of course equivalent to the differential $f_{*,A}$.

**(g)** Show that $I$ is a regular value of $f$. Conclude that $O(n)$ is a compact submanifold and find its dimension.

**(h)** Use problem 1b to find the tangent space $T_I O(n)$.

**(i)** **(*bonus*)** Show that $\mathrm{SL}_n(\mathbb{R})$ is also a submanifold with 2 connected components and find the tangent space $T_I \mathrm{SL}_n(\mathbb{R})$. In the case when $n = 2$, show that each component is diffeomorphic to $S^1 \times \mathbb{R}^2$ (implying that it's non-compact!).

## Problem 4

Let $M$ be a smooth manifold and let $X$ be a section over the tangent bundle $TM$, which we also call a vector field. For a chart $(U, \phi = (x^1, ..., x^n))$, we can write $X$ in components:

$$X_p := X(p) = \sum_i X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

where $X^i$ are functions on $U$. We then have the following smoothness criterion: $X$ is a smooth section if and only if the component functions $X^i$ are smooth for every chart (check lemma 14.1). In particular, this shows that the coordinate vector fields $\frac{\partial}{\partial x^i}$ are smooth as sections over $TU$.

Every map $F : \mathbb{R}^n \to \mathbb{R}^n$ defines a vector field $X$ on $\mathbb{R}^n$ in the following way: $X : \mathbb{R}^n \to T\mathbb{R}^n$ is the section over $T\mathbb{R}^n$ defined by $X_p = (p, D_{F(p)})$.

**(a)** Show that the vector field $X$ is smooth if and only if $F$ is smooth.

Let $X : M \to TM$ be a smooth vector field. Then it acts on functions in this way: for $f \in C^\infty(M)$, $X(f)$ is a function on $M$ defined by $X(f)(p) := X(p)(f)$. Another useful smoothness criterion of vector fields is: $X$ is smooth if and only if $X$ sends $C^\infty$ functions to $C^\infty$ functions (check proposition 14.3).

**(b)** For a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and its corresponding vector field $X$, describe how $X$ acts on functions by finding an explicit formula depending on $F$. Verify the second smoothness criterion: $F$ is smooth if and only if $X$ sends $C^\infty$ functions to $C^\infty$ functions.

Given an algebra $A$, a map $D : A \to A$ is a derivation if $D$ is linear and satisfies the Leibniz rule: $\forall f, g \in A, D(fg) = D(f)g + fD(g)$. Denote by $Der(C^\infty(M))$ the space of derivations on $C^\infty(M)$. The obvious addition and scalar multiplication (with scalars being in $\mathbb{R}$ or $C^\infty(M)$) on $Der(C^\infty(M))$ makes it a vector space over $\mathbb{R}$ and a module over the ring $C^\infty(M)$, just like the space of smooth sections $\Gamma(M)$. As described above, each smooth section $X$ acts as a map from $C^\infty(M)$ to $C^\infty(M)$; in fact, it is straightforward to see that $X : C^\infty(M) \to C^\infty(M)$ is indeed a derivation (verify this). This defines a map

$$\begin{aligned}\Phi : \Gamma(M) &\to Der(C^\infty(M)) \\ : \Big[X : M \to TM\Big] &\mapsto \Big[X : C^\infty(M) \to C^\infty(M)\Big]\end{aligned}$$

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**(c)** Show that $\Phi$ is linear with respect to the vector space structure and the module structure of $\Gamma(M)$ and $Der(C^\infty(M))$, and show it's injective.

*Remark:* Attempt to show surjectivity and notice that you need more machinery to show it.

**(d)** Let $X, Y \in Der(C^\infty(M))$. Show that $X \circ Y$ is not another derivation but $X \circ Y - Y \circ X$ is.

*Remark:* This is called the Lie bracket $[X, Y] := X \circ Y - Y \circ X$, which gives the space of vector fields an even more complex algebraic structure called a Lie algebra.

## Problem 5

We will generalize the regular level set theorem. We know that if $c$ is a regular value of a $C^\infty$ map $F : N \to M$, then the level set $F^{-1}(c)$ is submanifold of $N$ of codimension m. What if we replace $c$ with a submanifold $Z$ of $M$? Under what conditions is $F^{-1}(Z)$ a submanifold of $N$ and what is its dimension?

Solve problem 9.10 in sectioon 9.