MAT 367: Differential Geometry
Midterm
Thursday, June 17, 2021

Note: Submit by 11:30 AM through Crowdmark. No late submissions will be accepted. To make the test easier, problem 3 has been omitted.

Problem 1 [5]

Let $S = \{(x, y, z) : x^2 + y^2 = z^2 + 1\} \subset \mathbb{R}^3$. Let $\Big(U_{x>0}, \phi_{x>0} : (x, y, z) \mapsto (y, z)\Big)$ and $\Big(U_{y>0}, \phi_{y>0} : (x, y, z) \mapsto (x, z)\Big)$ be charts on S, where $U_{x>0} = \{(x, y, z) \in S : x > 0\}$ and $U_{y>0} = \{(x, y, z) \in S : y > 0\}$

(a) Compute $\phi_{x>0} \circ \phi_{y>o}^{-1} : \phi_{y>0}(U_{x>0} \cap U_{y>0}) \to \phi_{x>0}(U_{x>0} \cap U_{y>0})$ and verify that it's C^∞. Then write down a C^∞ atlas for S making it a smooth manifold of dimension 2 (you don't need to show that it is an atlas).

(b) Justify why S is a submanifold of $\mathbb{R}^3$ by finding an adapted chart relative to S near $(1, 0, 0)$.

Problem 2 [5]

Let M be a manifold of dimension n. Suppose $\{X_1, ..., X_n\}$ is a basis for $\mathfrak{X}(M)$ with respect to the module structure. Show that $\{X_{1p}, ..., X_{np}\}$ is a basis for T_pM for all $p \in M$.

Problem 4 [10]

Let $S(k, n) = \{A \in \mathrm{MAT}_{n\times k} : A^T A = I\}$ be the space of orthonormal k-frames in $\mathbb{R}^n$.

(a) Show that $S(k, n)$ is a compact submanifold of $\mathrm{MAT}_{n\times k}$ and find its dimension.

(b) For any $A \in S(k, n)$, find the tangent space $T_A S(k, n)$ as a subset of $\mathrm{MAT}_{n\times k}$.

(c) Define the function $f \in C^\infty(S(k, n))$ by $f(A) = \|A\|^2 = \sum_{i,j=1}^{i=n,j=k} A_{ij}^2$. For a vector field $X \in \mathfrak{X}(S(k, n))$, find an expression for $X(f)$.

(d) ***(bonus)*** Find a smooth submersion from $S(2, 4)$ to the Grassmanian $G(2, 4)$.

Problem 5 [10]

Define the map $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u,v) = \Big(\cos v(2+\cos u), \sin v(2+\cos u), \sin u\Big)$
Define the map $f : \mathbb{R} \to S^1$ by $f(x) = (\cos x, \sin x)$.

(a) Show that F is an immersion.

(b) Show that the map $g := f \times f : \mathbb{R}^2 \to T^2$ defined by $g(x,y) = (f(x), f(y))$ is a surjective submersion.
Hint: it suffices to show that f is a surjective submersion.

(c) Show that F induces a smooth map $\bar{F} : T^2 \to \mathbb{R}^3$ satisfying $F = \bar{F} \circ g$. Conclude that $\bar{F}$ is an embedding of the torus in $\mathbb{R}^3$.

Problem 6 [20]

Are the following true or false? Justify your answer briefly.
There are 10 questions, 3 marks each; 20 is the maximum mark (excluding the bonus).

(a) An injective submersion is a local diffeomorphism.

(b) Let S be a submanifold of M. For any $f \in C^\infty(S)$, there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}\Big|_S = f$.

(c) Let $F : M \to N$ be a smooth immersion. Then the global differential $F_* : TM \to TN$ is injective.

(d) Let M and N be manifolds and let π_M and π_N be the projection of $M \times N$ into M and N respectively. Then for any $(p,q) \in M \times N$, $T_{(p,q)}\big(M \times N\big) = \text{Kernel}(\pi_{M*}) \oplus \text{Kernel}(\pi_{N*})$

(e) Let S be a submanifold of M and let $X \in \mathfrak{X}(M)$ be a vector field tangent to S. Then for any $f \in C^\infty(M)$ that is constant on S, $X(f)|_S = 0$.

(f) The graph of $f(x) = |x|$ is an immersed submanifold of $\mathbb{R}^2$.

(g) Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two maximal atlases on M that are not equal. Then the union is another maximal atlas for M.

(h) If 0 is a critical value for $F : M \to \mathbb{R}^k$, then the level set $F^{-1}(0)$ is not a submanifold of M.

(i) For any smooth curve γ on M, γ' is a curve on TM.

(j) If $\mathfrak{X}(M)$ admits a basis $\{X_1, ..., X_n\}$ with respect to the module structure, then TM is diffeomorphic to $M \times \mathbb{R}^n$.

(k) **(*bonus* [2])** For any smooth map $F : N \to M$, there exists a manifold Q, an embedding $G_1 : N \to Q$, and a submersion $G_2 : Q \to M$ such that $F = G_2 \circ G_1$.

Problem 7: *bonus* [2]

For any chart (U, ϕ) on M, show that $\mathfrak{X}(U)$ is an infinite dimensional with respect to the vector space structure.

Note: You can get up to 56/50 in this test.