- Course evaluation
- Assignment 7 & essay are left to submit. ↖ email it to me.
- OH
- Mock exam.

We defined a manifold with boundary.

We defined an interior point as a point $p \in M$ s.t. $\exists$ a chart (U, ϕ) near p s.t.

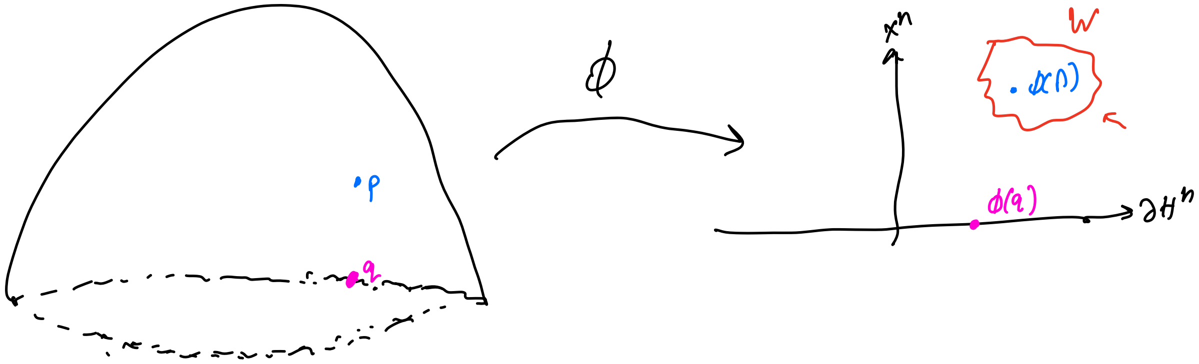
$$\phi(p) \in (\mathbb{H}^n)^\circ \quad (x_n(p) > 0)$$

We defined a boundary point as a point $p \in M$ s.t. $\exists$ a chart (U, ϕ) near p s.t.

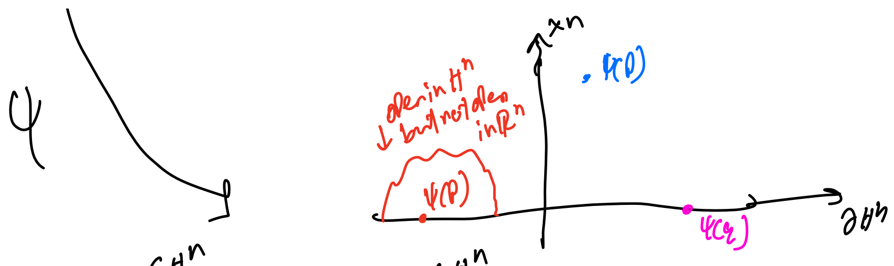
$$\phi(p) \in \partial \mathbb{H}^n \quad (x_n(p) = 0)$$

Proposition: These concepts are well defined & independent of coordinates. They partition M into $M^\circ := \{\text{interior points}\}$ and $\partial M := \{\text{boundary points}\}$

Smooth invariance of domain: If $f: U \to S$ is a diffeomorphism where $S \subseteq \mathbb{R}^n$ is arbitrary and $U \subseteq \mathbb{R}^n$ is open, then S is open in $\mathbb{R}^n$.



If $\psi(p) \in \partial H^n$,



$$\psi\circ\phi^{-1} : \overset{\subseteq H^n}{\phi(U\cap V)} \longrightarrow \overset{\subseteq H^n}{\psi(U\cap V)}$$ is a diffeomorphism

$$: \underset{\text{open in } \mathbb{R}^n}{W} \longrightarrow \underset{\text{not open in } \mathbb{R}^n}{\phi(W)}$$

So contradiction.

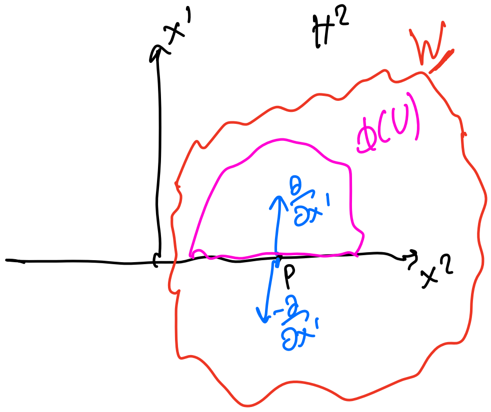
Things that carry over to manifolds with boundary:

Let M be a manifold with boundary.

1) **Def:** $f: M \to \mathbb{R}$ is a C^∞ if for any chart (U,ϕ) $f\circ\phi^{-1} : \underset{\subseteq H^n,\text{ open in } H^n}{\phi(U)} \to \mathbb{R}$ is C^∞ in the standard sense

(C^∞ means $\exists$ smooth extension on a nbhd of $\phi(U)$ in $\mathbb{R}^n$)

2) Tangent vectors.



$$T_pH^2 := \left\{ v: C_p^\infty(H^2) \to \mathbb{R} \;\middle|\; v \text{ is a point derivation} \right\}$$

$$= \text{span}\left\{ \frac{\partial}{\partial x^1}\Big|_p , \frac{\partial}{\partial x^2}\Big|_p \right\}$$

$$= T_p\mathbb{R}^2$$

For a manifold with boundary. Let (U, ϕ) be a chart near $p \in \partial M$

Then $T_pM := \{ v : C_p^\infty(M) \to \mathbb{R} \mid v \text{ is point derivation} \}$

$$= \text{span}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$$

Then TM and $\mathfrak{X}(M)$ are defined in the same way.

Also distributions and orientation are defined in the same way.

3) T_p^*M is defined in the same way.
And so is $\Lambda^k(T^*M)$ and $\Omega^k(M)$

4) embedded / regular submanifolds are defined in the same way.

↳ could be with or w/o boundary.

If $S \cap \partial M = \phi$, then it is a manifold
→ with boundary $\partial S = S \cap \partial M$ } True or [✓ False]

Thm: Let M be an n-dim manifold with boundary.
Then ∂M is $n-1$ dim submanifold of M and is without boundary.

Proof: $(U, \phi = (x^1, \dots, x^n))$ chart on M near $p \in \partial M$

$\Rightarrow$ $U \cap \partial M$ is defined as the vanishing of the last coordinate.

$\Rightarrow$ ∂M is a $n-1$ dim submanifold of M with chart

$(U \cap \partial M, \phi_{\partial M} = (y^1, \dots, y^{n-1}))$

where $y^i = x^i \circ i$ and $i : \partial M \hookrightarrow M$

Let $\{(U_p, \phi_p) \mid p \in \partial M\}$ be adapted charts relative to ∂M that cover ∂M.

Then $\{(U_p \cap \partial M, \phi_{p_{\partial M}}) \mid p \in \partial M\}$ is a C^∞ atlas for ∂M

Since $\phi_{p_{\partial M}}(U_p \cap \partial M)$ is open subset of $\mathbb{R}^{n-1}$, ∂M is a manifold w/o boundary.

$$\Rightarrow \partial^2 M = \phi$$

∎

We make the usual abuse of notation: $i_{*,p}(T_p \partial M)$ "=" $T_p \partial M$

Let (U, ϕ) be a chart near $p \in \partial M$.

Then $T_p \partial M \underset{\curvearrowleft\ \curvearrowright}{=} \text{span}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^{n-1}}\Big|_p \right\} \subseteq T_p M$

W/o abuse of notation: $i_{*,p}(T_p \partial M) = \text{span}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^{n-1}}\Big|_p \right\} \subseteq T_p M$

$T_p \partial M = \text{span}\left\{ \frac{\partial}{\partial y^1}\Big|_p, \dots, \frac{\partial}{\partial y^{n-1}}\Big|_p \right\}$

where $\underline{y^i = x^i \circ i}$ and $i : \partial M \hookrightarrow M$

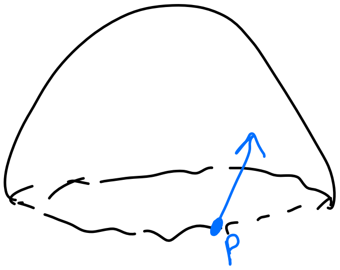
and $i_{*,p}\left(\frac{\partial}{\partial y^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p$

Let $P \in \partial M$

We say $X_P \in T_PM$ is inward pointing if $X_P \notin T_P\partial M$ and $\exists c : [0, \varepsilon] \to M$ s.t. $c(0) = P$, $c'(0) = X_P$

↳ equivalent to: $X_P = a^i \frac{\partial}{\partial x^i}\Big|_P$ where $a^n > 0$

We say $X_P \in T_PM$ is outward pointing if $-X_P$ is inward pointing.



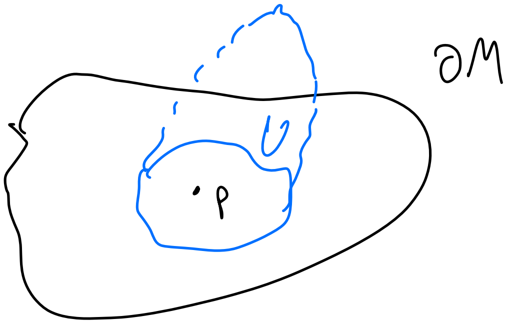
A vector field along ∂M is a map $X : \partial M \to TM$

X is C^∞ if for every chart (U, ϕ), $X_q = a^i(q) \frac{\partial}{\partial x^i}\Big|_q$ $\forall q \in U$ where $a^i \in C^\infty(U \cap \partial M)$

It is outward pointing if $a^n(q) < 0$ $\forall q \in U$.
(if X_q is outward pointing $\forall q \in \partial M$)

Proposition: On a manifold with boundary, $\exists$ C^∞ outward pointing vector field along ∂M.

Proof:



Let (U,ϕ) be a chart, then $X = \frac{-\partial}{\partial x^n}$ is a C^∞ outward pointing vectorfield along $U \cap \partial M$.

(use partition of unity)

Orientation of ∂M:

Proposition: Let M be an oriented n-manifold with boundary. If ω is an orientation form (nowhere vanishing n-form that is consistent with the fixed orientation on M), and X is a C^∞ outward pointing vectorfield along ∂M, then $\iota_X \omega$ is a smooth nowhere vanishing $n-1$ form on ∂M. So ∂M is orientable and that orientation is called the boundary orientation (or the induced orientation).

Proof: $\iota_X \omega \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}} \right)$

$= \omega \left(X, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}} \right)$

$$= \omega\left(g\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}}\right)$$

$$= f\, dx^1\wedge\cdots\wedge dx^n\left(g\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}}\right)$$

$$= fg(-1)^{n-1} \neq 0$$

∎

Exc. Show that the induced orientation on ∂M doesn't depend on X and so depends only on the orientation of M.

Let (U,ϕ) be a chart near p in the oriented atlas of M.

Then $dx^1\wedge\cdots\wedge dx^n$ is an orientation form on U.

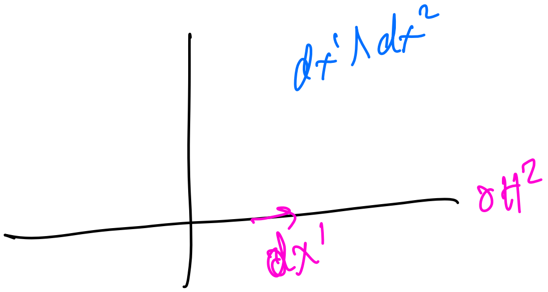
Let $X = \frac{-\partial}{\partial x^n}$, Then $\iota_{\frac{-\partial}{\partial x^n}}(dx^1\wedge\cdots\wedge dx^n) = (-1)^n dx^1\wedge\cdots\wedge dx^{n-1}$

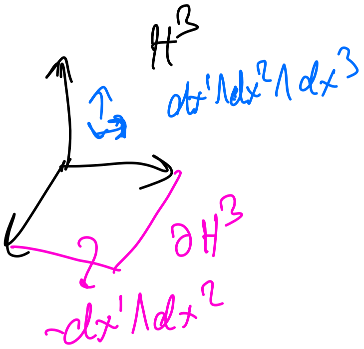
So $(U\cap\partial M, \psi = ((-1)^n x^1, \ldots, x^{n-1}))$ is a chart in the oriented atlas of ∂M wrt the boundary orientation.

If $\{(U_\alpha, \phi_\alpha = (x^1_\alpha, \ldots, x^n_\alpha))\}$ is an oriented atlas for M,

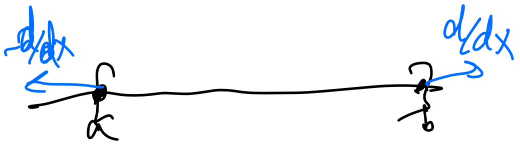
Then $\{(U_\alpha\cap\partial M, \psi_\alpha = ((-1)^n x^1_\alpha, \ldots, x^n_\alpha))\}$ is an oriented atlas for ∂M that giving ∂M the boundary orientation

on H^2





Let $M = [a,b]$ equipped with the standard orientation, then $\partial M = \{a,b\}$



The boundary orientation on M is

$a \longmapsto -1$

$b \longmapsto +1$

10 mins break

(Please fill out the course evaluation)

Integration

- ⊛ Role of alternating k-tensors in the theory of integration
- ⊛ Def of Integration on manifolds
- ⊛ Stokes Thm.

⊛ We cannot integrate functions on manifolds in a coordinate independent way:

Let $f: \mathbb{R}^2 \to \mathbb{R}$, $\quad M = \overline{B_1(0)}$

Then $\text{Integral}(f) = \int_M f := \underbrace{\int_M f(x,y)\, dx\, dy}_{\text{is not coordinate independent}}$ using The standard coordinate system

Let $\phi : (x,y) \longmapsto (\tilde{x}, \tilde{y})$ be a diffeomorphism

$$\int_{\phi(M)} f(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} = \int_M f(x,y)\, |\det(D\phi)|\, dx\, dy$$

$$\neq \int_M f(x,y)\, dx\, dy$$

⊛ Recall $\alpha \in \Lambda^k(V^*)$ can be interpreted as a "signed k-dim volume meter".

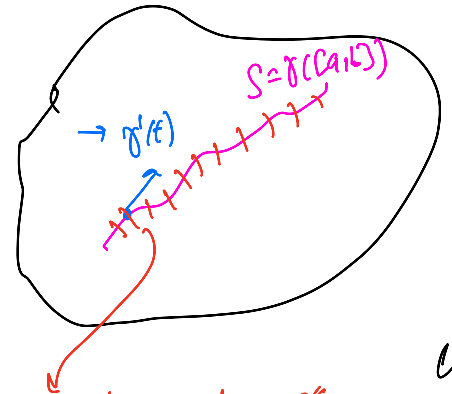
$\alpha : (v_1, \dots, v_k) \longmapsto$ signed volume of $P_{v_1, \dots, v_k}$

$:= \{ \sum t_i v_i \mid t_i \in [0,1] \}$

$\subseteq \operatorname{span}\{v_1, \cdots, v_k\}$

Ⓧ $k=1$ case

Let M be a manifold and let $w \in \Omega^1(M)$ be compactly supported



Let $\gamma : [a,b] \to M$ be an embedded curve ($\gamma'(t) \neq 0 \ \forall t$) (oriented by γ.)

for $P \in S$,

$w_p : v \longmapsto w_p(v)$ defines a signed length meter.

Approximate "signed length" of a segment is $w_p(\gamma'(t_i))\,(t_{i+1} - t_i)$, So Approx "signed" length of the curve is $\sum_i w_p(\gamma'(t_i))\,(t_{i+1} - t_i)$

Define "signed length of S" wrt w $:= \lim_{|P| \to 0} \sum_i w_p(\gamma'(t_i))\,(t_{i+1} - t_i)$

$$= \int_a^b w_{\gamma(t)}(\gamma'(t))\,dt$$

So given a signed length meter $w \in \Omega^1(M)$,

"signed length of curve S" wrt w $:= \underbrace{\int_a^b w_{\gamma(t)}(\gamma'(t))\,dt}_{\uparrow \text{ independent of the parametrization.}}$

If $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to M$ is another embedding s.t. $\tilde{\gamma}(\tilde{a}, \tilde{b}) = S$ with the same orientation as γ ?

Then $\int_{\tilde a}^{\tilde b} \omega_{\tilde\gamma(t)}(\tilde\gamma'(t))\,dt = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt$

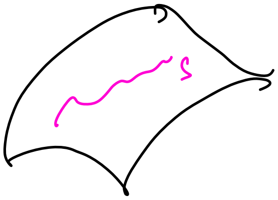
$\gamma^{-1}\circ\tilde\gamma : (\tilde a,\tilde b)\to(a,b)$ is a ↗ diffeomorphism

We define $\int_S \omega := \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt$

↗ this well defined.
depends only on S and ω.

⊛ General $1\le k\le n$

Let $\omega\in\Omega^k(M)$ be compactly supported and let S be a oriented k-dim submanifold



For simplicity, suppose ~~S~~ $\mathrm{supp}(\omega)$ can be covered with one oriented chart:

$\phi : S \to \mathbb{R}^k$

Then $\phi^{-1} : (t_1,\dots,t_k) \longmapsto \phi^{-1}(t_1,\dots,t_k)$

↗ parametrization of S.

"signed k-dim Volume of S" $=$
wrt ω

$$\int_{\phi(S)} \omega_{\phi^{-1}(t_1,\dots,t_k)}\Big(\underbrace{\phi^{-1}_{*,(t_1,\dots,t_k)}\frac{\partial}{\partial r^1}}_{\frac{\partial}{\partial x^1}\big|_{\dots}}\Big|_{(t_1,\dots,t_k)},\dots,\phi^{-1}_{*,(t_1,\dots,t_k)}\frac{\partial}{\partial r^k}\Big)\,dr^1\dots dr^k$$

$$= \int_{\phi(S)} \omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\right)\Big|_{\phi^{-1}(x^1,\dots,x^k)} dx^1 \dots dx^k$$

Let $\psi : S \to \mathbb{R}^k$ be another oriented chart $(p \mapsto (y^1(p), \dots, y^k(p)))$

Then $\int_{\phi(S)} \omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\right)\Big|_{\phi^{-1}(x^1,\dots,x^k)} dx^1 \dots dx^k$

$y \frac{\partial y^i}{\partial x^i} \frac{\partial}{\partial y^i}$

$$= \int_{\phi(S)} \omega\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}\right)\Big|_{\phi^{-1}(x^1,\dots,x^k)} \det(D\psi\circ\phi^{-1})\, dx^1 \dots dx^k$$

$$= \int_{\psi(S)} \omega\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}\right)\Big|_{\psi^{-1}(y^1,\dots,y^k)} \cancel{\det(D\psi\circ\phi^{-1})}\ \cancel{\left|\det D(\phi\circ\psi^{-1})\right|}_{>0}\ dy^1 \dots dy^k$$

$$= \cancel{\pm} \int_{\psi(S)} \omega\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}\right)\Big|_{\psi^{-1}(y^1,\dots,y^k)} dy^1 \dots dy^n$$

$\therefore$ Given a k-form $\omega \in \Omega^k(M)$,

$$\int_S \omega := \int_{\phi(S)} \omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\right)\Big|_{\phi^{-1}(x^1,\dots,x^k)} dx^1 \dots dx^k$$

← independent of the oriented

7

↙ Chart used.

"signed k-dim Volume of S"
wrt w

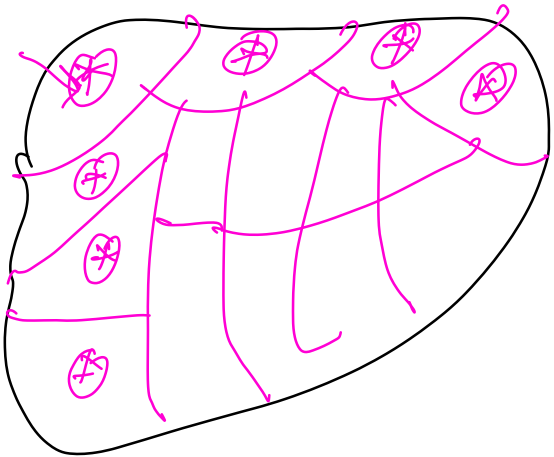
be compactly supported

If $k=n$ and $w \in \Omega^n(M)$ and $S=M$ (assume also $\text{supp}(w)$ is covered by 1 chart)

Then "signed Volume of M" wrt w $:= \int_M w$

if $k=0$, $S=\{\overset{+1}{p_i}, \overset{-1}{q_i} \mid i \in I\}$
let $f \in : S \to \mathbb{R}$, Then $\int_S f = \Sigma f(p_i) - \Sigma f(q_i)$

What if $\text{supp}(w)$ cannot be covered by 1-chart?



↙ compactly supported.

let $w \in \Omega^n(M)$ where
M is oriented manifold

Let $\{(U_\alpha, \phi_\alpha)\}$ be an oriented atlas

Let $\{\rho_\alpha\}$ be a partition of unity.

Then $\rho_\alpha w$ is compactly supported in U_α since $\text{supp}(\rho_\alpha w)$
$\subseteq \text{supp}(\rho_\alpha)$

$\subseteq U_\alpha$

And so we can already make sense of

$$\int_{\underset{M}{\cancel{U_\alpha}}} \rho_\alpha \omega := \int \underset{\rho_\alpha(x^1,\dots,x^n)}{\downarrow} \omega\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right)\Big|_{\phi^{-1}(x^1,\dots,x^n)} dx^1\dots dx^n$$

We now define $\displaystyle\int_M \omega := \sum_\alpha \int_M \rho_\alpha \omega$

Proposition: 1) $\int_M \omega$ is independent of Partition of unity and the oriented atlas.

2) Sum is finite ($\{U_\alpha\}$ is open cover supp(ω))

3) $\displaystyle\int_{-M} \omega = -\int_M \omega$

4) $\forall a \in \mathbb{R}$, $\omega, \eta \in \Omega^n(M)$ that are compactly supported:

$$\int_M a\omega + \eta = a\int_M \omega + \int \eta$$

$\int_M : \Omega^n(M) \longrightarrow \mathbb{R}$ is $\mathbb{R}$-linear.

1st

5) Let $F: N \to M$ & let $\omega \in \Omega^n(M)$ compactly supported

Then $\int_M \omega = \int_N F^*\omega$

Ex

6) If M can be covered by 1 chart U up to a measure 0 set,

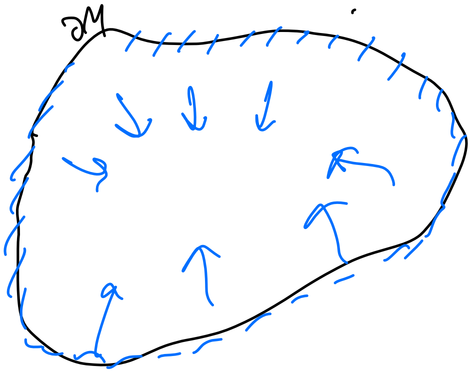
Then $\int_M \omega = \int_U \omega$

Stokes Theorem: Let M be an oriented manifold with boundary, and let ∂M be the boundary with the boundary orientation.

Let $\omega \in \Omega^{n-1}(M)$ be compactly supported.

Then $\int_M \underline{d\omega} = \int_{\partial M} i^*\omega$ where $i: \partial M \hookrightarrow M$

" $\int_{\partial M} \omega$



$\int$ exact n-form is completely determined by information on ∂M

$\int_a^b f'(x)\,dx = f(b) - f(a)$

Suppose $\underset{\text{with standard orientation}}{M=[a,b]}$. Then $\partial M=\{a,b\}$ with the boundary orientation

$$a \longmapsto -1$$
$$b \longmapsto +1$$

Let $f\in\Omega^0(M)$. Then $\int_a^b f'(x)\,dx = \int_M df = \int_{\partial M} f = -f(a)+f(b)$

Corollary: FTC.

Proof: you proved it on $\mathbb{R}^n$ and H^n in 257. Let $\{\rho_\alpha\}$ be partition of unity subordinate to $\{U_\alpha\}$ ↖ oriented atlas

$$\int_{\partial M}\omega =$$

$$= \sum_\alpha \int_{\underset{\partial U_\alpha}{\partial M}} \rho_\alpha\omega$$

$$= \sum_\alpha \int_{\Phi_\alpha(\partial U_\alpha)} \Phi_\alpha^{-1*}\rho_\alpha\omega$$

↖ on $\mathbb{R}^{n-1}$

$$= \sum_\alpha \int_{\Phi_\alpha(U_\alpha)} d(\Phi_\alpha^{-1*}\rho_\alpha\omega)$$

using Stokes Thm on H^n

$$= \sum_\alpha \int_{\Phi_\alpha(U_\alpha)} \Phi_\alpha^{-1*}(d\rho_\alpha\omega)$$

$$= \sum_\alpha \int_{U_\alpha\,M} d(\rho_\alpha\omega)$$

$$= \int_M \sum_\alpha d(\rho_\alpha\omega)$$

$$= \int_M d\left(\sum \rho_\alpha \omega\right)$$

$$= \int_M d\omega$$

$$\underset{\sim}{\omega} = x\,dy \wedge dz - y\,dx \wedge dz \pm z\,dx \wedge dy$$

2-form on S^2

$$\int_{S^2} \omega = \int_{B_1(0)} \overbrace{d\omega}^{dx \wedge dy \wedge dz} = \int_{B^1} dx\,dy\,dz$$

$$= \frac{4}{3}\pi r^3 \quad \nwarrow$$

Post-Lecture Practice Questions

1) Do the exercises above

2) Define $f : \mathbb{H}^2 \to \mathbb{R}$ by $f(x^1, x^2) = x^1 x^2$

show that f is C^∞ standard. (Find an extension $\tilde{f} \in C^\infty(U)$ where $U \supseteq \mathbb{H}^2$ and is open s.t. $\tilde{f}|_{\mathbb{H}^2} = f$

3) let $f \in C^\infty(\mathbb{H}^2)$.

Define $\left.\frac{\partial f}{\partial x^1}\right|_{(0,0)} := \left.\frac{\partial \tilde{f}}{\partial x^1}\right|_{(0,0)}$ where $\tilde{f}$ is an extension of f near $(0,0)$.

Show that $\left.\frac{\partial f}{\partial x^1}\right|_{(0,0)}$ is independent of the extension.

4)

Show that $X_p \in T_pM$ is inward pointing iff on every chart (U,ϕ) near p, $X_p = a^i \frac{\partial}{\partial x^i}\Big|_p$ where $a^n > 0$.

5) Show that $X : \partial M \to TM$ is C^∞ iff $\forall p \in \partial M$, $\exists$ nbhd $U \subseteq M$ of p and $\tilde{X} \in \mathfrak{X}(U)$ s.t.

$$\tilde{X}\big|_{\partial M \cap U} = X\big|_{\partial M \cap U}$$

6) There is a natural isomorphism

$$\Phi : \Omega^0(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n)$$
$$f \mapsto f\, dx^1 \wedge \cdots \wedge dx^n$$

that allow for a well defined notion on integration of functions:

let $f \in \Omega^0(\mathbb{R}^n)$ with compact support. Then we define:

$$\int_{\mathbb{R}^n} f := \int_{\mathbb{R}^n} f\, dx^1 \wedge \cdots \wedge dx^n$$

We have this because $\mathbb{R}^n$ comes with a standard coordinate system and has a standard nowhere vanishing n-form.

On a manifold, for every $w \in \Omega^n(M)$ that is nowhere vanishing, comes with a notion of integration of functions on M (this will depend on w)

Show that for every fixed isomorphism $\Phi : \Omega^0(M) \to \Omega^n(M)$, we have a notion on ~~of~~ integration on M that depends on Φ.

7)

Define a 2-form $w \in \Omega^2(\mathbb{R}^3)$ by

$$w = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

a) compute w in spherical coordinates (r, φ, θ) defined by

$$(x, y, z) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\theta)$$

b) Compute dw in both cartesian & spherical coordinates and verify that both expression represent the same 3-form

c) Compute $i^* w$ to S^2 where $i : S^2 \hookrightarrow \mathbb{R}^3$ using coordinates (φ, θ) on the open subset where these coordinates are defined

d) Show that $i^* w$ is nowhere vanishing 2-form and use stokes theorem to compute $\int_{S^2} i^* w$. Show it's not exact.

8) let $w = \frac{x\,dy - y\,dx}{x^2 + y^2} \in \Omega^2(\mathbb{R}^2 \setminus \{0\})$ be the closed form defined in assignment 6. Compute $\int_{S^1} i^* w$ where $i : S^1 \hookrightarrow \mathbb{R}^2$ using stokes thm and show that implies $i^* w$ is not exact.

9) let $P, Q \in C^\infty(\overline{B_1(0)})$ be functions on closed unit ball $\overline{B_1(0)} \subseteq \mathbb{R}^2$

~~Sho~~ Use stokes thm to show that

$$\int_{\overline{B_1(0)}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{S^2} P\,dx + Q\,dy$$

↑ Green's Theorem.

10) Let $w = (z - x^2 - xy)\, dx \wedge dy - dy \wedge dz - dz \wedge dx$

Compute $\int_D i^* w$ where $i : D \hookrightarrow \mathbb{R}^3$
and $D = \{ (x,y,z) \in \mathbb{R}^3 / x^2 + y^2 \leq 1,\ z = 0 \}$

11) Problem 22-7 - 22.11
12) Problem 23.3

13) What is wrong with the following argument.

Let $B_1(0)$ be the open unit ball which is an n-dim manifold without boundary.

$$\frac{4}{3}\pi = \int_{B_1(0)} dx \wedge dy \wedge dz = \frac{1}{3} \int_{B_1(0)} d\left[x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \right]$$

$$= \frac{1}{3} \int_{\partial B_1(0)} \left[x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \right]$$

by stokes Thm

$$= 0 \qquad \text{since } \partial B_1(0) = \emptyset$$

so $\frac{4\pi}{3} = 0$

14) Recall for $X \in \mathfrak{X}(\mathbb{R}^{n+1})$, $\nabla \cdot X \in C^\infty(\mathbb{R}^{n+1})$ satisfying
$\mathcal{L}_X (dx^1 \wedge \dots \wedge dx^n) = \nabla \cdot X \, dx^1 \wedge \dots \wedge dx^n$

Show that $\int_{\overline{B_1(0)}} \nabla \cdot X \; dx^1 \wedge \cdots \wedge dx^n = \int_{S^n} \langle X, N \rangle \, \iota_N (dx^1 \wedge \cdots \wedge dx^n)$

where $N = \sum_{i=1}^{n+1} x^i \frac{\partial}{\partial x^i}$

(Use stokes thm and Cartan's magic formula)