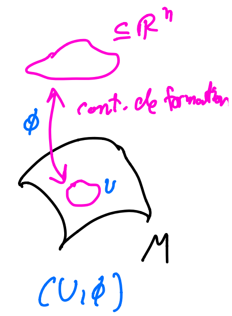
- OH
- OH for the TAs

} 2 hours of OH + Piazza

- Assignment 1: * updated version
  * Crowdmark
- Tutorials

M is a set

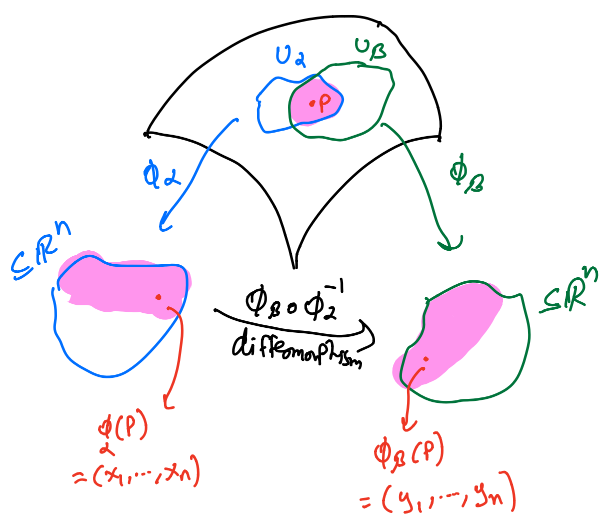
$\Downarrow$ Equipped it with a topology 1)... 2)... 3)...



Topological manifold

$\Downarrow$ Equipped it with a C^∞ atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$

Smooth manifold



Recall:

· **Def:** A smooth manifold M of dim n is a topological manifold of dim n equipped with a C^∞ atlas.

Remark: The C^∞ atlas gives the topological manifold a smooth structure by giving rise to a notion of smooth functions on M defined later. We denote the set of smooth functions on M by $C^\infty(M)$.

Question: Given a smooth manifold M with a C^∞ atlas $\mathcal{A}$, is there a different C^∞ atlas $\mathcal{A}'$ that makes M the same smooth manifold? or makes M a different smooth manifold? (Can $\mathcal{A}'$ give M the same (or a different) smooth structure)

$$\left(C^\infty_{\mathcal{A}}(M) = C^\infty_{\mathcal{A}'}(M) \right)$$

Def: A chart (U, ϕ) is C^∞ compatible with an atlas $\mathcal{A}$ if (U, ϕ) is C^∞ compatible with every chart in $\mathcal{A}$.

If so, $\mathcal{A} \cup \{(U, \phi)\}$ is another C^∞ atlas.

Def: $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^∞ compatible if all charts in $\mathcal{A}_1$ are C^∞ compatible with $\mathcal{A}_2$.

If so, $\mathcal{A}_1 \cup \mathcal{A}_2$ is another C^∞ atlas

Informal Proposition (Proven later): $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^∞ compatible iff they equip a topological manifold M with the same smooth structures.

Ex: $\mathbb{R}^n$ is a smooth manifold of dim n.

$$\left.\begin{aligned} \mathcal{A}_1 &= \{(\mathbb{R}^n, Id)\} \\ \mathcal{A}_2 &= \{(B_1(x), Id) : x \in \mathbb{R}^n\} \end{aligned}\right)$$ 2 different atlas equipping $\mathbb{R}^n$ with the same smooth structure.

Also, $\mathcal{A}_1 \cup \mathcal{A}_2$ is another C^∞ atlas compatible with $\mathcal{A}_1$ and $\mathcal{A}_2$.

Define a relation between C^∞ atlases:
$\mathcal{A}_1 \sim \mathcal{A}_2$ if $\mathcal{A}_1$ is C^∞ compatible with $\mathcal{A}_2$.

Exc: Show this an equivalence relation

Each equivalence class $[\mathcal{A}]$ associates M with a unique smooth structure

$$\left\{ [\mathcal{A}] \right\} \xleftrightarrow[\text{onto}]{1-1} \left\{ \begin{array}{c} \text{smooth} \\ \text{structures} \\ \text{on } M \end{array} \right\}$$

Def: A C^∞ maximal atlas $\mathcal{M}$ is a C^∞ atlas that's not contained in a larger atlas.

(if $\mathcal{A}$ is an atlas s.t. $\mathcal{M} \subseteq \mathcal{A}$, then $\mathcal{M} = \mathcal{A}$)

$\boxed{\text{Existence \& Uniqueness Lemma}}$: Any C^∞ atlas on M is contained in a unique maximal atlas.

(In each $[\mathcal{A}]$, $\exists!$ representative that is maximal)

Outline of Proof: Let $\mathcal{A}$ a C^∞ atlas

Define $\mathcal{M} := \cancel{\bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'}$ $\quad (\mathcal{M} \in [\mathcal{A}])$

$$:= \boxed{\mathcal{A} \cup \left\{ (U, \phi) : \begin{array}{l} \text{where } (U,\phi) \text{ is } C^\infty \\ \text{compatible with } \mathcal{A} \end{array} \right\}}$$

Show 1) $\mathcal{M}$ is a C^∞ atlas

2) $\mathcal{M}$ is maximal

3) There is no other maximal atlas.

$\boxed{\text{Less ambiguous Def of smooth manifold}}$: A smooth manifold of dim n is a topological manifold of dim n equipped with a maximal atlas.

$$\left\{ \text{maximal atlases} \right\} \longleftrightarrow \left\{ [\mathcal{A}] \right\} \longleftrightarrow \left\{ \text{smooth structures on } M \right\}$$

(↗, M is a fixed topological manifold)

smooth manifold is indeeeed a generalization ☺

Example

1) $\mathbb{R}^n$ ✓

2) A k-dim manifold in $\mathbb{R}^n$ is a smooth of dim k.

If M is a k-dim manifold in $\mathbb{R}^n$, then

$\forall p \in M$, $\exists f_p : \underset{\subseteq \mathbb{R}^n}{V} \longrightarrow \underset{\substack{\subseteq M \\ \text{open neighbhd of } p}}{U_p}$

satisfies
1) ...
2) ...
3) ...

Let $\mathcal{A} = \left\{ (U_p, f_p^{-1}) : p \in M \right\}$

exc: show $\mathcal{A}$ is a C^∞ atlas.

Remark: $\mathcal{A}$ gives rise to the same notion of smooth functions as the notion introduced in MAT 257.

3) Let V be a vectorspace over $\mathbb{R}$ with dim n.

Is V is a smooth manifold?