MAT 367: Differential Geometry
Assignment #2
Due on Sunday May 30, 2021 by 11:59 pm

Note: This assignment covers material from lectures and sections 5,6,7 and 8.1.

Problem 1

In this problem, you will find 2 different atlases on S^n and show that they are equivalent.

(a) Problem 5.3 in section 5.

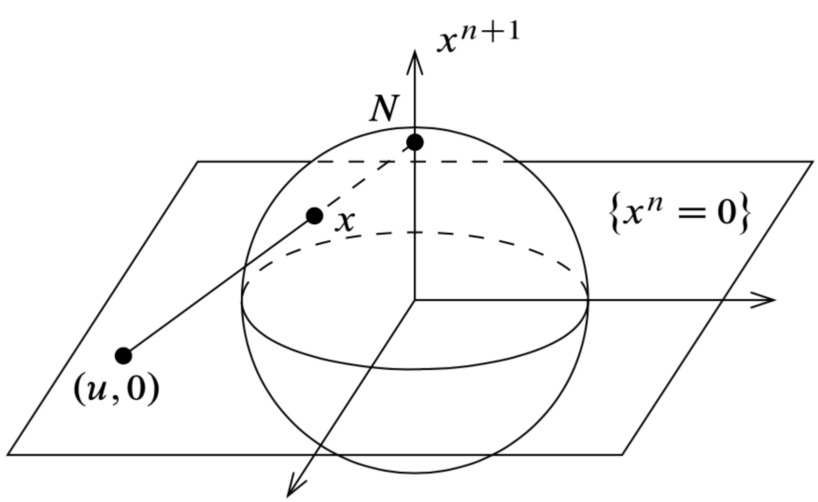
(b) Extend the idea in the previous problem to find an atlas for S^n. (Just write down what the atlas is; you don't need to show that it's an atlas since it's just the same idea as in (a)).

Let N be the north pole $(0, 0, ..., 1) \in S^n \subseteq \mathbb{R}^n$, and let S denote the south pole $(0, 0, ..., -1) \in S^n$. Define the stereographic projection $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, ..., x^{n+1}) = \frac{(x^1, ..., x^n)}{1 - x^{n+1}}$$

Let $\tilde{\sigma}(x) := -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(c) For any $x \in S^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersect the linear subspace $\{x^{n+1} = 0\}$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same linear subspace.



(d) Show that the atlas $\{(S^n \setminus \{N\}, \sigma), (S^n \setminus \{S\}, \tilde{\sigma})\}$ defines a C^∞ atlas on S^n.

(e) ***bonus*** Show that this smooth structure is the same as the one in part(b).

Problem 2

In this problem, you will construct a smooth atlas on an arbitrary vector space making it a smooth manifold, and you will study some important examples.

Let V be a vector space of dimension n. Fix a basis $\{v_1, ..., v_n\}$. Let $\Phi : V \to \mathbb{R}^n$ be the unique linear map satisfying $\Phi(v_i) = e_i$ for $i = 1, ..., n$, where e_i is the ith unit vector on $\mathbb{R}^n$. We can then equip V with the norm $\|\cdot\| : V \to \mathbb{R}$ defined by $\|v\| = \|\sum_{i=1}^n a_i v_i\| := \sqrt{\sum_{i=1}^n a_i^2}$. This makes $(V, \|\cdot\|)$ a normed vector space.

(a) Use the norm to construct a topology on V.

Hint: Now that we have a norm, we have the notion of an open ball centred at a vector v with radius r.

Remark: This topology is called the norm topology on $(V, \|\cdot\|)$.

(b) Notice that you used the norm to define the topology above, and the norm depended on the basis fixed at the beginning. Show that if you change the norm, the topology stays the same. (i.e. the topology defined above is the only norm topology one can have on V)

Hint: You can use the fact that any two norms on a finite dimensional vector space are equivalent.

(c) Show that Φ is a homeomorphism, and find a C^∞ atlas on V that makes it a smooth manifold of dimension n.

Now we ask if this is the only smooth structure we can have on V defined in this way. Fix another basis $\{w_1, ..., w_n\}$ on V and define $\Phi' : V \to \mathbb{R}^n$ as before. Using Φ', we have another C^∞ atlas on V as done in (c).

(d) Show that this atlas is C^∞ compatible with the one achieved in (c) (i.e. this atlas equips V with the same smooth structure).

Remark: We call this the standard smooth structure on V.

(e) Given two vector spaces V and W, denote by $L(V, W)$ the space of linear maps from V to W. Show that any $f \in L(V, W)$ is smooth. Conclude that if the dimensions are the same, then a bijective linear map $f \in L(V, W)$ is a diffeomorphism.

(f) Since $L(V, W)$ is a vector space itself, it can be equipped with the standard smooth structure making it a smooth manifold. Show that $L(V, W)$ is diffeomorphic to $\mathrm{MAT}_{m\times n}(\mathbb{R})$, where $\dim(V) = n$ and $\dim(W) = m$.

Hint: This is easier than it looks. Construct a bijective linear map $f \in L\Big(L(V,W), \mathrm{MAT}_{m\times n}(\mathbb{R})\Big)$ and use part (e).

Problem 3

The Grassmannian $G(k, n)$ is the generalization of $\mathbb{R}P^n$; it's the space of k-dimensional planes in $\mathbb{R}^{n+1}$ passing through the origin. You will prove in this problem that $G(k, n)$ is a smooth manifold of dimension $k(n - k)$.
Solve problem 7.8 in section 7.

Problem 4

Let M be a k-dim manifold in $\mathbb{R}^n$ as defined in the first assignment. Fix a point $p \in M$ and a coordinate map $\phi : V \to U$, where V is an open subset of $\mathbb{R}^k$ and V is a neighbourhood of p in M. (ϕ satisfies the three properties listed in assignment 1). The standard derivative of ϕ at p from multivariable calculus is defined as the linear map $D\phi|_q : T_q\mathbb{R}^k \to T_p\mathbb{R}^n$, where $\phi(q) = p$. It is common to define the tangent space of M at p, denoted by T_pM, as a subset of $T_p\mathbb{R}^n$; more precisely, we define $T_pM := D\phi|_q\left(T_q\mathbb{R}^k\right) \subseteq T_p\mathbb{R}^n$.

(a) Show that T_pM is independent of the choice of coordinate map. This shows that the definition of the tangent space is well defined.

One can alternatively define tangent vectors as derivations. We say a map $D : C^\infty_p(M) \to R$ is a derivation if it's linear and satisfies the Leibniz rule. Denote $\mathcal{D}_p$ to be the space of derivations of M at p, which is a vector space under the standard addition and scalar multiplication of maps from $C^\infty_p(M)$ to $\mathbb{R}$.

(b) Write down the natural isomorphism between T_pM and $\mathcal{D}_p$. (You don't need to prove it's an isomorphism since the idea is similar to what we did in lectures)

Hint: Associate each vector $v \in T_pM$ to a particular derivation. In lectures, we did this for $M = \mathbb{R}^n$.

Another alternative way of defining the tangent space is through an equivalence relation defined on the space of curves. Let A be the space of smooth curves $\gamma : (-\epsilon, \epsilon) \to M$ satisfying $\gamma(0) = p$. We define the following equivalence relation on A: $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for any $f \in C^\infty_p(M)$. An alternative definition of the tangent space is $\mathcal{V}_pM := A/\sim$.

(c) Define addition and scalar multiplication on $\mathcal{V}_pM$ making it a vector space over $\mathbb{R}$.

(d) Show that there is a natural isomorphism between T_pM and $\mathcal{V}_pM$.

(e) ***bonus*** Note that the definition above carries over directly to abstract smooth manifolds. Let $F : N \to M$ be a smooth map between smooth manifolds. If we adopt the above definition of the tangent space, how can we define the differential of F? Show that your proposed definition is consistent with the standard derivative in the case when $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$.

Hint: Recall that the differential of F will be a map $F_* : \mathcal{V}_pN \to \mathcal{V}_{F(p)}M$.

Problem 5

A ring is a set R together with two operators $+, * : R \times R \to R$: which we denote as addition and multiplication respectively, satisfying the following conditions:

- Addition and multiplication are associative: $\forall a, b, c \in R, a + (b + c) = (a + b) + c$ and $a * (b * c) = (a * b) * c$.
- Addition is commutative: $\forall a, b \in R, a + b = b + a$.
- There exists an element $0 \in R$, which we denote by the additive identity, such that $\forall a \in R, a + 0 = a$.
- For every $a \in R$, there exists an element $-a \in R$, which we denote by the additive inverse of a, such that $a + (-a) = 0$.
- For every $a, b, c \in R$, $a * (b + c) = (a * b) + (a * c)$ and $(b + c) * a = (b * a) + (c * a)$.

An algebra A over a field K is a ring that is also a vector space over K such that the ring multiplication satisfies the homogeneity condition: $\forall a, b \in A$, and $r \in K$, $r(a * b) = (ra) * b = a * (rb)$. Thus, an algebra has three operations: addition and multiplication of a ring and a scalar multiplication of a vector space.
If A and A' are algebras over K, we define an algebra homomorphism as a map $L : A \to A'$ that preserves the algebraic structure of A and A'. More precisely, L is linear and satisfies $L(a * b) = L(a) * L(b)$ for all $a, b \in A$.
Standard addition, multiplication and scalar multiplication of functions makes $C^\infty(M)$ an algebra over $\mathbb{R}$, where M is a smooth manifold.

(a) Define carefully addition, multiplication and scalar multiplication in $C_p^\infty(M)$ making it an algebra over $\mathbb{R}$.

Given a topological space M, denote by $C(M)$ the algebra of continuous functions $f : M \to \mathbb{R}$. Consider a continuous map between topological spaces $F : N \to M$. This induces a map (the pull back) $F^* : C(M) \to C(N)$ defined by $F^*(f) = f \circ F$.

(b) Show that F^* is an algebra homomorphism.

(c) Let $F : N \to M$ be a map of smooth manifolds. The pull back could also be thought of as a map $F^* : C^\infty_{F(p)}(M) \to C^\infty_p(N)$ defined by $F^*([f]) = [f \circ F]$. Convince yourself that this map is well defined and is an algebra homomorphism (you don't need to show that as it's similar to what you did in (b)). Show that F is smooth if and only if $F^*(C^\infty_{F(p)}(M)) \subseteq C^\infty_p(N)$ $\forall p \in N$.

Remark: We will prove later that any smooth function defined on a small neighbourhood of p can be extended to a smooth function defined on N. It will then follow that F is smooth if and only if $F^*(C^\infty(M)) \subseteq C^\infty(N)$.

(d) If F is a homeomorphism, show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(M)$ to $C^\infty(N)$.

Remark: This result shows that in a certain sense, the entire smooth structure of a smooth manifold M is encoded in the subset $C^\infty(M) \subseteq C(M)$. In fact, one can take an algebraic approach to smooth manifolds and define a smooth structure on a topological manifold M as a subalgebra of $C(M)$ satisfying certain properties. In other words, one can define a smooth structure by choosing which functions to be declared smooth.

Problem 6 *bonus*

In this problem, you will show that for every topological manifold that admits a smooth structure, it also admits uncountably many other different smooth structures.

(a) Given $\alpha > 0$, define the function F_α from the open ball B_1 centered at 0 in $\mathbb{R}^n$ to itself by $F_\alpha(x) = |x|^{\alpha-1}x$. Show that F_α is a homeomorphism and is a diffeomorphism if and only if $\alpha = 1$.

(b) Let M be a topological manifold of positive dimension that admits a smooth structure. Use the function above to find uncountably many distinct smooth structures.