Let S be a topological space and let $\sim$ be an equivalence relation.

$\pi : S \to S/\sim$

$\sim$ is an open equivalence relation is open if π is an open map

(equivalently)

if $\bigcup_{x \in U} [x]$ is open in S whenever U is open in S.

Thm: Suppose $\sim$ is open equiv relation. Then $S/\sim$ is Hausdorff iff the graph $R := \{ (x,y) \in S \times S \mid x \sim y \}$ of $\sim$ is closed in $S \times S$.

(S doesn't need to be Hausdorff)

etc

Thm: Suppose $\sim$ is an open equiv. relation. If S is second countable, $S/\sim$ is second countable.

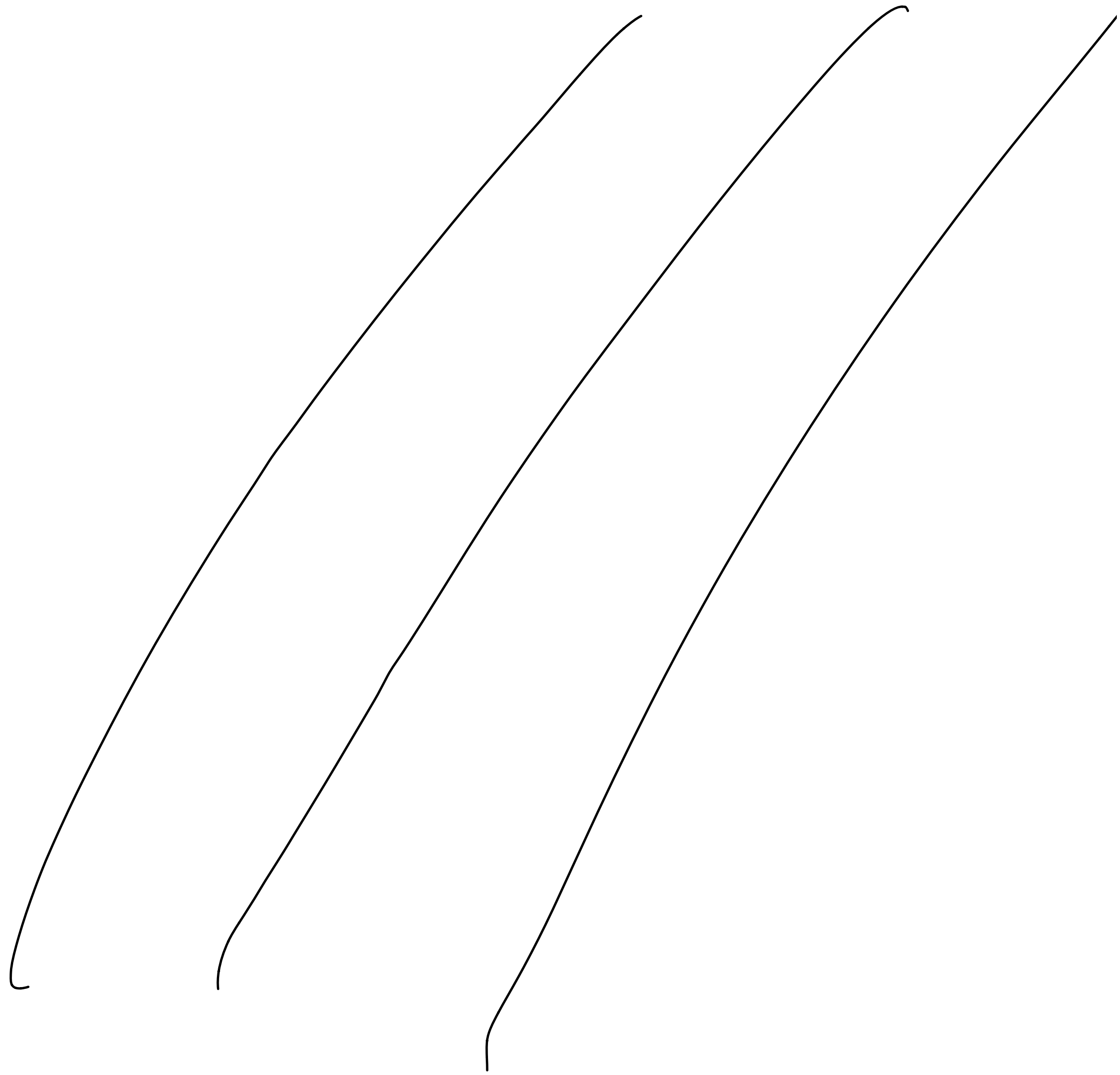
Proof: If $\{ B_\alpha \}$ is a countable basis for S, then $\{ \pi(B_\alpha) \}$ is a countable basis for $S/\sim$.

(check)

($\Leftarrow$ is false)

Check locally Euclidean on a case by case basis.

Exercise: Define rigorously the topological spaces cylinder, torus, Möbius strip, Klein bottle as quotient spaces of the square on $\mathbb{R}^2$.



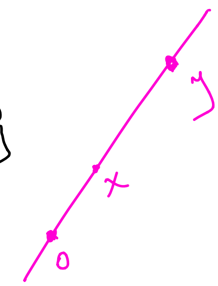
Specific Imp Example:

The Real Projective space

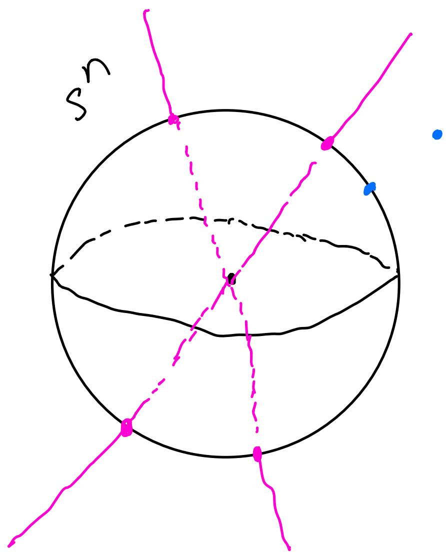
$\mathbb{RP}^n$ is going to be the space of lines in $\mathbb{R}^{n+1}$ passing through the origin.

Define an equivalence relation on $\mathbb{R}^{n+1}\setminus\{0\}$ by

$x \sim y$ if $x = ty$ for some $t \in \mathbb{R}\setminus\{0\}$



Define $\mathbb{R}P^n := \left(\mathbb{R}^{n+1}\setminus\{0\}\right)/\sim$



Any line passing through the origin can be associated to a pair of antipodal points on S^n.

This motivates the following:

Define an equivalence relation on S^n by

$x \sim y$ if $x = \pm y$

Then we have an obvious bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$

let $f : \boxed{\mathbb{R}^{n+1}\setminus\{0\}} \longrightarrow \boxed{S^n}_{\sim}$, $f(x) = \dfrac{x}{\|x\|}$

Proposition:

This function induces a homeomorphism: $\bar{f} : \mathbb{R}P^n \longrightarrow S^n/\sim$

Proof:

$$\mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{f} S^n \xrightarrow{\pi} S^n/\sim$$

$$\hookrightarrow \mathbb{R}P^n \to S^n/\sim$$

etc

So we conclude $\boxed{\mathbb{R}P^n \cong S^n/\sim}$

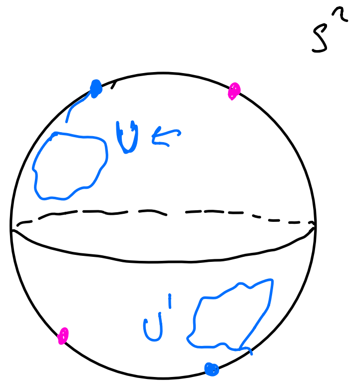
Is $\mathbb{R}P^n$ compact?

$\pi : S^n \to S^n/\sim$

is cont is compact

we could say more:

Consider $\mathbb{R}P^2$
$\cong \boxed{S^2/\sim}$

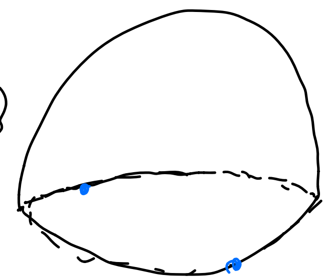


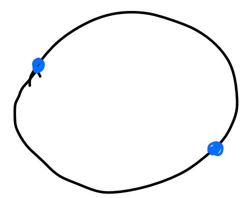
$\bigcup_{x\in U} [x] = U \cup U'$ is open

Define the upper hemisphere

$H^2 = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1, z\geq 0\}$

$D^2 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$





Note that $H^2 \cong D^2$. How?

let $\varphi : H^2 \to D^2$, $\varphi = ?$

let $\psi : D^2 \to H^2$, $\psi = ?$

Then φ, ψ are cont and are inverses of each other

Define $\sim$ on H^2 by identifying antipodal points on the equator.

Define $\sim$ on D^2 by identifying antipodal points on ∂D^2

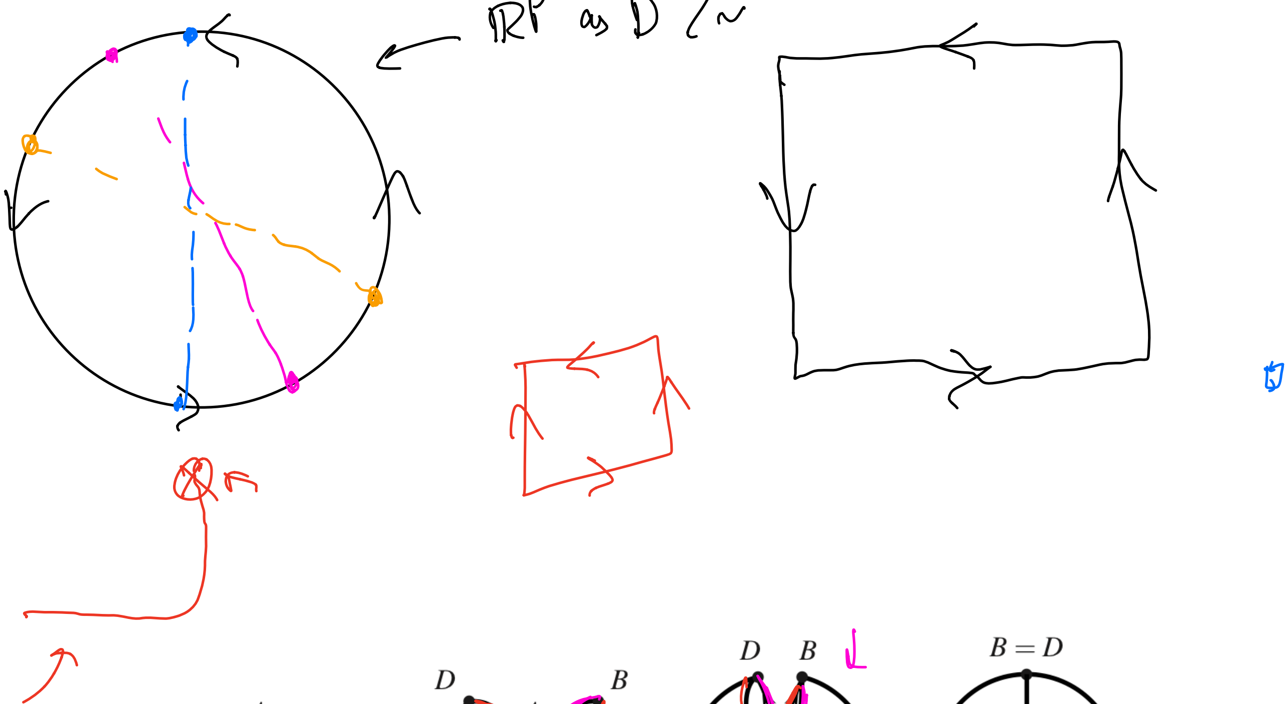
Then φ and ψ induce homeomorphisms $\overline{\varphi} : H^2/\sim \;\longrightarrow D^2/\sim$

$\overline{\psi} : D^2/\sim \;\longrightarrow H^2/\sim$

We have the following sequence of homeomorphisms:

$$\mathbb{RP}^2 \cong S^2/\sim \;\cong H^2/\sim \;\cong D^2/\sim$$

$\mathbb{RP}^2$ as $D^2/\sim$



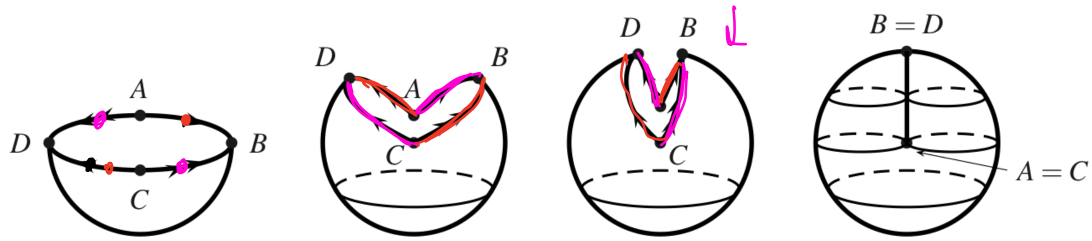


Fig. 7.8. The real projective plane immersed as a cross-cap in $\mathbb{R}^3$.

We wts 1) equivalence relation of $\mathbb{RP}^n$ is open

$\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$ is open map

2) $\mathbb{RP}^n$ is Hausdorff second countable

3) $\mathbb{RP}^n$ admits a C^∞ atlas and so is a smooth manifold.

1) π is open iff if U is open set in $\mathbb{R}^{n+1}\setminus\{0\}$,

then $\bigcup_{x\in U}[x]$ is open in $\mathbb{R}^{n+1}\setminus\{0\}$

$= \bigcup_{t\in\mathbb{R}\setminus\{0\}} tU$ which is open

2) Second Countable: ✓ $\mathbb{R}^{n+1}\setminus\{0\}$ is second countable ✓

Hausdorff: let $R=\{(x,y)\in S\times S : x\sim y\}$ The graph of the equivalence relation where $S=\mathbb{R}^{n+1}/\{0\}$

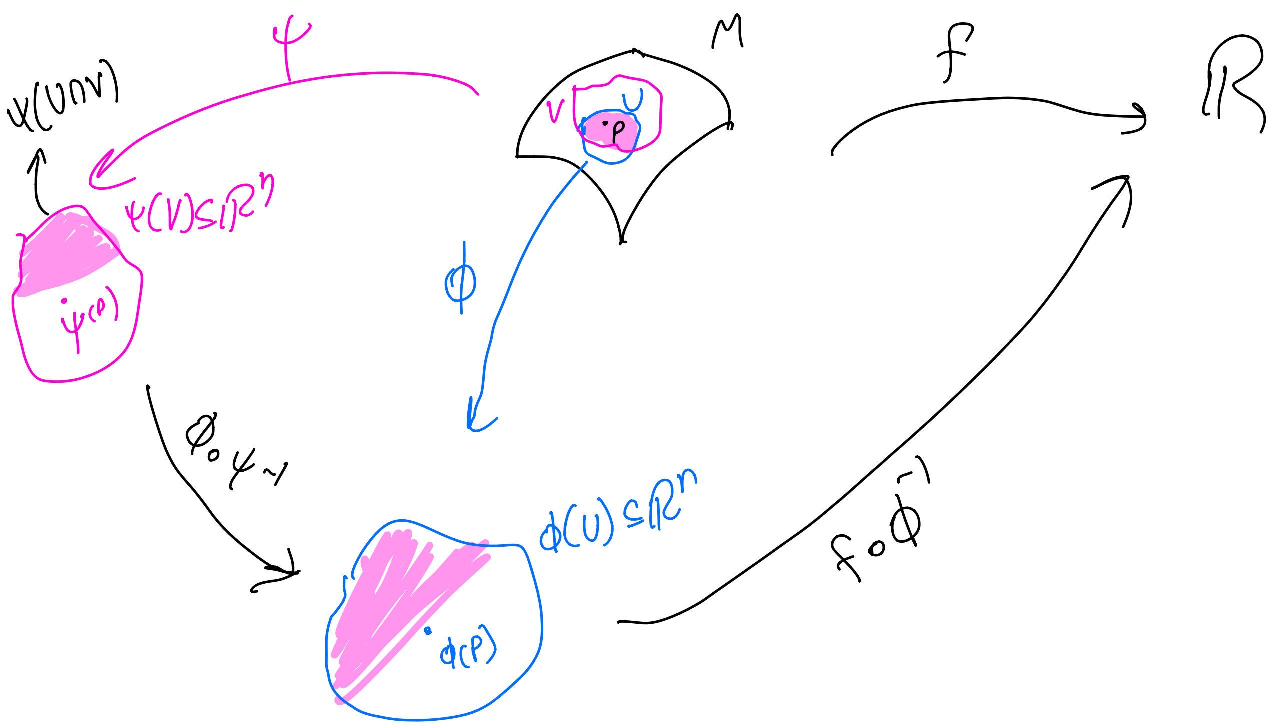
so $R=\left\{(x,tx) : x\in\mathbb{R}^{n+1}\setminus\{0\},\ t\in\mathbb{R}\setminus\{0\}\right\}$ which is closed ✓ ☺

$\boxed{\text{3) In tutorial.}}$

Take the first step in developing Calculus on smooth manifolds

Definition of a smooth function

Let M be a smooth manifold.
and let $f : M \to \mathbb{R}$ ←



<u>Def</u>: f is C^∞ at p if $\exists$ a chart (U, ϕ) containing p s.t. $f\circ\phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞ at $\phi(p)$.

↑ regular C^∞.

$\Rightarrow$ f is cont at p since $f = (f\circ\phi^{-1})\circ\phi$ on U

Is this definition well defined?

Let (V, ψ) be another chart near p. in the maximal atlas on M.

$$f \circ \psi^{-1} = \underbrace{(f \circ \phi^{-1})}_{C^\infty \text{ at } p} \circ \underbrace{(\phi \circ \psi^{-1})}_{C^\infty \text{ at } p} : \psi(U \cap V) \to \mathbb{R}$$

$\Rightarrow f \circ \psi^{-1}$ is C^∞ at p.

"f is C^∞ at p" is independent of the chart.