- Assignment 1 due Friday (20% off per day of lateness)
- OH tomorrow 2-4.

Examples of Smooth Manifolds

#1) $\mathbb{R}^n$ ✓

#2) k-dim manifolds in $\mathbb{R}^n$ ✓

#3) Let $A \subseteq M$ be an open subset of a smooth manifold M of dim n

If $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is a C^∞ atlas on M, then

then $\mathcal{A}_A = \{(U_\alpha \cap A, \phi_\alpha|_{U_\alpha \cap A})\}$ is a C^∞ atlas on A, making A a smooth manifold of dim n.

#3.5) Let V be a V.S. of dim n. You will prove in Assignment 2.

#4) Level sets.

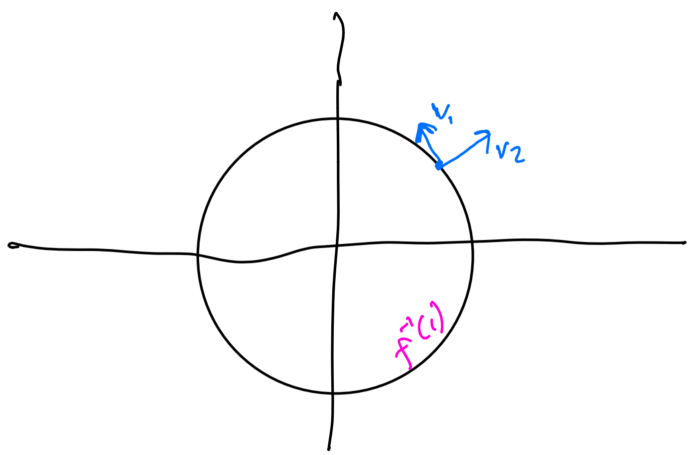
let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be $f(x,y) = x^2+y^2$

you see $f^{-1}\{1\} = S^1$

$D_{v_1} f = v_1 \cdot \nabla f = 0$

$D_{v_2} f = v_2 \cdot \nabla f \neq 0$

So as you go in direction of v_2, f moves away from 1.



we have a family of disjoint 1-dim manifolds covering $\mathbb{R}^2 \setminus \{0\}$

This is called a foliation and we say $\{f^{-1}(r) : r>0\}$ is a 1-dim foliation of $\mathbb{R}^2 \setminus \{0\}$.

Those 1-dim manifolds are called the leaves of the foliation.

$n > k$

$f : \mathbb{R}^n \to \mathbb{R}^k$

If $Df|_{f^{-1}(c)}$ is of full rank, Then $f^{-1}\{c\}$ is a smooth manifold of dim $n-k$

We generalize this example:

let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞ function

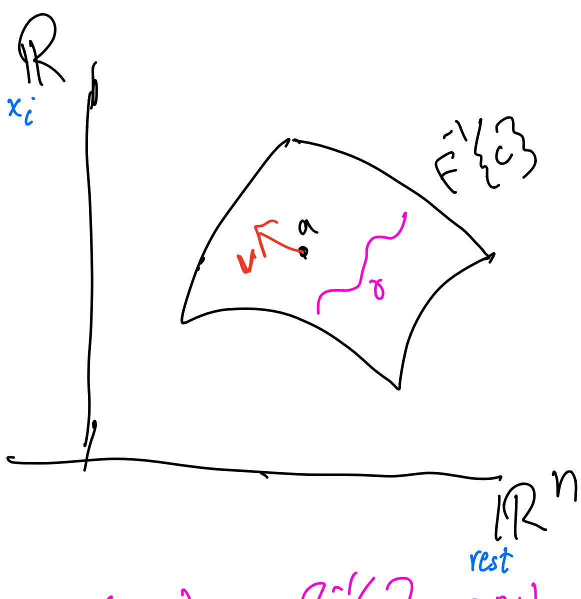
Let $c \in \mathbb{R}$ s.t. $F^{-1}(\{c\}) \neq \phi$

Suppose also that $\nabla F|_p \neq 0 \quad \forall p \in F^{-1}(\{c\})$

for example: $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by
$F(x) = \|x\|^2$

$\{F^{-1}\{r\} : r > 0\}$ make an n-dim foliation of $\mathbb{R}^{n+1}/\{0\}$

Then $S^n = F^{-1}\{1\}$ and $\nabla F|_{F^{-1}\{1\}} \neq 0$



Let $a \in F^{-1}\{c\}$

$0 \neq \nabla F|_a = \left[\frac{\partial F}{\partial x_1} \cdots \frac{\partial F}{\partial x_{n+1}} \right]$

$T_a F^{-1}\{c\} \subseteq \{\lambda \nabla F|_a : \lambda \in \mathbb{R}\}^\perp$

$= \{v : \nabla f = 0\}$

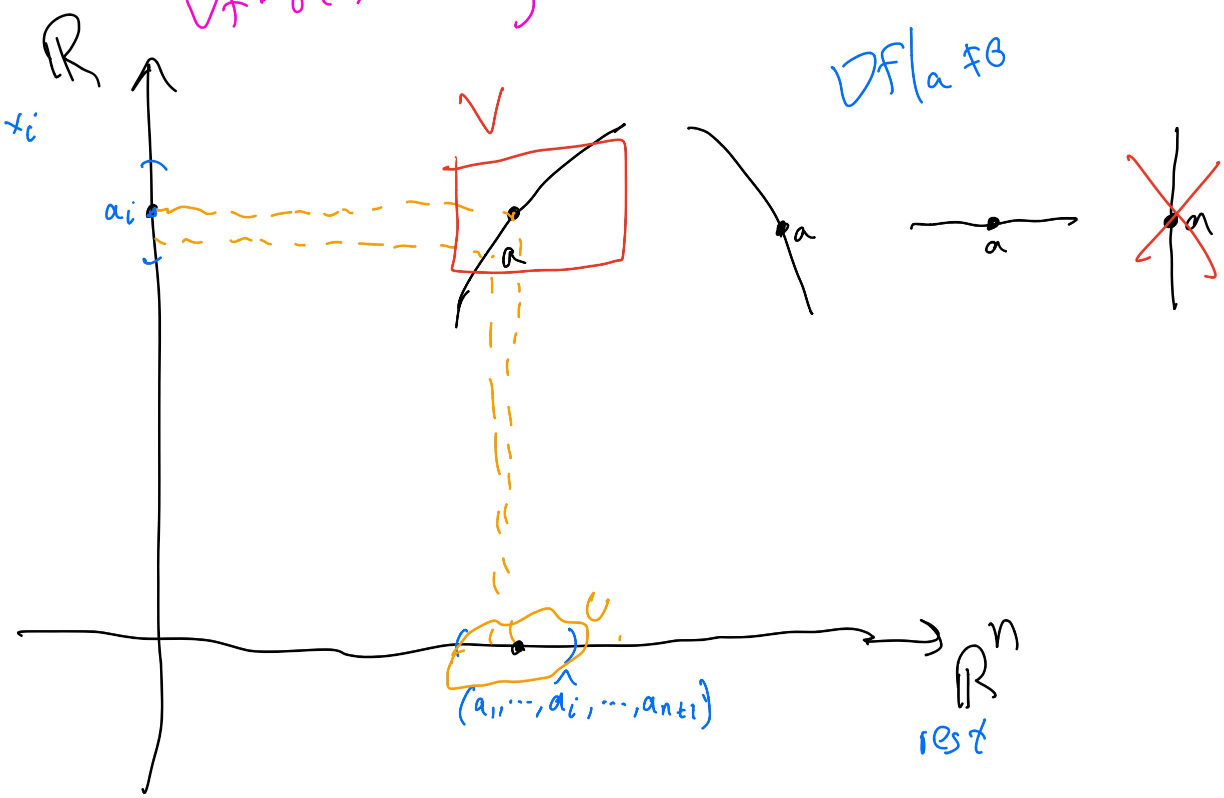
let's say $\left.\frac{\partial F}{\partial x_i}\right|_a \neq 0$

$\gamma : (-1,1) \to F^{-1}\{c\} \subseteq \mathbb{R}^{n+1}$

$\left.\frac{d}{dt}\right| [F \circ \gamma(t)] = 0$

$\gamma'(0) \in$

$t=0$ $\nabla f \cdot \gamma'(0)$



$$F(x_1,\dots,x_i,\dots x_{n+1}) = c$$

x_i can be expressed locally in terms of the rest

By Implicit function Theorem: $\exists\, U$ neighbhd of $(a_1,\dots,\hat{a}_i,\dots a_{n+1})$ in $\mathbb{R}^n$

and a unique smooth function $g: U \longrightarrow \mathbb{R}$

satisfying: 1) $g(a_1,\dots\hat{a}_i,\dots a_{n+1}) = a_i$

2) $F(x_1, \dots, g(x_1, \dots \hat{x_i}, \dots x_{n+1}), \dots, x_{n+1}) = c$

$$\forall (x_1, \dots, \hat{x_i}, \dots x_{n+1}) \in U$$

meaning that $\Gamma_g = \{ (x_1, \dots, g(x_1, \dots, \hat{x_i}, \dots x_{n+1}), \dots x_{n+1}) \in \mathbb{R}^{n+1} ;$

$(x_1, \dots \hat{x_i}, \dots x_{n+1}) \in U \}$

$= V \cap F^{-1}(\{c\})$ where V is neighbhd of a in $\mathbb{R}^{n+1}$

Conclusion:

if $\nabla F|_a \neq 0 \ \forall a \in F^{-1}\{c\}$

then $F^{-1}\{c\}$ is locally the graph a function!!

what would the charts be?

coordinate neighbhd of a — coordinate map

The Chart near a : $(V \cap F^{-1}(\{c\}), \phi)$ where

$$\phi : V \cap F^{-1}(\{c\}) \longrightarrow U$$

defined by $\phi(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x_i}, \dots, x_{n+1})$

with inverse $\phi^{-1} : (x_1, \dots, \hat{x_i}, \dots, x_{n+1}) \longmapsto$

$(x_1, \dots, g(x_1, \dots, \hat{x_i}, \dots, x_{n+1}), \dots, x_{n+1})$

Clear ϕ and ϕ^{-1} cont & so

$(V \cap F^{-1}(c), \phi)$ is a chart.

Since we could do this for each $a \in F^{-1}\{c\}$,

Consider collection of charts $\mathcal{A} := \{(V_a, \phi_a) : a \in F^{-1}\{c\}\}$

is this an atlas?

Check the transition maps: $\phi_a \circ \phi_b^{-1} : \phi_b(V_{ab}) \to \phi_a(V_{ab})$

$$\phi_a \circ \phi_b^{-1} : (x_1, \dots, x_i, \dots, \hat{x}_j, \dots, x_{n+1}) \longmapsto$$

$$(x_1, \dots, \hat{x}_i, \dots, g_b(x_1, \dots \hat{x}_j, \dots x_{n+1}), \dots x_{n+1})$$

which is C^∞ so $\mathcal{A}$ is a C^∞ atlas on $F^{-1}\{c\}$ proving it's a smooth manifold of dim n.

$n \to k$

#5) Products of manifolds.

Recall from Appendix A: The product topology.

Let $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ be two topological spaces.

Consider the collection $\mathcal{B} = \{U \times V : U \in \mathcal{T}_M, V \in \mathcal{T}_N\}$
this forms a basis for a topology on $M \times N$ called the product topology
where $A \subseteq M \times N$ is open if it's a union of elements in $\mathcal{B}$.

exercise: verify this.

Proposition: Product of Hausdorff and second countable spaces is Hausdorff and second countable.

atlas on M (dim m)

atlas on N (dim n)

Given a chart $(U,\phi) \in \mathcal{A}_M$ and $(V,\psi) \in \mathcal{A}_N$,

$(U\times V, \phi\times\psi)$ is clearly a chart on $M\times N$ where

$$\phi\times\psi\,(p,q) = (\phi(p), \psi(q)) \in \mathbb{R}^{m+n}$$

All such charts make a C^∞ atlas on $M\times N$ [making it a smooth manifold of dim $m+n$.]

So $M\times N$ is a topological space

1) Hausdorff
2) Second countable
3) admits a C^∞ atlas

$\Rightarrow$

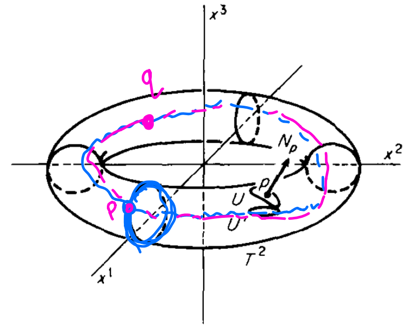
Examples:

S^1 is a smooth manifold
$\mathbb{R}$ is " "

$S_1\times\mathbb{R}$ is also a smooth manifold of dim 2

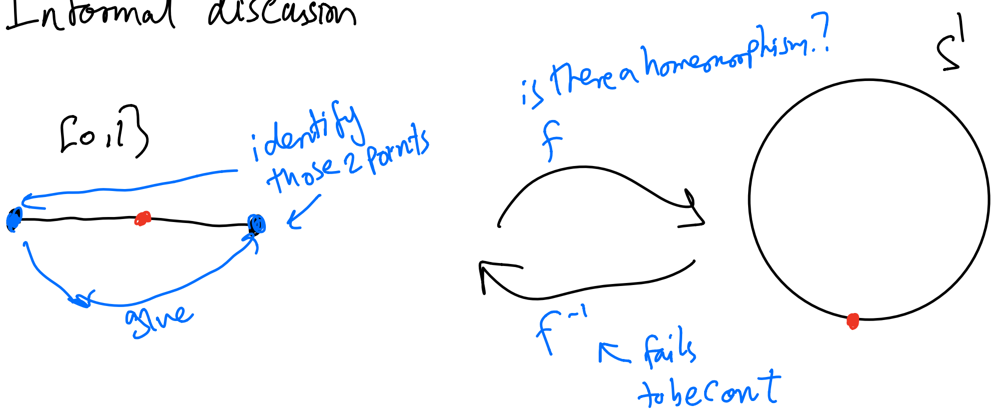
$S_1 \times S_1$ is a smooth manifold of dim 2.

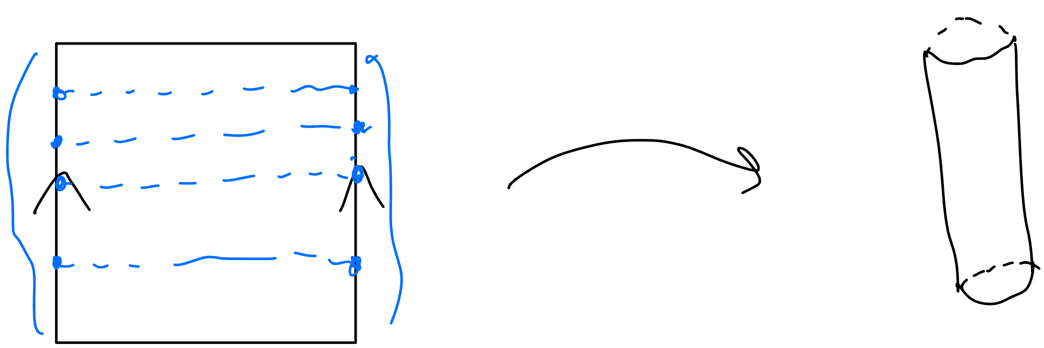
$(p,q) \in S_1 \times S_1$

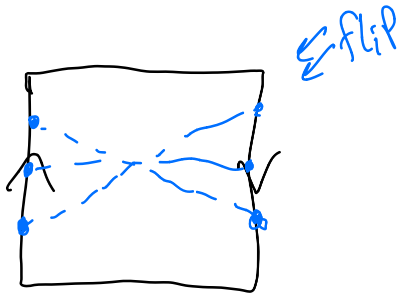


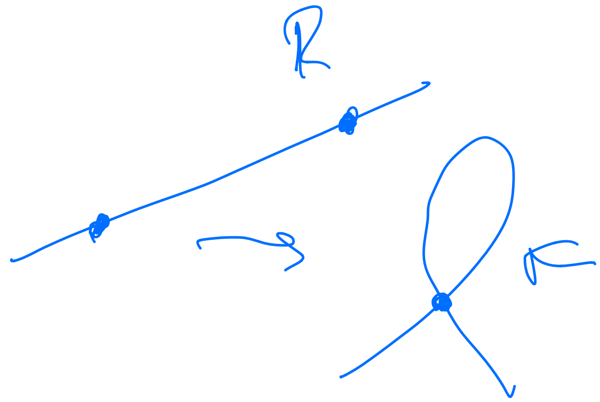
Creating more spaces by Gluing manifolds to each other

Informal discussion









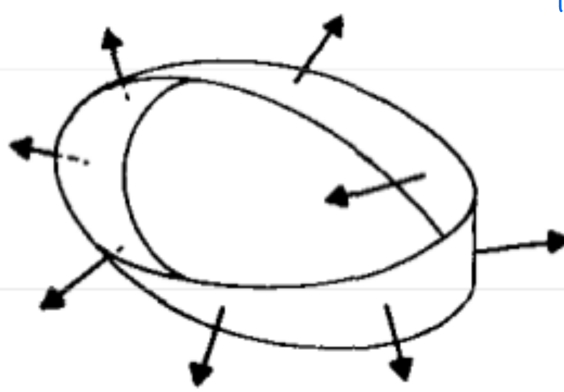
does it have 2 sides?

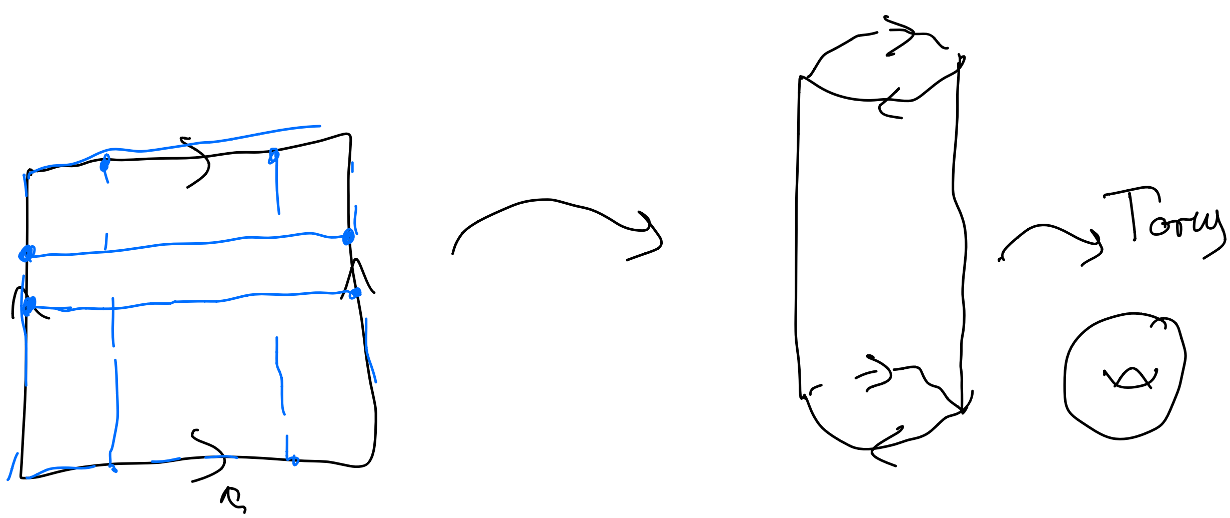
No!

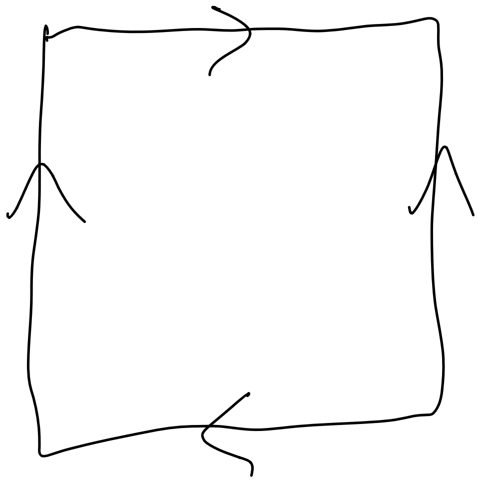
non orientable manifold

On surfaces on $\mathbb{R}^3$, this equivalent to

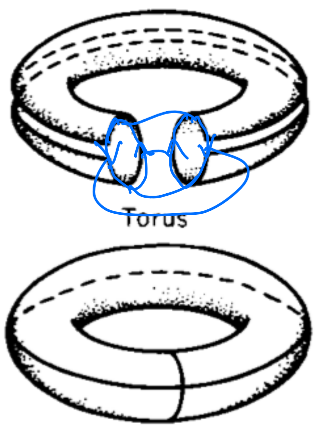
$\nexists$ unit normal vector field that is cont.



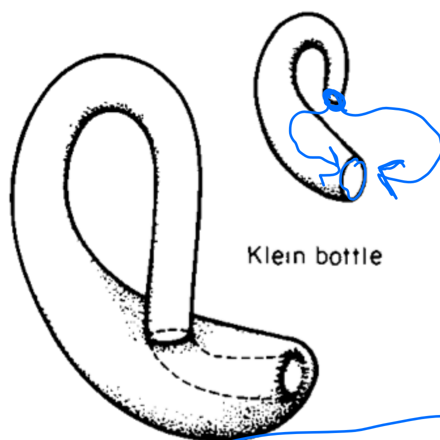




Klein Bottle



(c)



(d)

∇f is the vectorfield uniquely determined by:

$\forall v,\ D_v f = v \cdot \nabla f$

$$\lim_{\|h\|\to 0} \frac{f(x+h)-f(x)-L|_x(h)}{\|h\|} = 0$$

$$L = \boxed{Df} = \boxed{\nabla f}$$

Glueing Manifolds: Rigorously

Quotient space.

Let S be a set. Let $\sim$ be an equivalence relation

we can talk about equivalence classes $[x]$ where $x \in S$

Define $S/\sim = \{[x] : x \in S\}$ as the quotient space.

which comes with $\pi : S \to S/\sim$ defined by
$\pi(x) = [x]$

Suppose S is a topological space. What is the natural topology on $S/\sim$?

It's the one that makes π continuous. Which one?
Choose the finest topology making π continuous

Declare $U \subseteq S/\sim$ to be open if $\pi^{-1}(U)$ is open in S.
Verify this forms a topology on $S/\sim$. It's called the quotient topology

If we have $\tilde{f} : S \to Y$ where Y is some topological space.
and f is constant on equivalence classes
(so $f(x) = f(y)$ whenever $x \sim y$)

This induces a function $\bar{f} : S/\sim \to Y$
defined by $\bar{f}([x]) = f(x)$

This well-defined map because f is constant on equivalence classes

If we have $\bar{f} : S/\sim \to Y$

Then:

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[rr, bend right, "f = \pi \circ \bar{f}"'] & S/\sim \arrow[r, "\bar{f}"] & Y \end{tikzcd}$$

called lift of $\bar{f}$

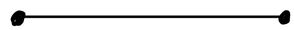
In other words: f and $\bar{f}$ are defined in a way such that this diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Proposition: $f : S \to Y$ is cont iff $\bar{f} : S/\sim \to Y$ is cont.

Example:

$[0,1]$



define the relation: a) $x \sim x \quad \forall x \in [0,1]$

b) $x \sim y \quad \forall x, y \in \{0,1\}$

which is an equivalence relation.

Then $I/\sim$ is the same S^1.

Define $\underbrace{f : I \to S^1}$, $\underbrace{f(t) = (\cos 2\pi t, \sin 2\pi t)}$

which induces $\bar{f} : I/\sim \to S^1$, $\bar{f}([x]) = f(x)$ $\Big\}$? $f(0) = f(1)$

is this function a homeomorphism?

first, since f is cont, $\hat{f}$ is cont thanks to the above proposition.

Second, $\hat{f}$ is bijective.

Lemma: If $\hat{f}$ is a continuous bijection from a compact space X to a Hausdorff space Y, Then $\hat{f}^{-1}$ is continuous.

Proof: To check $\hat{f}^{-1}$ is cont, it's sufficient that $\hat{f}$ is a closed map.

let A is closed in X, Then A is compact. Then $\hat{f}(A)$ is compact.

Since Y is Hausdorff and $\hat{f}(A)$ is compact in Y, we get $\hat{f}(A)$ is closed. $\blacksquare$

S^1 is Hausdorff. Why is $I/\sim$ compact? $\pi : I \to I/\sim \Rightarrow \pi(I)$ is compact since I is compact.

$\Rightarrow \hat{f}^{-1}$ is continuous.

When is $S/\sim$ Hausdorff and Second countable

Hausdorff:

Let $S/\sim$ be a quotient space.
with the projection map $\pi: S \to S/\sim$

Let us find a necessary condition:
Recall a singleton in a Hausdorff space is closed.

so? for any $p \in S$, $\pi(p)$ is closed in $S/\sim$ if its Hausdorff.

If so, then $\underbrace{\{[p]\}}_{\subseteq S/\sim}$ is closed in $S/\sim$

$\Rightarrow \underbrace{[p]}_{\subseteq S}$ is closed in S.

<u>Example</u>: define a relation on $\mathbb{R}$ identifying $(0,\infty)$ so $\mathbb{R}/\sim$ cannot be Hausdorff since $\{\pi(1)\}$ is not closed in $\mathbb{R}/\sim$.

$[1] = (0,\infty)$ which is not closed in $\mathbb{R}$.

$S/\sim$ is Hausdorff $\Rightarrow [p]$ is closed in S $\forall p \in S$.

<u>Plan</u>: Find sufficient conditions under which the quotient space is Hausdorff.

$\{\text{Conditions}\} \Rightarrow S/\sim$ is Hausdorff.

Def: An equivalence relation is open if the projection map $\pi : S \to S/\sim$ is an open map.

so $\pi(U)$ is open in $S/\sim$ if U is open in S.

or equivalently:

If U is open in S,

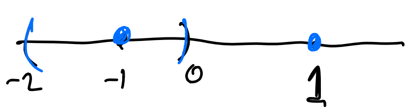
then $\displaystyle\bigcup_{x\in U} [x] \subseteq S$ is open.

$$\left(\begin{array}{ll} \pi(U) \text{ is open in } S/\sim & \text{iff} \\ \pi^{-1}(\pi(U)) \text{ is open in } S & \text{iff} \\ \displaystyle\bigcup_{x\in U}[x] \text{ is open} & \end{array}\right)$$

Example: Consider the relation on $\mathbb{R}$ defined by identifying -1 with 1.

Let $\pi : \mathbb{R} \to \mathbb{R}/\sim$ be the projection map.

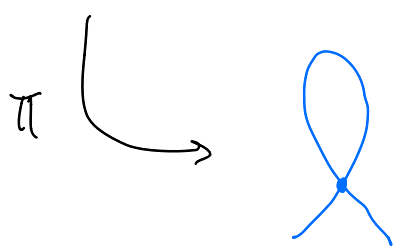
Is π open?



$(-2,0)$ is open but

$\displaystyle\bigcup_{x\in(-2,0)} [x] = (-2,0) \cup \{1\}$ is not open.

So π is not an open map.



not a topological manifold

Def $R := \{ (x,y) \in S \times S : x \sim y \}$ called the graph of the equivalence relation.

Thm: Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\sim$ is Hausdorff iff the graph R of $\sim$ is closed in $S \times S$.

$\sim$ is open
R is closed $\implies$ $S/\sim$ is Hausdorff

Proof:

R is closed in $S \times S$

$\Leftarrow\!\Rightarrow$

leave it as an exercise.