1) Assignment 3. (due June 11, 11:59 PM)

2) Termtest (Mock Termtest on June 16)

3) Post-lecture Practice questions.
(finger exercises)

Submanifolds

We start by studying more the algebraic properties of F_*.

Let $F: N \to M$ be a C^∞ map

$$\mathrm{rk}\,F(p) := \mathrm{rank}\left(F_{*,p} : T_pN \to T_{F(p)}M\right)$$
$$= \dim\left(F_{*,p}(T_pN)\right)$$
$$= \mathrm{rank}\left(D(\psi\circ F\circ\phi^{-1})\big|_{\phi(p)}\right) = \mathrm{rank}\left(\left[\frac{\partial F^i}{\partial x^j}\Big|_p\right]\right)$$

<u>Def</u>: F is an immersion at p if $F_{*,p}$ is injective. ← open condition

F is an immersion if $F_{*,p}$ is injective $\forall p\in N$
(we say F has constant rank n)

<u>Def</u>: F is a submersion at p if $F_{*,p}$ is surjective ← open condition.

F is a submersion if $F_{*,p}$ is surjective $\forall p\in N$.
(we say F has constant rank m)

<u>Proposition</u>: F is a local diffeomorphism at p iff F is a submersion and an immersion at p.

Ex: Canonical immersion $i : \mathbb{R}^n \to \mathbb{R}^m$, where $n < m$

$: (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^n, 0, \cdots, 0)$

Check i is an open map

i is smooth.

$$i_* : \frac{\partial}{\partial x^i}\Big|_p \mapsto \frac{\partial}{\partial x^i}\Big|_{i(p)} \qquad i = 1, \cdots, n$$

$$[i_*] = Di = \begin{bmatrix} I_{n \times n} \\ 0 \end{bmatrix} \text{ injective } \forall p$$

Ex: Canonical submersion $\pi : \mathbb{R}^n \to \mathbb{R}^m$, where $n > m$

$: (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^m)$

i is smooth.

$$[i_*] = Di = \begin{bmatrix} I_{m \times m} & 0 \end{bmatrix} \text{ which is surjective } \forall p.$$

The immersion Theorem: Let $F : N \to M$ be an immersion, let $p \in N$.

$\exists\ (U, \phi)$ near p on N and (V, ψ) near $F(p)$ on M s.t.

$$\psi \circ F \circ \phi^{-1} : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^n, 0, \cdots, 0)$$

Corollary: An immersion is locally injective.

Compare F_* with F

<u>Submersion Theorem</u>: Let $F : N \to M$ be a submersion. Let $p \in N$

$\exists$ - - - - - - - - -

$$\psi \circ F \circ \phi^{-1} : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^m)$$

<u>Corollary</u>: A submersion is an open map.

More generally:

<u>Constant rank theorem</u>: Let $F : N \to M$ be C^∞ map with constant rank r. Let $p \in N$.

$\exists$ - - - - -

s.t. $\psi \circ F \circ \phi^{-1} : (x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^r, 0, \cdots, 0)$

(Proven in Appendix).

<u>Def</u>: Let $F : N \to M$ be a C^∞ map. We say F is an embedding if it's an immersion that is homeomorphic onto its image,

1) F is an immersion ✓
2) $F : \underline{N} \to \underline{F(N)}$ is a homeomorphism

equipped with subspace topol.

(topological embedding

Ex1: Canonical immersion

$\Longrightarrow$ Daniel's Theorem:

An immersion is locally an embedding

Ex2: Let M be a k-dim manifold in $\mathbb{R}^n$ (as defined in Assignment 1)

Then the inclusion map $i: M \hookrightarrow \mathbb{R}^n$ is an embedding.

equipped with smooth structe

equipped with smooth structure

$i: p \longmapsto p$

subspace topology

Proof: Clearly $i: M \to i(M)$ is a homeomorphism onto its image (because M is equipped with the subspace topology)

Let $f: \underset{\text{open}}{V \subseteq \mathbb{R}^k} \to \underset{\text{open in } M}{U}$ be a map satisfying the 3 properties in Assignment 1, then (U, f^{-1}) is a chart

Coordinate representation of i is

$$\mathrm{Id} \circ i \circ f : U \to \mathbb{R}^n$$
$$: x \mapsto f(x) \quad \text{which is } C^\infty$$

and $D(\mathrm{Id} \circ i \circ f) = Df$ is injective by the third property. So i is an immersion

$\boxtimes$

<u>Remark</u>: The smooth structure of M inherited the smooth structure on $\mathbb{R}^n$ in the following sense:

If $f : \mathbb{R}^n \to \mathbb{R}$ is C^∞,
Then $f|_M = f \circ i : M \to \mathbb{R}$ is C^∞

<u>Def #1</u>: $S \subseteq M$ is an embedded submanifold of M if it's a manifold s.t. $i : S \hookrightarrow M$ is an embedding.

<u>Ex1</u>: k-dim manifolds in $\mathbb{R}^n$.

Ex2 : Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$.

Then $\Gamma_f = \{ (x, f(x)) : x \in \mathbb{R} \} \subseteq \mathbb{R}^2$ is a smooth manifold with C^∞ atlas

$$\{ (\Gamma_f, \pi|_{\Gamma_f}) \} \quad \text{where } \pi : (x,y) \mapsto x$$

Is this an embedded submanifold of $\mathbb{R}^2$?

Is $i : \Gamma_f \hookrightarrow \mathbb{R}^2$ an embedding?

topological embedding? ✓

Note that

$$\mathrm{Id} \circ i \circ \pi|_{\Gamma_f}^{-1} : x \mapsto (x, |x|)$$

which is not C^∞ so i is not an embedding.

Ex3 : Graph of a smooth function.

Let $F: N \to M$ be a C^∞ map.

Then $\Gamma_F = \{ (p, F(p)) \in N \times M \mid p \in N \} \subseteq N \times M$

is an embedded submanifold of $N \times M$.

Exc

Ex 4: open subsets $U \subseteq M$ are embedded submanifolds of the same dimension.

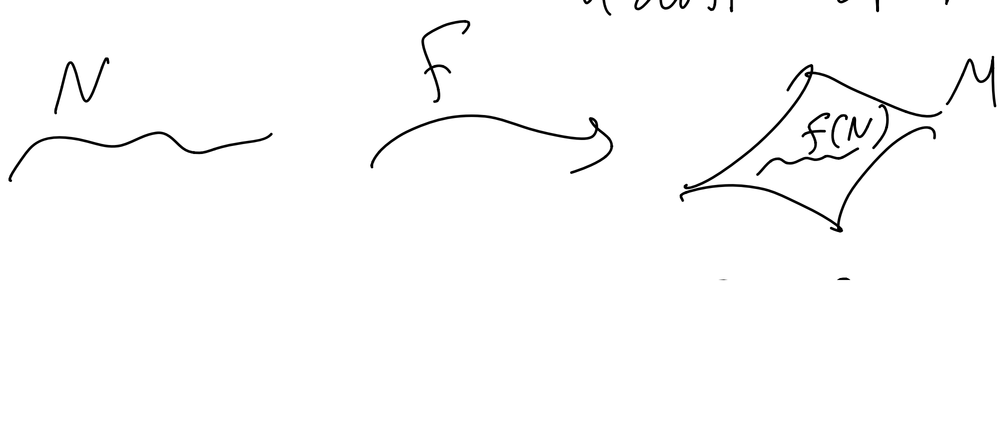
(they are the only ones)

etc

Ex 5: Let $f: N \to M$ be an embedding

Then $\exists!$ smooth structure on $f(N)$ s.t. $f(N)$ is an embedded submanifold with the property

$f: N \to f(N)$ is a diffeomorphism.



For every diffeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^3$,

$f(S^2)$ is a submanifold of $\mathbb{R}^3$

that is diffeomorphic to S^2

$S^2 \cong$ ellipse

$f(x,y,z) = (ax, by, cz)$

f restricts to an embedding.

In fact, Every manifold is a embedded submanifold of a Euclidean space.

Whitney's Embedding Thm:

For any smooth manifold M of dim n,

$\exists\ F: M \to \mathbb{R}^{2n}$ that is an embedding

one you studied in 257

so $F(M)$ is a submanifold of $\mathbb{R}^{2n}$ diffeomorphic to M

$\boxed{\text{Sharp!!}}$

why?
$\mathbb{RP}^2$ & Kleinbottle
are 2-dim manifolds that
cannot be embedded in $\mathbb{R}^3$

Post-lecture Practice questions:

1) do the exercises above?

2) Show $F:(x,y,z) \to (ax, by, cz)$
restricts to an embedding on S^2.
Assignment 3 problem 1c then implies that the ellipse $F(S^2)$ is diffeomorphic to S^2.

3) show $\mathrm{rk}\, F: N \to \mathbb{R}$ is continuous iff F is of constant rank.

4) for $p \in N$, show that $\{p\} \times M$ is an embedded submanifold of $N \times M$.

5) If $S \subseteq M$ is an embedded submanifold. Show that there is no other smooth structure on S making S a submanifold of M.

Pictures are fine.

6) Think of an immersion that is not an embedding.
Think of an injective immersion that is not an embedding.

7) Suppose $F: N \to M$ is C^∞ and $F(N) \subseteq S$ where S is a submanifold.
Show that $F: N \to S$ is also C^∞.

8) Problem 11.1, 11.2, 11.3

9) F is an immersion $\implies$ F is injective ?
F is a submersion $\implies$ F is surjective ?
F is a local diffeomorphism $\implies$ F is bijective ?