MAT 367: Differential Geometry
Midterm
Thursday, June 18

Note: Submit by 11:30 AM through Crowdmark. No late submissions will be accepted.

Problem 1 [5]

Let $S = ([0,1) \times \{0\}) \cup (\{0\} \times [0,1)) \subseteq \mathbb{R}^2$ equipped with the subspace topology.

(a) Show that S is a smooth manifold of dimension 1.

(b) Find a function $f \in C^\infty(\mathbb{R}^2)$ such that $f|_S \notin C^\infty(S)$. Conclude that S is not a submanifold of $\mathbb{R}^2$.

Problem 2 [10]

Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be the function defined by

$$F(x, y, z, w) = (x + y^2 + z^2 + w^2 + y, x + y)$$

(a) Show that $(1, 0)$ is a regular value and conclude that $M := F^{-1}((1, 0))$ is a submanifold of $\mathbb{R}^4$.

(b) Show that M is diffeomorphic to S^2.

(c) For $p = (0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in M$, find a basis for T_pM expressed as a linear combination of the coordinate vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}$.

Problem 3 [5]

Let S be a submanifold of a manifold M and let $f \in C^\infty(S)$. Show that for all $p \in S$, there exists a neighbourhood W_p of p in M and a smooth function $\tilde{f}_p : W_p \to \mathbb{R}$ such that $\tilde{f}_p\big|_{W_p \cap S} = f|_{W_p \cap S}$.

Problem 4 [10]

(a) Let $f : S^n \to \mathbb{R}$ be a smooth function. Show that there exists $p \in S^n$ such that $v(f) = 0$ for all $v \in T_pS^n$.

(b) Let $f : S^1 \to \mathbb{R}$ be a function such that $p = (\cos f(p), \sin f(p))$ for any $p \in S^1$. Show that f cannot be smooth.
(*bonus* [2]) Show that f cannot be continuous.

Problem 5 [20]

Are the following true or false? Justify your answer briefly.
3 marks each; 20 is the maximum mark (excluding the bonus).

(a) For any germ $[f] \in C_p^\infty(M)$, there exists a function $\tilde{f} \in C^\infty(M)$ such that $[f] = [\tilde{f}]$.

(b) The map $F : \mathbb{R}^n \to \mathbb{R}P^n$ defined by $F(x^1, ..., x^n) = [x^1, ..., x^n, 2]$ is a diffeomorphism onto its image.

(c) T_pM is an n dimensional submanifold of TM that is diffeomorphic to $\mathbb{R}^n$. ($\dim M = n$).

(d) Let $M \times N$ be the product manifold of M and N and let $\pi : M \times N \to M$ be the projection map. Then for any $(p, q) \in M \times N$, $\text{Kernel}(\pi_{*,(p,q)})$ is isomorphic to T_pM.

(e) An injective submersion from M to M is an embedding.

(f) If $F : M \to M$ has constant rank $r < \dim M$, then F cannot be injective.

(g) Let $(U, \phi = (x^1, ..., x^n))$ be a chart on a manifold M. Then the coordinate vector fields $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$ is a basis for $\mathfrak{X}(U)$ making it a vector space over $\mathbb{R}$ of dimension n.

(h) Let γ be a curve on M and let $X \in \mathfrak{X}(M)$. If $f \in C^\infty(M)$ is constant on γ, then $X(f) = 0$ on γ.

(i) Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth embedded curve on M that lies in a coordinate open set U. There exists a smooth vector field $X \in \mathfrak{X}(M)$ such that $X_{\gamma(t)} = \gamma'(t)$ for all $t \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$.

(j) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two maximal atlases on M. If $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \phi$, then $\mathcal{M}_1 = \mathcal{M}_2$.

(k) **(*bonus* [2])** If there exists a surjective smooth immersion from N to M, then their dimensions are the same.

Problem 6: *bonus* [2]

Find an example of a smooth manifold $S \subseteq M$ equipped with the subspace topology that preserves smooth functions ($f|_S \in C^\infty(S)$ whenever $f \in C^\infty(M)$), but is not a submanifold of M.

Note: You can get up to 56/50 in this test.