MAT 367: Differential Geometry
Assignment #1
Due on Friday May 14, 2021 by 11:59 pm

Note: This assignment covers material from Week #1 (lectures, mandatory readings, and corresponding sections from the book).

Problem 1

Recall the definition of a k-dim manifold in $\mathbb{R}^n$. $M \subset \mathbb{R}^n$ is a k-dim manifold if $\forall p \in M$, there exists a neighbourhood U of p in M and an open set $V \subset \mathbb{R}^k$, and a map $f : V \to U$ satisfying:

- f is C^∞.
- f is bijective and continuous, and $f^{-1} : U \to V$ is continuous. (f is a homeomorphism).
- The derivative $Df|_q$ is injective $\forall q \in V$.

(a) What is meant by "a neighbourhood of p in M"?

(b) Informally explain why the above definition will be too general if the last condition is not included.

Remark: You can answer by giving an example of a set satisfying the above conditions except for the third, and justifying informally why we don't want to consider it as k-dim manifolds in $\mathbb{R}^n$.

We can extend the definition of smooth in the following way. A map $g : A \to \mathbb{R}^k$ is "smooth" on an arbitrary set $A \subset \mathbb{R}^n$ if it can locally be extended to a smooth function; more precisely, g is "smooth" if $\forall p \in A$, there exists a neighbourhood W of p in $\mathbb{R}^n$ and a smooth map $\tilde{g} : W \to \mathbb{R}^k$ such that $\tilde{g}|_{W \cap A} = g$. Given arbitrary sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$, we say $g : A \to B$ is a "diffeomorphism" if it's a "smooth" homeomorphism with a "smooth" inverse.

(c) Let M be a k-dim manifold in $\mathbb{R}^n$ and let $f : V \to U$ be a map where $V \subset \mathbb{R}^k$ is open and $U \subset M$ is open in M. Show that if f is a "diffeomorphism", then f satisfies the three conditions listed in the definition of a manifold above.

Remark: The other direction requires more advanced tools since you need to prove the existence of a smooth extension. (requires bump functions).

Consider the following new attempt of defining a surface in $\mathbb{R}^3$: $S \subset \mathbb{R}^3$ is a surface if it's the graph of a smooth function. More precisely, this means that there exists a smooth function $f : U \to \mathbb{R}$ on an open set $U \subset \mathbb{R}^2$ such that $S = \Gamma_f := \{(x, y, f(x, y)) : (x, y) \in U\}$ up to reordering of the coordinates.

(d) What is wrong with that definition? Is it too general or too specific?

Hint: Can you find a surface that doesn't satisfy the definition above, or a set that satisfies the definition above but is not a surface?

(e) Correct the above attempt by slightly modifying it, and then show it's equivalent to the definition described in the beginning of Problem 1.

Hint: Consider a set $S \subset \mathbb{R}^3$ that is *locally* the graph of a smooth function.

(f) ***(bonus)*** Let S be a surface in $\mathbb{R}^3$, and let $f : U \to S$ be an injective map defined on an open set $U \subset \mathbb{R}^2$ satisfying the first and third condition of the definition of a surface . Show that f is a homeomorphism onto its image.

Remark. This shows that if we *a priori* know that S is a surface, we don't need to check that f^{-1} is continuous, provided that the other conditions hold.

Problem 2

(a) Prove proposition A.42 in Appendix A.10.

(b) Let M be a topological space. Prove that M has k connected components if and only if there exists a continuous surjective function $f : M \to \{1, 2, ..., k\}$ such that for $i = 1, 2, ..., k$, $f^{-1}(i)$ are connected and pairwise disjoint.
(The set $\{1, 2, ..., k\}$ is equipped with the discrete topology).

(c) Show that the number of connected parts is a topological invariant. More precisely, if M and N are topological spaces that are homeomorphic, then M has k connected components if and only if N also does.

(d) Use this to show that $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ equipped with the subspace topology that it inherits from $\mathbb{R}^2$ is *not* a topological manifold.

(e) **(*bonus*)** Show that a topological manifold has at most countably many connected components.

Hint: M is second countable.

Problem 3

Let $M \subset \mathbb{R}^2$ be the boundary of a square with vertices $(\pm 1, \pm 1)$.

$$M = \left\{(x,y) \in \mathbb{R}^2 : |x| \leq 1 \text{ and } |y| \leq 1, \text{ with } |x| = 1 \text{ or } |y| = 1\right\}$$

Decide whether or not the charts (U, ϕ) and (V, ψ) given as

$$U = \{(x,y) \in M : y > -1\}, \quad \phi : U \to \mathbb{R}, \quad (x,y) \mapsto \frac{x}{1+y}$$
$$V = \{(x,y) \in M : y < 1\}, \quad \psi : V \to \mathbb{R}, \quad (x,y) \mapsto \frac{x}{1-y}$$

define a C^∞ atlas on M. Justify your answer.

(Hint: Compute $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ on their domain.)

Problem 4

In this problem, we will look at an example of a non-Hausdorff "topological manifold" called "the real line with 2 origins".

(a) Problem 5.1 (a) in section 5.

(b) Problem 5.1 (b) in section 5.

(c) Find a sequence $\{x_n\}_{n=1}^\infty$ in the real line with 2 origins that converges to two different points.

(d) Show that this doesn't happen in Hausdorff spaces. Namely, show that in a Hausdorff space, if a sequence $\{x_n\}_{n=1}^\infty$ converges to p and q, then $p = q$.

Problem 5

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Show that the graph of f

$$\Gamma_f := \left\{(x, f(x)) \in \mathbb{R}^2 : x \in \mathbb{R}\right\}$$

is a topological manifold of dimension 1.

(b) Is it a smooth manifold? Justify your answer.

(c) Is it a 1-dim manifold in $\mathbb{R}^2$ as defined in problem 1? Justify your answer.