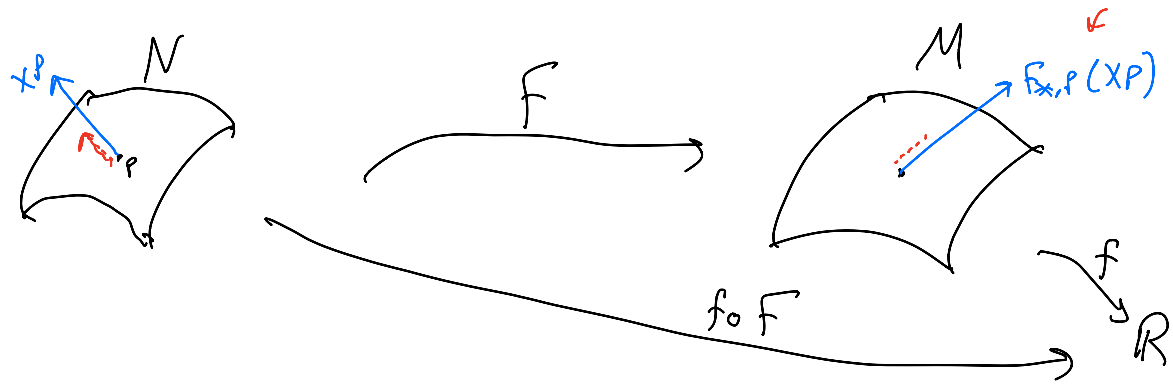
Let $F: N \to M$ be C^∞ map

Then $F_{*,p}: T_pN \to T_{F(p)}M$



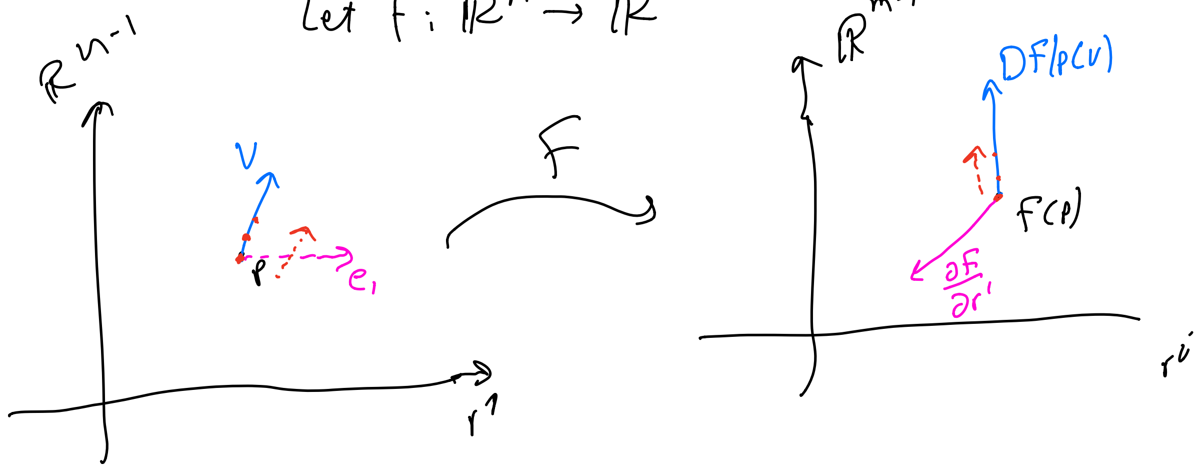
$F_{*,p}$ is defined in this way;

For $X_p \in T_pN$, $F_{*,p}(X_p)$ is the vector in $T_{F(p)}M$ satisfying:

for $f \in C_p^\infty(M)$, $F_{*,p}(X_p)(f) = X_p(f \circ F)$

Check that F_* is a generalization of DF in Euclidean space

Let $F: \mathbb{R}^n \to \mathbb{R}^m$



Old definition:

$$DF|_p : \overset{(old)}{T_p \mathbb{R}^n} \longrightarrow \overset{(old)}{T_{F(p)} \mathbb{R}^m}$$

$$DF|_p = \left[\frac{\partial F}{\partial r^1}\Big|_p \cdots \frac{\partial F}{\partial r^i}\Big|_p \right]$$

Let $v \in \sum_{i=1}^{n} v^i e_i$, Then $DF|_p(v) = \sum_{i=1}^{n} v^i \frac{\partial F}{\partial r^i}\Big|_p$

New definition: $F_{*,p} : \overset{(new)}{T_p \mathbb{R}^n} \longrightarrow \overset{(new)}{T_{F(p)} \mathbb{R}^m}$

D_p $\quad D_{F(p)}$

Instead of v, we are looking at

$D_v : g \in C_p^\infty(\mathbb{R}^n) \longmapsto \nabla g|_p \cdot v$

we want this diagram to commute.

$$\begin{tikzcd}
\overset{(old)}{T_p \mathbb{R}^n} \arrow[r, "DF|_p"] \arrow[d, "\Phi"'] & \overset{(old)}{T_{F(p)} \mathbb{R}^m} \arrow[d, "\Psi"] \\
D_p \arrow[r, "F_{*,p}"] & D_{F(p)}
\end{tikzcd}$$

Let $\Phi : T_p \mathbb{R}^n \to D_p$, $\Psi : T_{F(p)} \mathbb{R}^m \to D_{F(p)}$

$: v \longmapsto D_v \qquad : w \longmapsto D_w$

WTS: for $v \in T_p\mathbb{R}^n$,

$$F_{*,p}(D_v) = D_{DF|_p(v)}$$

$$\left(F_{*,p}(\Phi(v)) = \Psi(DF|_p(v)) \right.$$

Let $f \in C^\infty_{F(p)}(\mathbb{R}^m)$, then

$$D_{DF|_p(v)}(f) = \nabla f|_{F(p)} \cdot (DF|_p v)$$

$$= (\nabla f \circ F)|_p \cdot v$$

$$= D_v(f \circ F)$$

$$= F_{*,p}(D_v)(f)$$

$$\Rightarrow D_{DF|_p(v)} = F_{*,p}(D_v)$$

Chain rule: Let $F: N \to M$, $G: M \to P$ be C^∞ maps.

Let $p \in N$

$$\begin{tikzcd}
\overset{\mathbb{R}^n}{T_pN} \arrow[r, "F_{*,p}", "Df|_p"'] \arrow[rr, bend right, "G_{*,F(p)}\circ F_{*,p}", "DG|_{F(p)}\,Df|_p"'] & \overset{\mathbb{R}^m}{T_{F(p)}M} \arrow[r, "G_{*,F(p)}", "DG|_{F(p)}"'] & \overset{\mathbb{R}^p}{T_{G\circ F(p)}P}
\end{tikzcd}$$

Since $G\circ F$ is C^∞

$$T_pN \xrightarrow[D(G\circ F)|_p]{(G\circ F)_{*,p}} T_{G\circ F(p)}P$$

Thm: Chain Rule

$$(G\circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$$

Proof: for any $f \in C^\infty_{G\circ F(p)}P$ and for any $X_p \in T_pN$

$$\left[G_{*,F(p)} \circ F_{*,p}\right](X_p)\ (f)$$
$$= G_{*,F(p)}\left(F_{*,p}(X_p)\right)\ (f)$$
$$= F_{*,p}(X_p)\ (f\circ G)$$
$$= X_p\left(f\circ G\circ F\right) = (G\circ F)_{*,p}(X_p)(f)$$

so $G_{*,F(p)} \circ F_{*,p} = (G \circ F)_{*,p}$ ∎

Consider $Id : M \to M$ (C^∞ map)

$$Id_{*,p} : T_pM \to T_pM$$
$$: X_p \mapsto X_p$$

for $f \in C_p^\infty(M)$ and for $X_p \in T_pM$,

$$Id_{*,p}(X_p)(f) = X_p(f \circ Id)$$
$$= X_p(f)$$

$$\Rightarrow Id_{*,p}(X_p) = X_p$$

Corollary: If $F : N \to M$ is a diffeomorphism.

Then $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism.

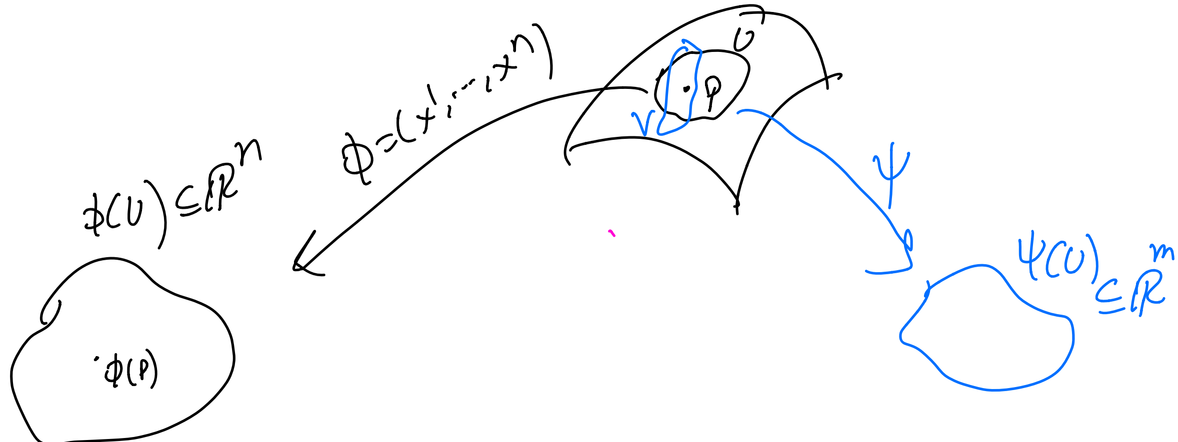
Proof: $Id_{*,p} = (F^{-1} \circ F)_{*,p} = (F^{-1})_{*,F(p)} \circ F_{*,p}$

$Id_{*,F(p)} = (F \circ F^{-1})_{*,F(p)} = F_{*,p} \circ F^{-1}_{*,F(p)}$

$\Rightarrow$ $F_{*,p}$ is invertible and so is an isomorphism and $\boxed{(F_{*,p})^{-1} = F^{-1}_{*,F(p)}}$. $T_pM := \underbrace{D\phi(\mathbb{R}^k)}_{n\text{-dim}}$

<u>Corollary</u>: $\dim M = \dim T_pM$

Let M be a manifold of dim n. (locally Euc of dim n)



Recall $\phi : U \to \phi(U)$ is a diffeomorphism

$\phi_{*,p} : \cancel{T_pU}^{\,T_pM} \to \cancel{T_{\phi(p)}}\phi(U)^{\,(\text{new})\, T_{\phi(p)}\mathbb{R}^n}$ is an isomorphism

<u>note</u>:

$\underline{C_p^\infty(U)} \cong \underline{C_p^\infty(M)}$

$\underline{[f]_U} \cong \underline{[f]_M}$

$T_pU \cong T_pM$

If $M=\mathbb{R}^n$,

$\underset{\uparrow \mathbb{R}^n}{\overset{(\text{old})}{T_pU}} = \underset{\nearrow \mathbb{R}^n}{\overset{(\text{old})}{T_p\mathbb{R}^n}}$

$\dim(T_pM) \overset{\phi_* \text{ is isomorphism}}{=} \dim(T_{\phi(p)}\overset{(\text{new})}{\mathbb{R}^n})$

$= \dim(\underbrace{T_{\phi(p)}\overset{(\text{old})}{\mathbb{R}^n}}_{\backslash\backslash\, \mathbb{R}^n})$

$\overset{\Phi: v \mapsto D_v \text{ is isomorphism}}{=} n$

$T_{\phi(p)}\overset{(\text{old})}{\mathbb{R}^n} = \mathbb{R}^n$

$\Rightarrow$ dimension of a manifold is well defined.

If (U,ϕ) and (V,ψ) are charts near p

s.t. $\phi : U \to \mathbb{R}^n$

$\psi : V \to \mathbb{R}^m$

ϕ_* is isomorphis

Then $n = \dim(\overset{(\text{new})}{T_{\phi(p)}}\mathbb{R}^n) \overset{?}{=} \dim(T_pM)$

ψ_* is isomorphism $\overset{?}{=} \dim(\overset{(\text{new})}{T_{\psi(p)}}\mathbb{R}^m)$

$= m$

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for $f \in C_p^\infty(M)$

Recall the partial derivative $\left.\dfrac{\partial f}{\partial x^i}\right|_p = \left.\dfrac{\partial(f\circ\phi^{-1})}{\partial r^i}\right|_{\phi(p)}$

Define $\left.\dfrac{\partial}{\partial x^i}\right|_p : C_p^\infty(M) \to \mathbb{R}$

$: f \longmapsto \left.\dfrac{\partial(f\circ\phi^{-1})}{\partial r^i}\right|_{\phi(p)}$

standard partial derivative

$\left.\dfrac{\partial}{\partial x^i}\right|_p$ is a derivation (exc)

$$\Rightarrow \frac{\partial}{\partial x^i}\Big|_p \in T_pM$$

<u>Proposition</u>: $\underbrace{\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}}$ is a basis for T_pM

<u>Proof</u>: On $\mathbb{R}^n$, $\{e_1, \dots, e_n\}$ make a basis for $\overset{(old)}{T_{\phi(p)}}\mathbb{R}^n$,

Since $\Phi : v \mapsto D_v$

$$\Rightarrow \{D_{e_1}, \dots, D_{e_n}\} \text{ is a basis for } \overset{(new)}{T_{\phi(p)}}\mathbb{R}^n \leftarrow D_{\phi(p)}$$

$$\frac{\partial}{\partial r^1}\Big|_p$$

$$\Big\downarrow$$

$$\left\{ \frac{\partial}{\partial r^1}\Big|_{\phi(p)}, \dots, \frac{\partial}{\partial r^n}\Big|_{\phi(p)} \right\}$$

insted $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2$, think $\frac{\partial}{\partial r^2}$

Since $\phi_{*,p}$ is an isomorphism

$$\phi_{*,p} : T_pM \longrightarrow \overset{(new)}{T_{\phi(p)}}\mathbb{R}^n$$

Then $\left\{ \phi^{-1}_{*,\phi(p)}\left(\frac{\partial}{\partial r^1}\Big|_{\phi(p)}\right), \ldots, \phi^{-1}_{*,\phi(p)}\left(\frac{\partial}{\partial r^n}\Big|_{\phi(p)}\right) \right\}$

is a basis for T_pM.

<u>WTS</u>: $\phi^{-1}_{*,\phi(p)}\left(\frac{\partial}{\partial r^i}\Big|_{\phi(p)}\right) = \frac{\partial}{\partial x^i}\Big|_p$

for $f \in C^\infty_p(M)$, $\phi^{-1}_{*,\phi(p)}\left(\frac{\partial}{\partial r^i}\Big|_{\phi(p)}\right)(f)$

$$= \frac{\partial}{\partial r^i}\Big|_{\phi(p)} (f \circ \phi^{-1})$$

$$= \frac{\partial}{\partial x^i}\Big|_p (f)$$

$\blacksquare$

<u>Def</u>: $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is the coordinate basis for T_pM wrt $(U, \phi = (x^1, \ldots, x^n))$

<u>Exc</u>: for any basis $\{v_1, \ldots, v_n\}$ of T_pM, $\exists (U, \phi)$ near p s.t. $v_i = \frac{\partial}{\partial x^i}\Big|_p$

Let $(V, \psi = (y^1, \cdots, y^n))$ be another chart near p,

then $\left\{ \frac{\partial}{\partial y^1}\Big|_p, \cdots, \frac{\partial}{\partial y^n}\Big|_p \right\}$ is another

coordinate basis.

Exc:
$$\frac{\partial}{\partial y^i} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$$

(Apply $x^i \in C_p^\infty(M)$ to both sides

and show $\frac{\partial x^i}{\partial x^j}\Big|_p = \delta^i_j$)

Let $F : N \to M$ be a C^∞ map

$F_{*,p} : T_pN \to T_{F(p)}M$

$$\begin{tikzcd} T_pN \arrow[r, "F_*"] \arrow[d, "\Phi_\gamma"'] & T_{F(p)}M \arrow[d, "\psi_\beta"] \\ \mathbb{R}^n \arrow[r, "{[F_*]^\beta_\gamma}"] & \mathbb{R}^m \end{tikzcd}$$

$m \times n$ matrix making diagram commute.

Let $(U, \phi = (x^1, \cdots, x^n))$ be chart near p

Let $(V, \psi = (y^1, \cdots, y^n))$ be chart near $F(p)$

We can write $F_{*,p}$ wrt $\gamma = \left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ and $\beta = \left\{ \frac{\partial}{\partial y^i}\Big|_{F(p)} \right\}$.

$$[F_{*,p}]_{\gamma}^{\mathcal{B}} = \left[\left[F_*\left(\frac{\partial}{\partial x^1}\right)\right]_{\mathcal{B}} \cdots \left[F_*\left(\frac{\partial}{\partial x^n}\right)\right]_{\mathcal{B}} \right]$$

$m \times n$ matrix

$m \times 1$ vector $\begin{bmatrix} a_1^i \\ \vdots \\ a_m^i \end{bmatrix}$ satisfying

$$F_*\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} a_j^i \left.\frac{\partial}{\partial y^j}\right|_{F(p)}$$

We compute the coefficients a_j^i. Apply y^k to both sides:

$$a_k^i = \sum_{j=1}^{n} a_j^i \left.\frac{\partial}{\partial y^j}\right|_{F(p)} (y^k)$$

$$= F_*\left(\frac{\partial}{\partial x^i}\right)(y^k)$$

$$= \left.\frac{\partial}{\partial x^i}\right|_p (y^k \circ F)$$

$$= \left.\frac{\partial (\psi \circ F \circ \phi^{-1})^k}{\partial r^i}\right|_{\phi(p)} = \left.\frac{\partial f^k}{\partial x^i}\right|_p$$

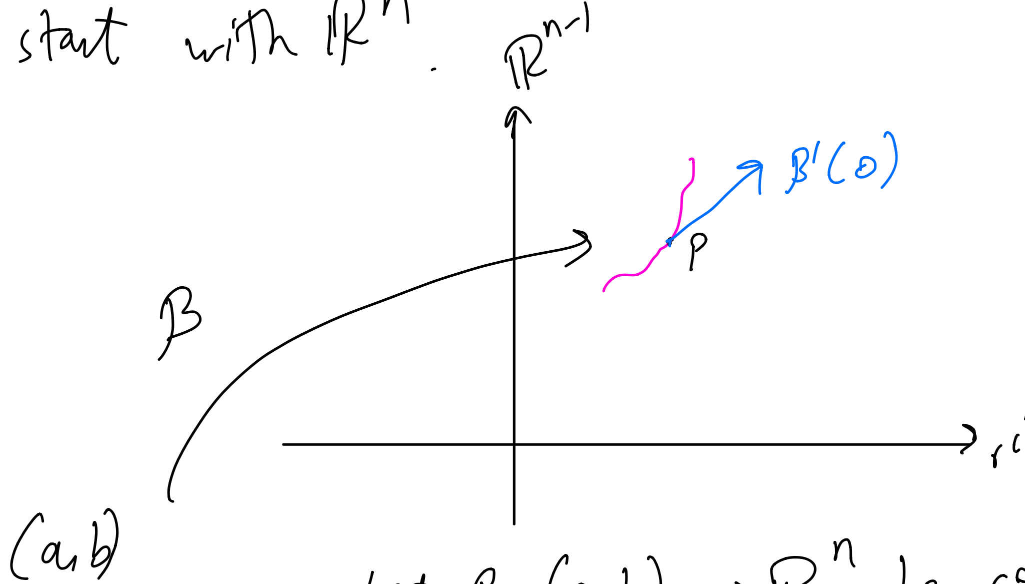
So $$\boxed{[F_{*,p}]_{\gamma}^{\beta} = D(\psi \circ F \circ \phi^{-1})}$$

Jacobian of F wrt the charts.

Curves on M

(how they give us a meaning to "direction")

We start with $\mathbb{R}^n$.



Let $\beta : (a,b) \to \mathbb{R}^n$ be a C^∞ map
s.t. $\beta(0) = p$.
(β is a smooth curve passing through p)

$$\beta : t \mapsto (r^1(t), \dots, r^n(t))$$

$$\overset{(old)}{D\beta\big|_{t_0}} : \overset{(old)}{T_{t_0}\mathbb{R}} \longrightarrow \overset{(old)}{T_{\beta(t_0)}\mathbb{R}^n}$$

$\nwarrow \mathbb{R}$ $\qquad\qquad\qquad \nwarrow \mathbb{R}^n$

$$: c \longmapsto D\beta|_{t_0}(c) = c \begin{bmatrix} r^{1\prime}(t_0) \\ \vdots \\ r^{n\prime}(t_0) \end{bmatrix}$$

$\left(\underset{\nearrow\, n\times 1 \text{ vector}}{D\beta|_{t_0}} \in L(\mathbb{R}, \mathbb{R}^n) = \mathbb{R}^n \right)$

$D\beta|_{t_0}$ we denote it by $\beta'(t_0)$ in multivariable course

Velocity vector $\beta'(t_0)$ of the curve β at P is defined as

$$\beta'(t_0) := D\beta|_{t_0}(1) = \begin{bmatrix} r^{1\prime}(t_0) \\ \vdots \\ r^{n\prime}(t_0) \end{bmatrix}$$

Now we use the new definition

$$\beta_{*,t_0} : \overset{(new)}{T_{t_0}\mathbb{R}} \longrightarrow \overset{(new)}{T_{\beta(t_0)}\mathbb{R}^n}$$

$$\left\{ c \frac{d}{dt}\Big|_{t_0} : c \in \mathbb{R} \right\} \qquad \operatorname{span}\left\{ \frac{\partial}{\partial r^i}\Big|_{\beta(t_0)} : i = 1, \cdots, n \right\}$$

$$: \; c\frac{d}{dt}\Big|_{t_0} \longmapsto \sum_{i=1}^{n} c\, r^{i\,\prime}(t_0)\frac{\partial}{\partial r^i}\Big|_{\beta(t_0)}$$

Velocity vector at $\beta(t_0)$ is then

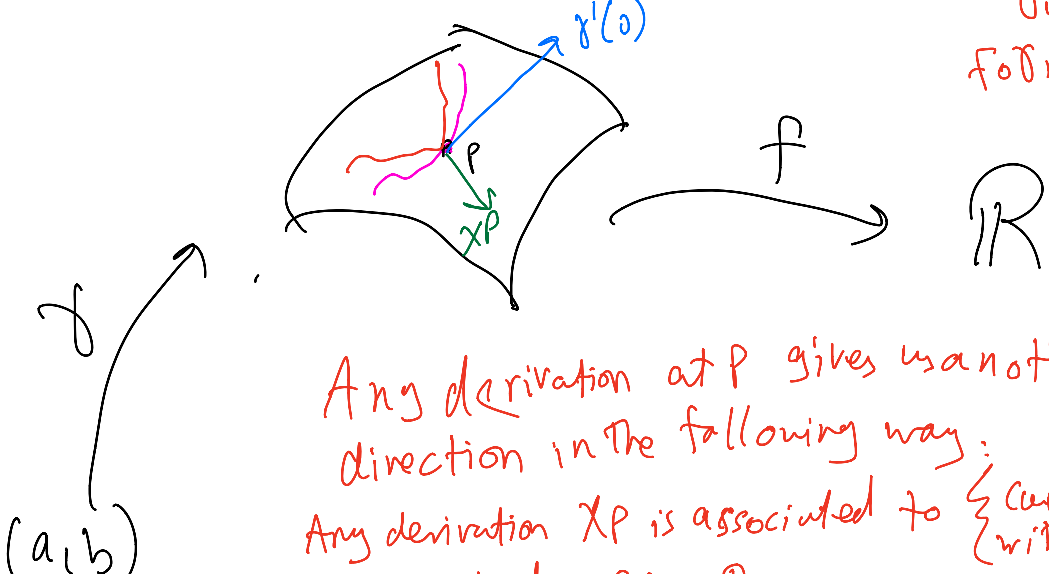
$$\beta'(t_0) = \sum_{i=1}^{n} r^{i\,\prime}(t_0)\frac{\partial}{\partial r^i}\Big|_{\beta(t_0)}$$

by above

$$= \underbrace{\beta_{*,t_0}\left(\frac{d}{dt}\Big|_{t_0}\right)}_{\text{will be used on manifolds.}}$$

Let M be a smooth manifold

$\gamma_1 \sim \gamma_2$ if $(f\circ\gamma_1)' = (f\circ\gamma_2)'$ $\forall f \in C^\infty_p$



Any derivation at p gives us a notion of "direction" in the following way: $[\gamma]$

Any derivation X_p is associated to $\{$curves passing p with velocity vector $X_p\}$ with the property that $\frac{d}{dt}\big|_{t=0} f\circ\gamma$ is the same for any such curve.

implies all those curves are passing through p in the same direction.

Let $\gamma : (a,b) \to M$ s.t. $\gamma(0) = p$

We define the velocity vector $\gamma'(t_0)$ of the curve γ at $\gamma(t_0)$ as:

$$\gamma'(t_0) := \gamma_{*t_0}\left(\frac{d}{dt}\Big|_{t_0}\right)$$

Let $f \in C_p^\infty(M)$,

Then $\gamma'(0)(f) = \gamma_{*0}\left(\frac{d}{dt}\Big|_{t_0}\right)(f)$

$$= \underbrace{\frac{d}{dt}\Big|_0 f \circ \gamma(t)}$$

standard derivative.

independent of the curve used (is the same for any other curve $\tilde{\gamma}$ satisfying $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(0) = \gamma'(0)$)

Proposition: for any $X_p \in T_pM$, $\exists \gamma : (a,b) \to M$ (C^∞) s.t. $\gamma(0) = p$, $\gamma'(0) = X_p$.

Let $F: N \to M$

be C^∞ map

$$F_{*,p}: T_pM \to T_{F(p)}N$$

I claim for any curve γ passing through p in the "direction" of X_p,

$F \circ \gamma$ is a curve passing through $F(p)$ in the "direction" of $F_{*,p}(X_p)$

"direction" of $F \circ \gamma$ $= F \circ \gamma\,'(0)$

$$= (F \circ \gamma)_{*,0}\left(\frac{d}{dt}\bigg|_{t=0}\right)$$

chain rule

$$= \left(F_{*,p} \circ \gamma_{*,0}\right)\frac{d}{dt}\bigg|_{t=0}$$

$$= F_{*,p}\left(\gamma_{*,0}\left(\frac{d}{dt}\Big|_{t=0}\right)\right)$$

$$= F_{*,p}\left(\gamma'(0)\right)$$

$$= F_{*,p}(X_p)$$

$$= \text{"direction" of } F_{*,p}(X_p)$$