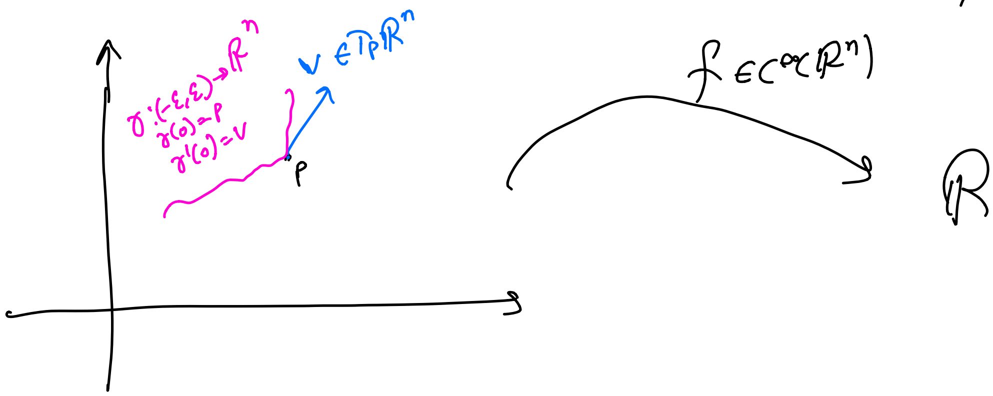
Attempt #2 in defining the tangent space:



Tangent space gives us a notion of direction away from a point p and that gives us a notion of directional derivative

For $v \in T_p\mathbb{R}^n$, we can talk about $D_v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$

defined by

$$D_v(f) = \frac{d}{dt}\Big|_{t=0} f \circ \gamma(t)$$
$$= Df|_p \cdot \gamma'(0)$$
$$= Df|_p \cdot v$$

For each vector $v \in T_p\mathbb{R}^n$, we associate it to a $D_v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$

Let $f : U \to \mathbb{R}$ (C^∞), U open neighbhd of p

Let $g : V \to \mathbb{R}$ (C^∞), V open neighbhd of p

we define an equivalence relation $\sim$:

$f \sim g$ if $\exists$ open neighbhd W of p s.t. $W \subseteq U \cap V$
and $f|_W = g|_W$

Notice that $D_v(f) = D_v(g)$

$$\left(\begin{array}{l} \text{let } h = f - g \text{, then } h|_W \equiv 0 \\ 0 = D_v h = D_v(f-g) = D_v f - D_v g \end{array} \right)$$

Define $C_p^\infty(\mathbb{R}^n)$ as the space of all equivalence classes

$[f]$ is the called the germ of f at p.

$D_v : \overset{C^\infty(U)}{C^\infty(\mathbb{R}^n)} \to \mathbb{R}$ is constant on equivalence classes, so it induces a map $D_v : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$

Algebra over $\mathbb{R}$

1) V.S. over $\mathbb{R}$ $\left(\begin{array}{l} [f]+[g] := [f+g] \\ c[f] := [cf] \end{array} \right)$

2) ring $\left(\begin{array}{l} [f]+[g] := [f+g] \\ [f][g] := [fg] \end{array} \right)$

D_v is not any map, it satisfies

1) linear wrt the V.S. structure of C_p^∞

2) satisfies the Leibniz rule

$$D_v(fg) = D_v(f)\, g(p) + f(p)\, D_v(g)$$

Def: A map $D : C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a derivation if it satisfies (1) and (2).

Let $\mathcal{D}_p = \left\{ D : C_p^\infty(\mathbb{R}^n) \to \mathbb{R} \;\middle|\; D \text{ is a derivation} \right\}$

Thm: 1) $\mathcal{D}_p$ is a V.S. over $\mathbb{R}$

2) The map $\Phi : T_p\mathbb{R}^n \to \mathcal{D}_p$

$: v \mapsto D_v$

is an isomorphism.

Proof: 1) etc

2) first Φ is linear.

$$
\begin{aligned}
\Phi(cv_1 + v_2)(f) &= D_{cv_1+v_2}(f) \\
&= Df \cdot (cv_1 + v_2) \\
&= c\,Df \cdot v_1 + Df \cdot v_2 \\
&= c\,D_{v_1}(f) + D_{v_2}(f) \\
&= c\,\Phi(v_1) + \Phi(v_2)
\end{aligned}
$$

Second, Φ is injective

Let $v \in T_p\mathbb{R}^n$ s.t. $\Phi(v) = D_v \equiv 0$

Consider the coordinate functions r^i

$$0 = D_v(r^i) = \sum_{j=1}^{n} v^j \underbrace{D_{e_j}(r^i)}_{\delta_j^i}$$

$$= v^i$$

$\Rightarrow v = 0$

Third, Φ is surjective. Let $D \in \mathcal{D}_p$.

WTS $\exists v \in T_p\mathbb{R}^n$ s.t. $D = D_v$

Lemma 1: Taylor's Theorem.

If $f \in C^\infty(U)$ where U is an open ball centered at P

Then $\exists g_i \in C^\infty(U)$ s.t.

1) $g_i(P) = \left.\frac{\partial f}{\partial r^i}\right|_P$

2) $f(x) = f(P) + \sum_{i=1}^{n}(x^i - p^i)\, g_i(x)$

Proof: $\{ \gamma(t) = p + t(x-p)$

Then $f(x) - f(p) = \displaystyle\int_0^1 \frac{d}{dt} f\circ\gamma(t)\, dt$

$$= \int_0^1 \sum_{i=1}^n \left.\frac{\partial f}{\partial r^i}\right|_{\gamma(t)} (x^i - p^i)\, dt$$

$$= \sum_{i=1}^n (x^i - p^i) \underbrace{\int_0^1 \left.\frac{\partial f}{\partial r^i}\right|_{\gamma(t)} dt}_{= g_i(x)}$$

let $g_i(x) := \displaystyle\int_0^1 \left.\frac{\partial f}{\partial r^i}\right|_{\gamma(t)} dt$

Also, $g_i(p) = \displaystyle\int_0^1 \left.\frac{\partial f}{\partial r^i}\right|_p dt = \left.\frac{\partial f}{\partial r^i}\right|_p \int_0^1 dt = \left.\frac{\partial f}{\partial r^i}\right|_p$ $\square$

Let $f \in C_p^\infty(\mathbb{R}^n)$, then

$$D(f) = D\left(f(p) + \sum_{i=1}^n (x^i - p^i)\, g_i(x)\right)$$

$$= D(\cancel{f(p)}) + \sum_{i=1}^{n} \Big[\underbrace{D(x^i - p^i)}_{Dx^i - \cancel{Dp^i}} \underbrace{g_i(p)}_{\frac{\partial f}{\partial x^i}\big|_p} + \cancel{(x^i - p^i)}\Big|_{x=p} D(g_i) \Big]$$

$$\left(\text{Recall } D_v(c) = 0 \quad \text{where } c \text{ is constant} \right)$$

$$\underbrace{Dc}_{0} \cdot v$$

<u>Lemma 2</u>: For any $D \in \mathcal{D}_p$, $D(c) = 0$ for any constant function c.

<u>Proof</u>:

$$D(c) = cD(1) = cD(1 \cdot 1)$$
$$= c[D(1)\,1 + 1\,D(1)]$$
$$= 2cD(1)$$
$$= 2D(c)$$

$$\Rightarrow D(c) = 0$$

$$\text{So } D(f) = \sum_{i=1}^{n} \underbrace{Dx^i}_{= v^i} \overbrace{\frac{\partial f}{\partial x^i}\Big|_p}^{D_{e_i}(f)} = \nabla f \cdot v = D_{\sum Dx^i e_i}(f)$$

where $v = \sum_{i=1}^{n} Dx^i e_i$

so $D = D_v$ for v

$\blacksquare$

So $\mathcal{D}_p$ is an equivalent definition of $T_p\mathbb{R}^n$.

Def of Tangent space on Manifolds

__Def__: Let M be a smooth manifold

we say $v: C_p^\infty(M) \to \mathbb{R}$ is a derivation at p

if v is linear and satisfies Leibniz rule
$(v(fg) = v(f)g(p) + f(p)v(g))$

We define the tangent space of M at p as

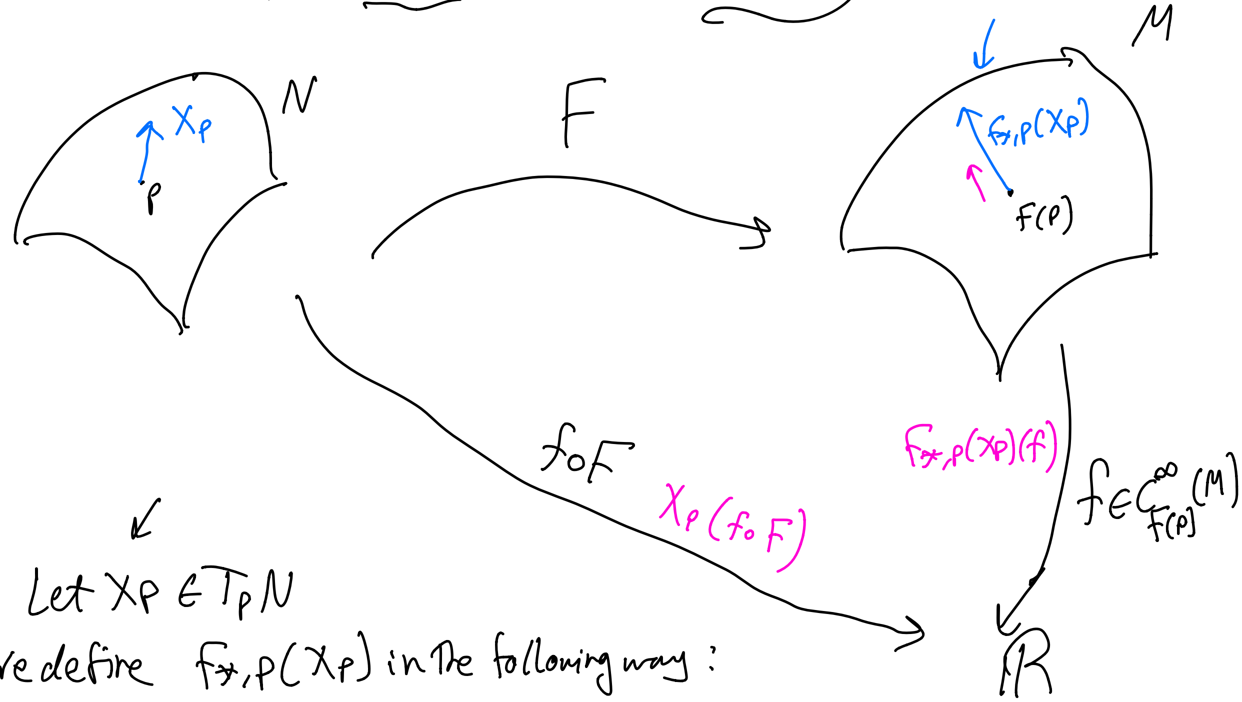
$$T_pM := \left\{ v: C_p^\infty(M) \to \mathbb{R} \;\middle|\; \begin{array}{c} v \text{ is a derivation} \\ \text{at } p \end{array} \right\}$$

Defining the derivative of a smooth map

Let $F: N \to M$ be a C^∞ map

We define its differential at p

$$F_{*,p} : \underbrace{T_p N} \longrightarrow \underbrace{T_{F(p)} M}$$



Let $X_p \in T_p N$
we define $F_{*,p}(X_p)$ in the following way :

for $f \in C^\infty_{F(p)}(M)$, $F_{*,p}(X_p)(f) = X_p(f \circ F)$

Note that $F_{*,p}$ is a linear map (exc)

We will show $F_{*,p} : D_p \to D_{F(p)}$ is the same as
$DF|_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$