-oH

- Assignment 2

Def: Let $(M, \mathcal{A})$ be a smooth manifold and let $f: M \to \mathbb{R}$. f is C^∞ at p if $\exists\ (U, \phi) \in \mathcal{A}$ near p s.t. $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞ at $\phi(p)$.

standard C^∞

Proposition (Smoothness Criterion): The following are equivalent.

1) f is C^∞ on M
2) There is an atlas $\mathcal{A}$ for M s.t. for any chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is C^∞.

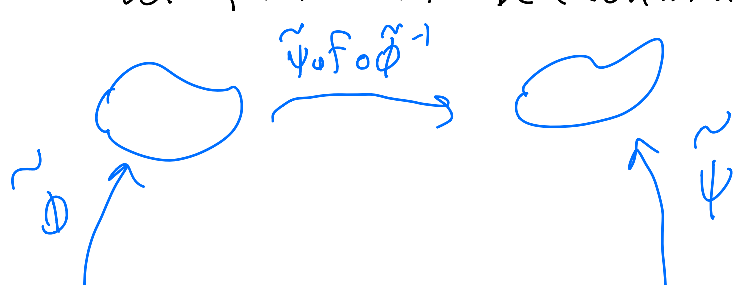
Check: Let $M = \mathbb{R}^n$ with $\mathcal{A} = \{(\mathbb{R}^n, Id)\}$.
Let $f: \mathbb{R}^n \to \mathbb{R}$.

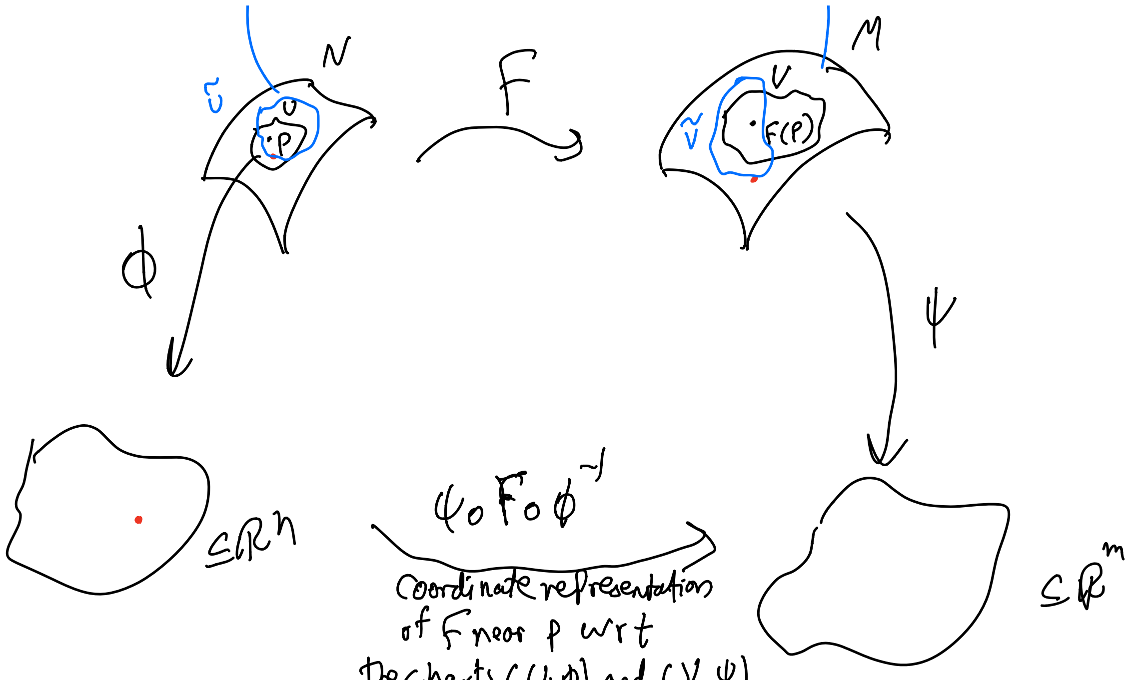
We want to check f is C^∞ $\nearrow$ $\Rightarrow$ f is C^∞ (standard)

f is C^∞ $\nearrow$ $\Rightarrow$ $f \circ Id^{-1}$ is C^∞ standard $\Rightarrow$ f is C^∞ (standard)

Remark: A maximal atlas on a topological manifold determines the space $C^\infty(M)$.

Let M and N be smooth manifolds.
Let $F: N \to M$ be a continuous map





coordinate representation of F near p wrt the charts (U,ϕ) and (V,ψ)

<u>Def</u>: F is C^∞ at p if $\exists (U,\phi) \in \mathcal{M}_N$ and $(V,\psi) \in \mathcal{M}_M$ s.t. $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$

standard

<u>Check that the def is independent of charts</u>:

$$\tilde{\psi} \circ F \circ \tilde{\phi}^{-1} = \underbrace{(\tilde{\psi} \circ \psi^{-1})}_{C^\infty} \circ (\psi \circ F \circ \phi^{-1}) \circ \underbrace{(\phi \circ \tilde{\phi})}_{C^\infty}$$

$$\Rightarrow \tilde{\psi} \circ F \circ \tilde{\phi}^{-1} \text{ is } C^\infty \text{ at } \tilde{\phi}(p)$$

Proposition (C^∞ criterion): Let $F : N \to M$ be a continuous map. Then the following are equivalent.

1) F is C^∞

2) $\exists \mathcal{A}_N$ for N and $\exists \mathcal{A}_M$ for M s.t.

$\forall (U, \phi) \in \mathcal{A}_N$ and $\forall (V, \psi) \in \mathcal{A}_M$,

$\psi \circ F \circ \phi^{-1}$ is C^∞

Check that this definition is a generalization:

Let $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$

$\mathcal{A}_N = \{(\mathbb{R}^n, Id)\}$ $\quad$ $\mathcal{A}_M = \{(\mathbb{R}^m, Id)\}$

Let $F: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function.

F is $C^\infty \Rightarrow Id \circ F \circ Id^{-1}$ is C^∞ (standard)

$\Rightarrow F$ is C^∞ (standard)

Proposition (C^∞ criterion): Let $F: N \to \mathbb{R}^m$, $F = (F^1, \dots, F^m)$

F is C^∞ iff F^i are C^∞.

Proposition: Let $F: N \to M$ and $G: M \to P$ be C^∞ maps, then $G \circ F$ is C^∞.

Def: $F: N \to M$ is a diffeomorphism if F is homeomorphism and C^∞, and F^{-1} is C^∞.

Proposition: Let (U, ϕ) be a chart on M, Then
$\phi : U \to \phi(U)$ is a diffeomorphism.

Proof: ϕ is a homeomorphism ✓

$\mathcal{A}_U = \{(U, \phi)\}$ is an atlas for U

$\mathcal{A}_{\phi(U)} = \{(\phi(U), Id)\}$ is an atlas for $\phi(U)$

$Id \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$ is $\overset{\text{standard}}{C^\infty} \Rightarrow \phi$ is C^∞

$\phi \circ \phi^{-1} \circ Id^{-1} : \phi(U) \to \phi(U)$ is $\underset{\text{standard}}{C^\infty} \Rightarrow \phi^{-1}$ is C^∞

Proposition: Let $F : U \to \mathbb{R}^m$ be a diffeomorphism on an open subset $U \subseteq M$. Then (U, F) is a chart for M.

exc

$\Longrightarrow$ A smooth manifold is locally diffeomorphic to $\mathbb{R}^n$.

Example of different smooth structure on $\mathbb{R}$

Let $\mathcal{A}_1 = \{(\mathbb{R}, Id)\}$ $x^{1/3}$ Call this $\mathbb{R}_1$

Let $\mathcal{A}_2 = \{(\mathbb{R}, \psi(x) = \cancel{x^3})\}$ Call this $\mathbb{R}_2$

mistake in lecture.

do these atlases give $\mathbb{R}$ the same smooth structure

Are the charts C^∞ compatible?

$Id \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}$ is not a diffeomorphims.
$: x \mapsto x^{1/3}$

Consider $f : \mathbb{R}_2 \to \mathbb{R}$ is smooth $\left(\begin{array}{l} f \circ \psi^{-1} : \mathbb{R} \to \mathbb{R} \\ : x \mapsto x \\ \text{which is standard } C^\infty \end{array} \right)$
$: x \mapsto x^{1/3}$

but $f : \mathbb{R}_1 \to \mathbb{R}$ is not smooth
$: x \mapsto x^{1/3}$

In fact $C^\infty(\mathbb{R}_1) \subsetneq C^\infty(\mathbb{R}_2)$

Is $\mathbb{R}_1$ diffeomorphic $\mathbb{R}_2$? Yes

mistake in lecture

Let $g : \mathbb{R}_2 \to \mathbb{R}_1$ is a diffeomorphism
$: x \mapsto \cancel{x^3}\ x^{1/3}$

Can we exist two non-diffeomorphic smooth structures on the same topological manifold?

S^7 (top. manifold)

28 nondiffeomorphic smooth structures
1 standard
27 are the exotic spheres.

for $n \le 3$, $\exists!$ smooth structure up to diffeomorphism

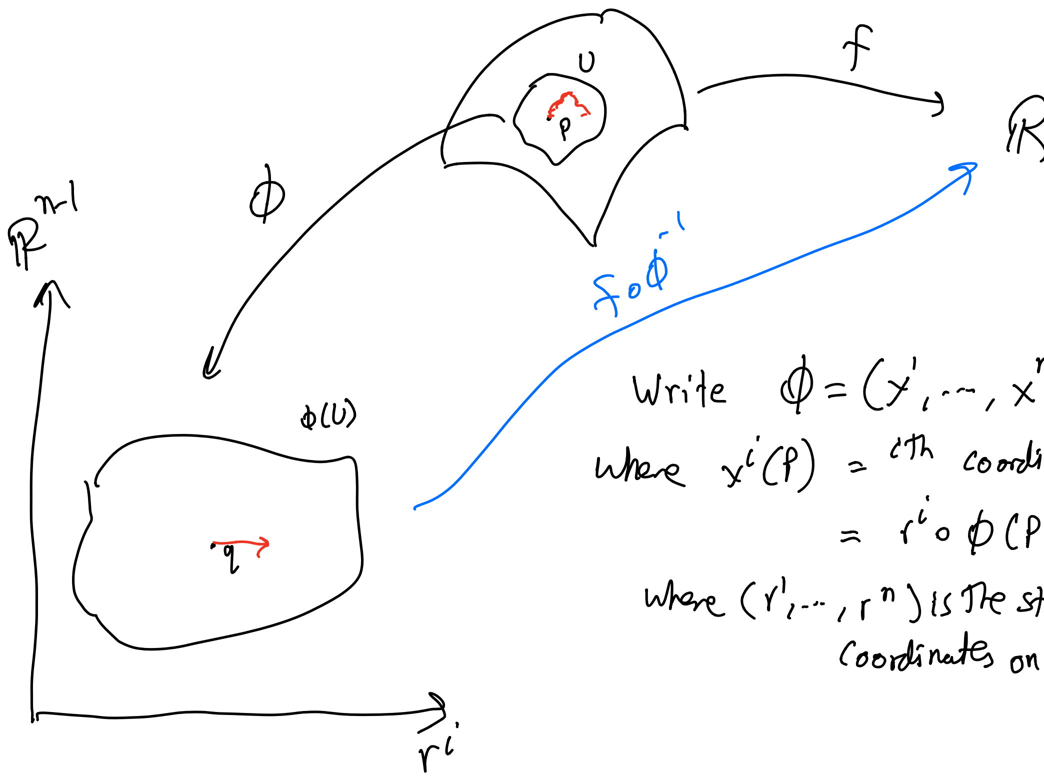
for $n > 4$, $\exists$ many smooth structure up to diff.

for $n > 4$ (compact), $\exists$ finitely many smooth structure up to diff.

$n = 4$: $\exists!$ smooth structure up to diff.

If $F: N \to M$ is C^∞, what is its derivative?
not defined yet.

Let $f: M \to \mathbb{R}$ be C^∞.



$x^i: U \to \mathbb{R}$

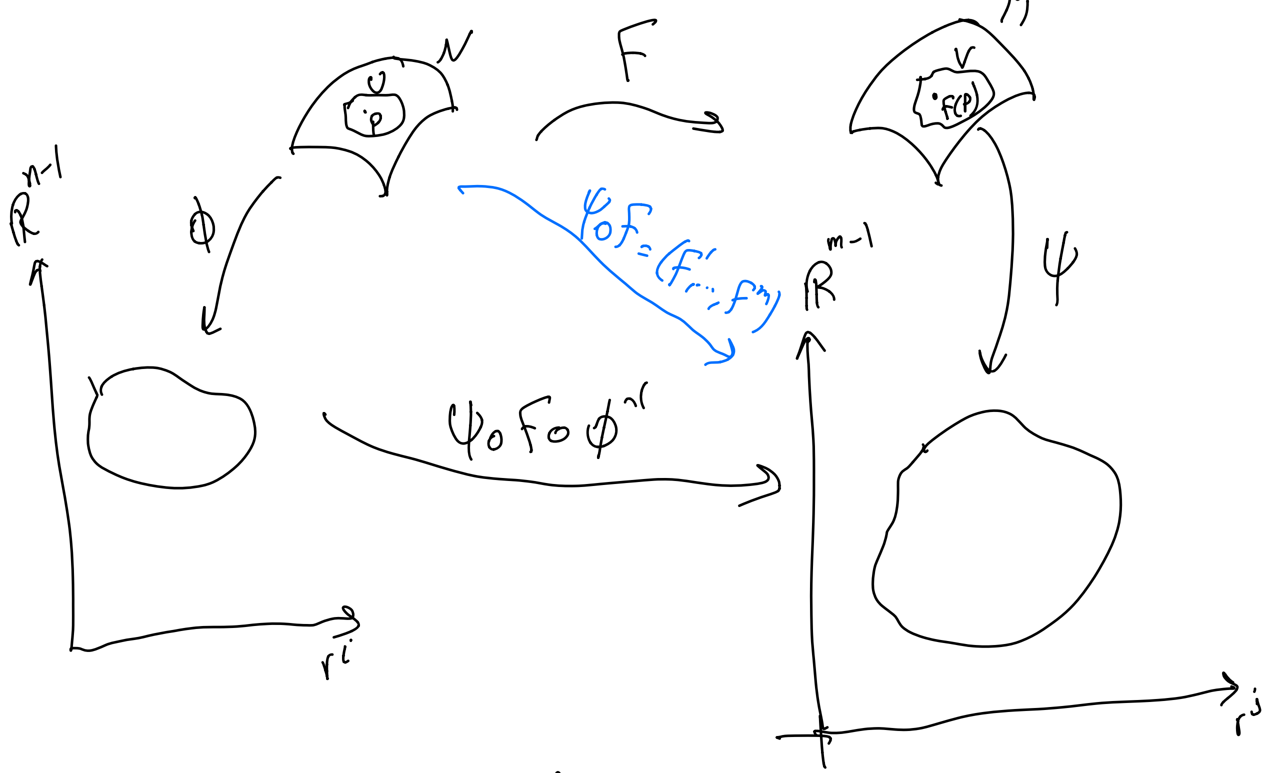
Write $\phi = (x^1, \cdots, x^n)$
where $x^i(p)$ = ith coordinate of p
$= r^i \circ \phi(p)$
where $(r^1, \cdots, r^n)$ is the standard coordinates on $\mathbb{R}^n$.

$(r^1, \cdots, r^n)$ are the coordinates on $\mathbb{R}^n$. $r^i: \mathbb{R}^n \to \mathbb{R}$

$$\left.\frac{\partial f}{\partial x^i}\right|_p := \left.\frac{\partial (f\circ\phi^{-1})}{\partial r^i}\right|_{\phi(p)}$$

← standard partial derivative on $\mathbb{R}^n$.

Let $F: N \to M$ be C^∞



Let $(U, \phi = (x^1, \dots, x^n))$ be a chart near p
and $(V, \psi = (y^1, \dots, y^m))$ be a chart near $F(p)$

We can write $\psi \circ F = (F^1, \dots, F^m)$
where $F^i = r^i \circ \psi \circ F : U \to \mathbb{R}$ which is C^∞

We can talk about

$$\frac{\partial F^i}{\partial x^j}\Big|_p = \frac{\partial (F^i\circ\phi^{-1})}{\partial r^j}\Big|_{\phi(p)} = \frac{\partial\left(r^i\circ(\psi\circ F\circ\phi^{-1})\right)}{\partial r^j}\Big|_{\phi(p)} = \frac{\partial(\psi\circ F\circ\phi^{-1})^i}{\partial r^j}$$

We call $\left[\frac{\partial F^i}{\partial x^j}\Big|_p\right] = \underset{\text{standard}}{D}(\psi\circ F\circ\phi^{-1})$ the Jacobian of F at p.

(dependent on the coordinate system).

If you have another chart $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$, then

$$\tilde{\psi}\circ F\circ\tilde{\phi}^{-1} = (\tilde{\psi}\circ\psi^{-1})\circ(\psi\circ F\circ\phi^{-1})\circ(\phi\circ\tilde{\phi}^{-1})$$

$$D[\cdots] = D[\cdots]\,D[\cdots]\,D[\cdots]$$

Lemma: The rank of $\left[\frac{\partial F^i}{\partial x^j}\Big|_p\right]$ is

is independent of coordinates and so is well defined.

It is called the rank of F at p.

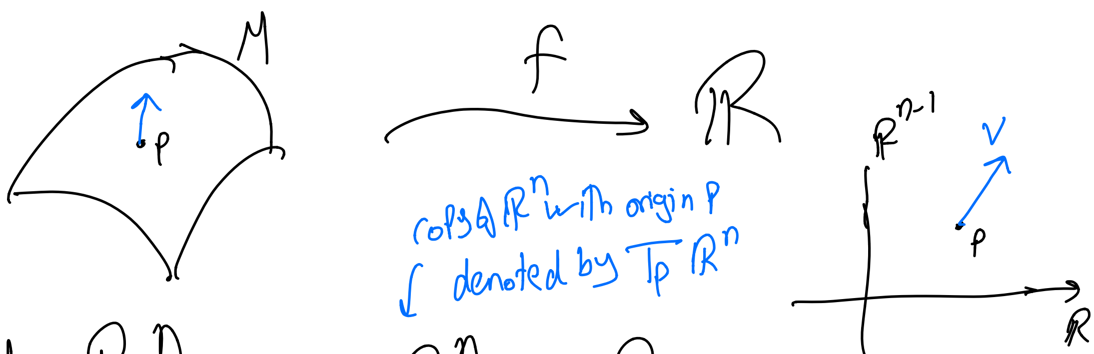
Now we have:

Inverse function Thm on manifold:

Let $F: N \to M$ be a smooth map ($\dim N = \dim M$).
If the Jacobian of F at p is invertible, then
$\exists U$ neighbhd of p and $\exists V$ neighbhd of $F(p)$ s.t.

$$F: U \to V \text{ is a diffeomorphism.}$$

What do we need to define the derivative?



copy of $\mathbb{R}^n$ with origin p
denoted by $T_p\mathbb{R}^n$

If $M = \mathbb{R}^n$, $Df: \mathbb{R}^n_p \to \mathbb{R}$

derivative of f in the direction of v $= Df(v) = \nabla f \cdot v$

we need a notion of tangent space on manifolds.

So $Df \in L(T_p\mathbb{R}^n, \mathbb{R})$

$Df : T_p\mathbb{R}^n \to \mathbb{R}$

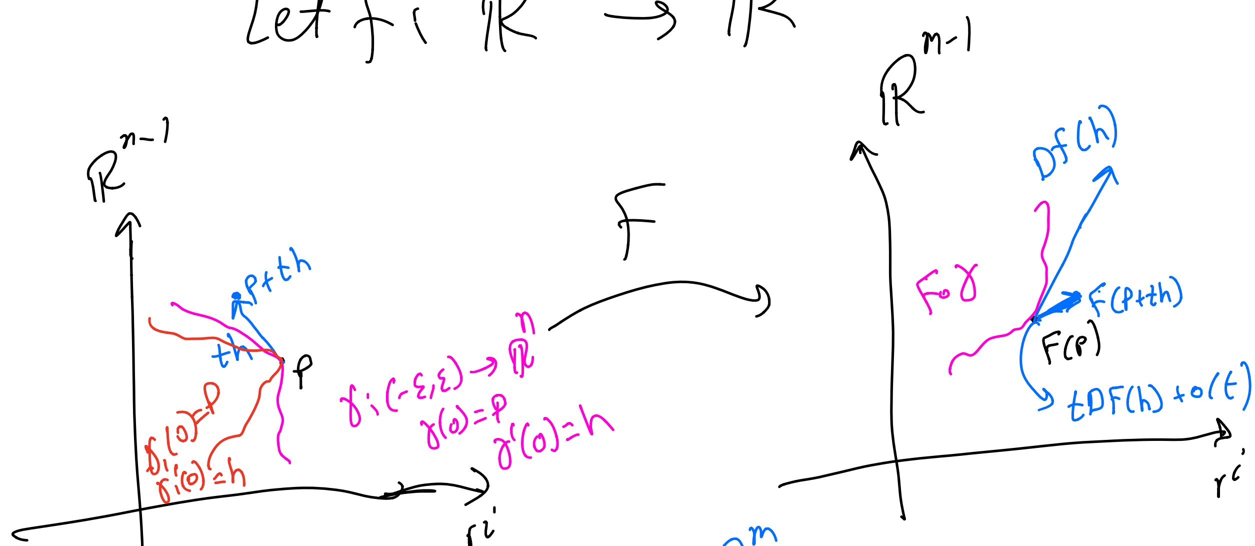
$: v \mapsto D_v f = \nabla f \cdot v$

↑ directional derivative

we also know Df is $1 \times n$ matrix

$Df = \left[\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right]$

Let $f: \mathbb{R}^n \to \mathbb{R}^m$



$Df: \overset{T_p\mathbb{R}^n}{\cancel{\mathbb{R}^n}} \to \overset{T_{F(p)}\mathbb{R}^m}{\cancel{\mathbb{R}^m}}$ uniquely determined by the following: for $h \in \mathbb{R}^n$, $Df(h) \in \mathbb{R}^m$ is the vector satisfying:

$$\left[F(p+th) - F(p) = t\,DF(h) + o(t) \right]$$

Magical thm: $DF|_p \in L(T_p\mathbb{R}^n, T_{F(p)}\mathbb{R}^m)$

is the $m \times n$ matrix $\left[\frac{\partial F^i}{\partial x^j}\Big|_p \right]$

$F \circ \gamma(0) = F(p)$

also independent of γ.

direction I go away from $f(P)$ as I go in the direction of h

$$= F \circ \gamma'(0)$$

$$= \frac{d}{dt}\Big|_{t=0} F \circ \gamma(t)$$

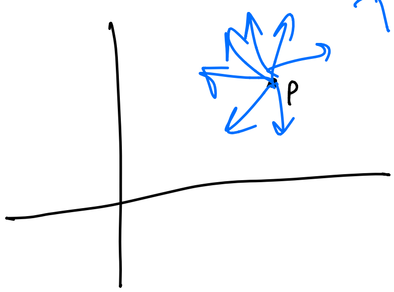
$$= DF|_p \cdot \gamma'(0)$$

$$= DF|_p(h) \quad !!$$

independent of the curve.

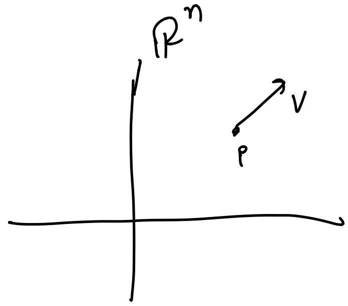
Define the Tangent space

$\mathbb{R}^n$



$T_p\mathbb{R}^n$
$(\mathbb{R}^n_p)$
$\mathbb{R}^n$

Attempt 1 :



Let $v \in T_p\mathbb{R}^n$

$\exists\ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ smooth

s.t. $\gamma(0) = p$

$\gamma'(0) = v$

Let $A = \{$ smooth curves passing through $p \}$

We define an equivalence relation on A .

$\gamma \sim \tilde{\gamma}$ if $\gamma'(0) = \tilde{\gamma}'(0)$?? ??

Fix this : $\gamma \sim \tilde{\gamma}$ if $f \circ \gamma'(0) = f \circ \tilde{\gamma}'(0)$ $\forall f \in C^\infty(M)$

We can think of v as $[\gamma]$

and so $A/\sim$ is a way to define the tangent space.

V.S.

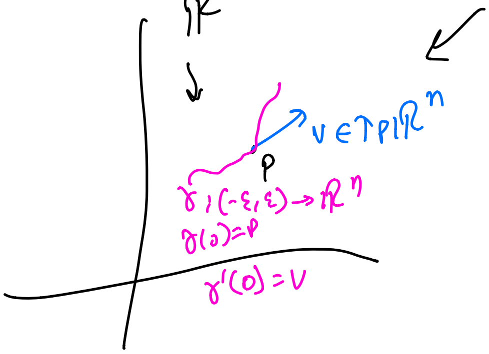
$: A/\sim \ \to T_p\mathbb{R}^n$

$: [\gamma] \mapsto \gamma'(0)$ is an isomorphism.

This definition carries over to Manifolds. Define T_pM as $A_p/\sim$

Attempt #2

$\mathbb{R}^n$



$$\mathbb{R}^n \xrightarrow{\ f\ } \mathbb{R}$$

let $f : \mathbb{R}^n \to \mathbb{R}$ That is C^∞.

We can talk about $$D_v f = \frac{d}{dt}\Big|_{t=0} f \circ \gamma(t)$$

directional derivative

$$= Df|_p \cdot v$$

independent of curve

We can associate v with $D_v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$

$: f \longmapsto D_v f$

D_v satisfies 1) linear wrt the V.S. of $C^\infty(\mathbb{R}^n)$

v.s over $\mathbb{R}$

2) Leibniz Rule wrt ring structure of $C^\infty(\mathbb{R}^n)$

$$D_v(fg) = f(p)\,D_v(g) + g(p)\,D_v(f)$$

$$\mathcal{D}_p := \left\{ D : C^\infty(\mathbb{R}^n) \to \mathbb{R} : \underline{D \text{ satisfies } 1 \text{ and } 2} \right\}$$

$$= \left\{ D_v : C^\infty(\mathbb{R}^n) \to \mathbb{R} \;\middle|\; v \in T_p\mathbb{R}^n \right\}$$

Think of $T_p\mathbb{R}^n$ as $\mathcal{D}_p$

__Thm:__ 1) $\mathcal{D}_p$ is a vectorspace over $\mathbb{R}$

2) $\Phi : T_p\mathbb{R}^n \to \mathcal{D}_p$ is an isomorphism

$: v \mapsto D_v$