(*) Course evaluations

(*) One question on the exam will come from the optional questions in Assignment 7.

Exam: Problem $\underbrace{1 \sim 5}_{\text{Choose 4 out of 5}}$, 6 , 7

6 ↑ True false

7 ↑ bonus

Recall: we defined orientation on vector space

$$\text{Orientation on } V := \left[\begin{matrix}\text{Choice of}\\ \text{an ordered}\\ \text{basis}\end{matrix}\right] \longleftrightarrow \left\{\begin{matrix}\text{Choice of}\\ \alpha \in \Lambda^n(V^*)/\{0\}\end{matrix}\right\}$$

$\alpha \sim \alpha'$ if $\alpha = a\alpha'$ for $a > 0$

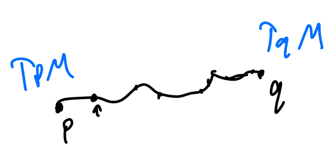
$$\left\{\begin{matrix}\text{orientations}\\ \text{on } V\end{matrix}\right\} = \left\{\begin{matrix}\text{Equivalence}\\ \text{classes of}\\ \text{ordered}\\ \text{basis}\end{matrix}\right\} \longleftrightarrow \left(\Lambda^n(V^*)/\{0\}\right)\Big\backslash\sim$$

Orientation on a manifold

A pointwise orientation on M is a choice of orientation on each T_pM.

We have $2^{|M|}$ choices of pointwise orientation.

I want to make a "smooth" choice of orientation on each T_pM



Def: An orientation on M is a pointwise orientation on M s.t. $\forall p \in M$, $\exists$ ordered local frame $X_1, \dots, X_n \in \mathfrak{X}(U)$ s.t.

$\{X_{1q}, \dots, X_{nq}\}$ is consistent with orientation specified on T_qM $\forall q \in U$.

(one can assume wlog that $X_1, \dots, X_n$ commutes)

exc

Equivalently: An orientation on M is a pointwise orientation on M s.t. $\forall p \in M$, $\exists$ a chart (U, ϕ) near p s.t.

$\left\{ \frac{\partial}{\partial x^1}\Big|_q, \dots, \frac{\partial}{\partial x^n}\Big|_q \right\}$ is consistent with orientation on T_qM $\forall q \in U$.

This gives rise to an "oriented atlas" that has the property: The transition maps satisfy $\det(D\psi \circ \phi^{-1}) > 0$ on $U \cap V$ for any 2-charts (U, ϕ), (V, ψ).

EXC

Equivalently: An orientation on M is a pointwise orientation on M that admits an oriented an atlas.

We can define an equivalence relation on the space of all oriented atlases: $\mathcal{A} \sim \mathcal{A}'$ if $\mathcal{A} \cup \mathcal{A}'$ is another oriented atlas.

(specifies the same orientation)

Each equivalence class represent an orientation on M

EXC
Equivalently: An orientation on M is a pointwise orientation s.t. $\forall p \in M$ $\exists$ a chart (U, ϕ) s.t. $dx^1_q \wedge \dots \wedge dx^n_q$ specifies the same orientation on T_qM $\forall q \in U$.

Def: M is orientable if it admits an orientation.
An oriented manifold is an orientable manifold together with an orientation.

(Mobius strip, Klein bottle, $\mathbb{R}P^{2n}$ are not orientable manifolds)

Proposition: An orientable manifold admits 2^c orientations where $c = \#$ of connected components.

Proof: let μ, ν be 2 orientations on M (μ_p, ν_p are orientations on T_pM)
($\mu_p = \nu_p$ or $\mu_p = -\nu_p$)

let $f: M \to \{\pm 1\}$
$$: p \mapsto \begin{cases} 1, & \mu_p = \nu_p \\ -1, & \mu_p = -\nu_p \end{cases}$$

connected

connected

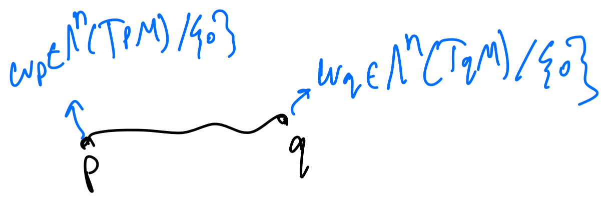
Let $p \in M$, let (U, ϕ) and (V, ψ) be charts near p consistent with μ and ν respectively.

Then $\det(D\psi \circ \phi^{-1}) \neq 0$ on $\phi(U \cap V)$.

Since $\Phi(U\cap V)$ is connected, $\det(D(\psi\circ\phi^{-1})) > 0$ or < 0 on $\phi(U\cap V)$

so $f = 1$ or -1 on $\phi(U\cap V)$

$\Rightarrow$ f is locally constant



Thm: A manifold is orientable iff $\exists$ a C^∞ nowhere vanishing n-form on M.

Proof: $(\Leftarrow)$

(Each nowhere vanishing n-form specifies an orientation $\Rightarrow$ M is orientable)

Let $\omega \in \Omega^n(M)$ be nowhere vanishing

Def an orientation on T_pM specified by ω_p which defines a pointwise orientation.

Let (U, ϕ) connected be a chart, Then $\omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) > 0$ or < 0 on U

Since $\omega\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) \neq 0$

Assume wlog that $\omega(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}) > 0$ on U

$\Rightarrow$ $\{\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q\}$ is consistent with the orientation on T_qM specified by ω_q $\forall q \in U$.

$\Rightarrow$ That pointwise orientation is an orientation.

($\Rightarrow$) Fix an orientation on M. Let $p \in M$.

Then $\exists$ (U, φ) near p s.t. $dx^1 \wedge \dots \wedge dx^n$ is consistent with the orientation.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$

Then define $\omega := \sum \rho_\alpha \, dx^1_\alpha \wedge \dots \wedge dx^n_\alpha$

Show ω is nowhere vanishing & is consistent with the orientation on M. Exc

Let M be an orientable manifold.
Let $\omega \in \Omega^n(M)$ be nowhere vanishing.
Then ω specifies an orientation as in the proof above.

We define a relation on nowhere where vanishing n-forms.

Let $\omega, \omega' \in \Omega^n(M)$ be nowhere vanishing:
$\omega \sim \omega'$ if $\omega = f\omega'$ for $f > 0$
(so if ω and ω' specify the same orientation).

{orientations} $\longleftrightarrow$ {eq classes of nowhere vanishing forms}

$\longleftrightarrow$ {eq classes of oriented atlas}

Manifolds with Boundary

The prototype of a manifold with boundary is

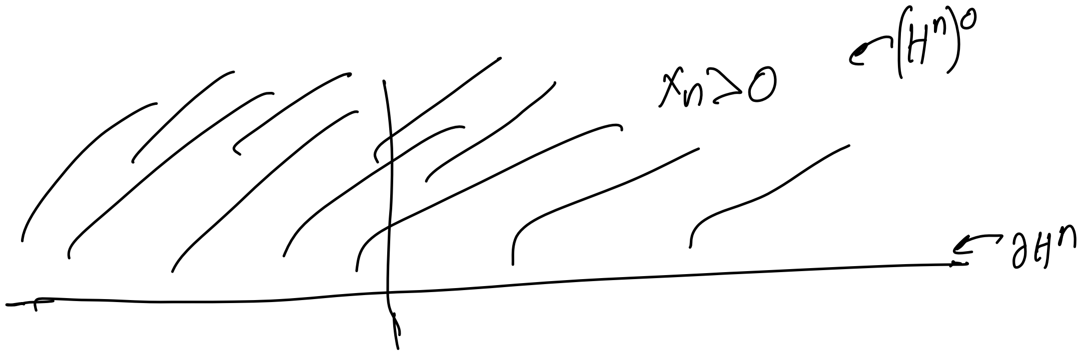
for $n \geq 2$: $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$ with the subspace topology

for $n=1$: $\mathbb{H}^1_+ = \{x \geq 0\}$ or $\mathbb{H}^1_- = \{x \leq 0\}$ for technical reasons

with the subspace topology

Points with $x_n > 0$ are called interior points of $\mathbb{H}^n$ denoted by $(\mathbb{H}^n)^o$.

Points with $x_n = 0$ are called boundary points of $\mathbb{H}^n$ (denoted by $\partial \mathbb{H}^n$)



<u>Def</u>: A topological n-manifold with boundary is second countable, Hausdorff topological space that is locally $\mathbb{H}^n$.

for $n \geq 2$, A chart (U, ϕ) is a homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{H}^n$

where U is open in M, and $\phi(U)$ is open in $\mathbb{H}^n$.
(for $n=1$, $\phi(U) \subseteq \mathbb{H}^1_+$ or $\mathbb{H}^1_-$)

A collection of charts $\{(U,\phi)\}$ is a C^∞ atlas if

they cover M and if for any 2 charts (U,ϕ), (V,ψ),

open in $\mathbb{H}^n$ open in $\mathbb{H}^n$

$$\psi\circ\phi^{-1} : \phi(U\cap V) \longrightarrow \psi(U\cap V)$$

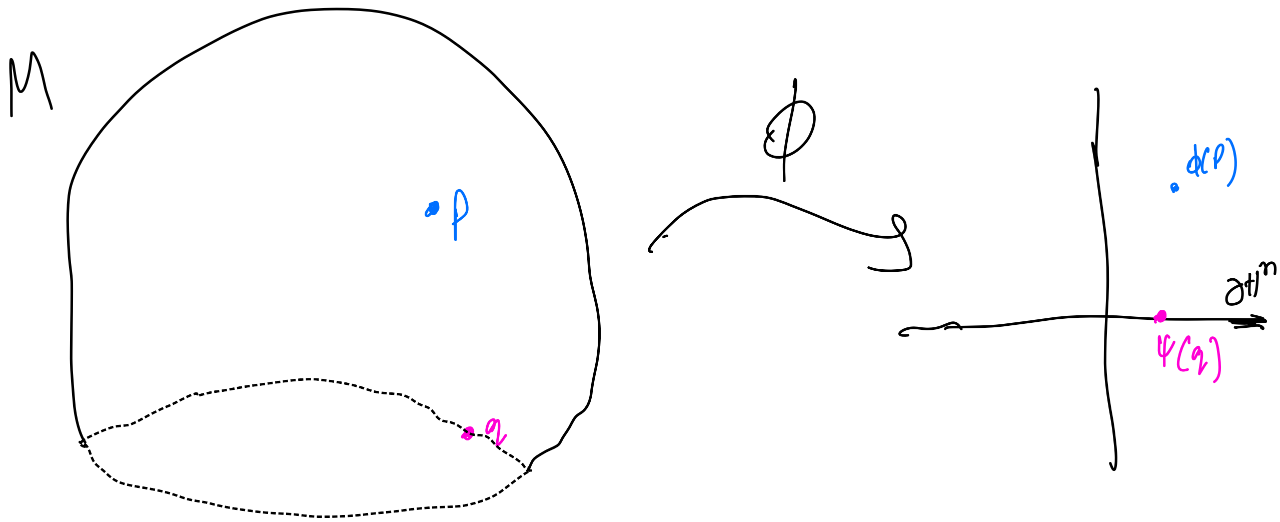
is a diffeomorphism

(smooth admits a smooth extension on an open subset in $\mathbb{R}^n$ containing $\phi(U\cap V)$)

Def: A smooth manifold with boundary is a topological manifold with boundary together with a maximal atlas.

A point $p\in M$ is an interior point if $\exists (U,\phi)$ near p s.t. $\phi(p) \in (\mathbb{H}^n)^\circ$

A point $p \in M$ is a boundary point if $\exists (U,\phi)$ near p s.t. $\phi(p) \in \partial \mathbb{H}^n$



<u>Proposition</u>: The notions of interior & boundary points is well defined and is independent of coordinates & partitions M into M° and ∂M.

(Corollary of smooth invariance of Domain)

Post-lecture Practice Questions

1) do the excercises above.
2) Let $F: \mathbb{R}^2 \to \mathbb{R}^2$

   $: (x,y) \mapsto (xe^y + y,\ xe^y + \lambda y)$

   a) for what λ is a diffeomorphism

   b) for which λ are the charts $(\mathbb{R}^n, Id)$ and $(\mathbb{R}^n, F)$ specify the same orientation. (i.e. $\det D(F \circ Id^{-1}) > 0$)

(we say F is orientation perserving if $\det DF > 0$)

3) Show $M = [a,b]$ is a manifold with boundary and $\partial M = \{a,b\}$.

Show that $\{([a,b], Id), ([a,b), Id)\}$ is an oriented atlas for $[a,b]$ for the standard positive orientation specified by the 1-form dx.

4) Let $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $df \neq 0$. Show that $f^{-1}(-\infty, 1]$ is a n-dim manifold $M \subseteq \mathbb{R}^n$ with boundary $\partial M = f^{-1}(1)$

Use this to show that $M = \overline{B_1(0)}$ the closed unit ball in $\mathbb{R}^n$ is a manifold with boundary and that $\partial M = S^2$.

5) do Problem 21.4, 21.5, 21.6, 21.10

6) Let $M = \overline{B_1} \setminus \{0\}$. Find the manif boundary ∂M and the topological boundary.

7) Let M be a manifold with bounday & N be a manifold. Then $M \times N$ is a manifold with boun and $\partial(M \times N) = \partial M \times N$

8) Only for fun:

<u>Non orientability of the Klein bottle</u> :

Let Γ be the group of diffeomorphisms of $\mathbb{R}^2$ generated by $\tau(x,y) = (x+1, y)$ and $\sigma(x,y) = (1-x, y+1)$

Then the Klein bottle $K := \mathbb{R}^2/\Gamma$ (quotient of $\mathbb{R}^2$ by the subgroup Γ of diffeomorphisms)

let $\pi: \mathbb{R}^2 \to K$ be the projection map. (which is a local diffeomorphism)

let ω be any smooth 2-form on K.

let $\tilde{\omega} := \pi^* \omega$.

a) Show that $\sigma^* \tilde{\omega} = \tilde{\omega}$

b) Show that $\tilde{\omega} = f\,dx \wedge dy$ where $f \in C^\infty(\mathbb{R}^2)$ satisfies $f \circ \sigma = -f$

c) Show that ω vanishes somewhere & Conclude K is not orientable.