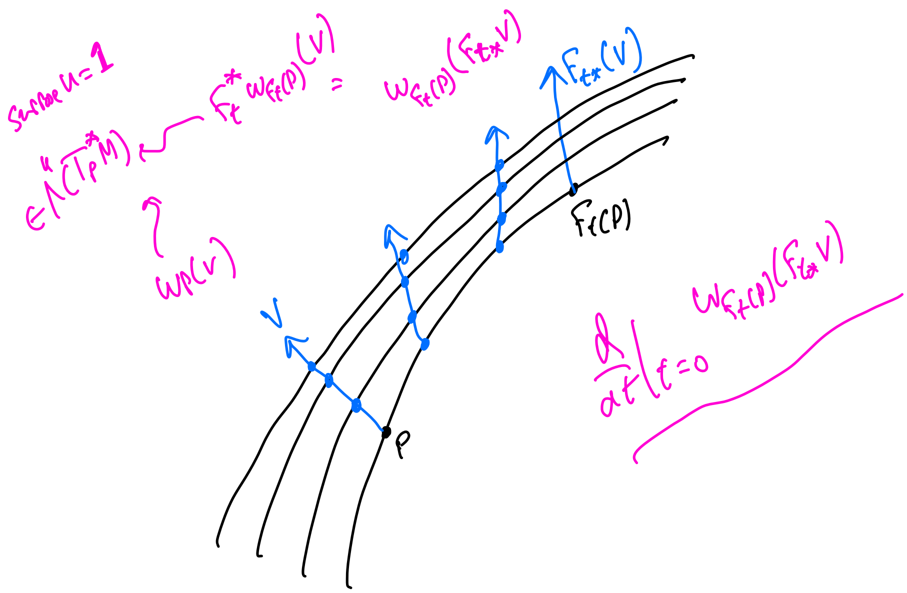
- ⊛ Assignment 6 is due Sunday August 8.
- ⊛ Assignment 7 is due Sunday August 13
  but no penalty until August 16
- ⊛ Post lecture practice questions more elementary.
- ⊛ Course evaluations

Recall we defined $\mathcal{L}_X$ on $\Omega^k(M)$, given $X \in \mathfrak{X}(M)$



$f \mapsto f'$

cure on $\Lambda^k(T_p^* M)$

$$\mathcal{L}_X \omega\Big|_p := \lim_{t \to 0} \frac{F_t^* \omega_{F_t(p)} - \omega_p}{t} = \frac{d}{dt}\Big|_{t=0} F_t^*\left(\omega_{F_t(p)}\right)$$

so for $v \in T_pM$ $\quad \mathcal{L}_X \omega(v) = \frac{d}{dt}\Big|_{t=0} \omega_{F_t(p)}(F_{t*}v)$

Equivalently, we can define $\mathcal{L}_X \omega|_p$ as the unique element in $\Lambda^k(T_p^* M)$ satisfying:
$$F_t^*(\omega_{F_t(p)}) = \omega_p + t\, \mathcal{L}_X \omega|_p + o(t)$$

Write it in coordinates:

$$\underbrace{\mathcal{L}_X}_{a_i \frac{\partial}{\partial x^i}} \omega\big|_p \left(\frac{\partial}{\partial x^I}\Big|_p\right) = \frac{d}{dt}\Big|_{t=0} F_t^*\Big(\overbrace{\omega_{F_t(p)}}^{b_J \circ F_t(p)\, dx^J_{F_t(p)}}\Big)\left(\frac{\partial}{\partial x^I}\Big|_p\right)$$

$$= \frac{d}{dt}\Big|_{t=0} b_J \circ F_t(p)\ \underbrace{\left(dF_t^{j_1} \wedge \cdots \wedge dF_t^{j_k}\right)_p \left(\frac{\partial}{\partial x^I}\Big|_p\right)}_{= \frac{\partial(F_t^{j_1} \cdots F_t^{j_k})}{\partial(x^{i_1} \cdots x^{i_k})}\big|_p}$$

$$= X(b_J)(p)\ \underbrace{\frac{\partial(x^{j_1} \cdots x^{j_k})}{\partial(x^{i_1} \cdots x^{i_k})}}_{\delta^J_I} + b_J(p) \frac{d}{dt}\Big|_{t=0} \frac{\partial(F_t^{j_1} \cdots F_t^{j_k})}{\partial(x^{i_1} \cdots x^{i_k})}\Big|_p$$

exc ...

$$= X(b_I)(p) + b_J(p)\Big[da_{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_k}\left(\frac{\partial}{\partial x^I}\Big|_p\right) + \cdots + dx^{j_1} \wedge \cdots \wedge dx^{j_{k-1}} \wedge da_{j_k}\left(\frac{\partial}{\partial x^I}\Big|_p\right)\Big]$$

$(*)$

If $X = \frac{\partial}{\partial x^1}$, $\mathcal{L}_X \omega\big|_p \left(\frac{\partial}{\partial x^I}\Big|_p\right) = \frac{\partial b_I}{\partial x^1}$ (Assignment 7)

$\Rightarrow$ $\mathcal{L}_X \omega$ in coordinates:

$$\mathcal{L}_X \omega = \underbrace{\mathcal{L}_X \omega\left(\frac{\partial}{\partial x^I}\right)}_{(*)\ \in C^\infty(U)} dx^I$$

$$\Rightarrow \mathcal{L}_X \omega \in \Omega^k(M).$$

Proposition: limit always exists & $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$

Theorem: Properties of $\mathcal{L}_X$

Let $X \in \mathfrak{X}(M)$

1) $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation:

meaning it's $\mathbb{R}$-linear and

$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$$

for $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$

(similar to $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is also a derivation on the Lie algebra $\mathfrak{X}(M)$)

$$\mathcal{L}_X(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0} f_t^*(\omega \wedge \eta)$$

$$= \frac{d}{dt}\Big|_{t=0} f_t^*\omega \wedge f_t^*\eta$$

etc

$$= \left(\frac{d}{dt}\Big|_{t=0} f_t^*\omega\right) \wedge \eta + \omega \wedge \left(\frac{d}{dt}\Big|_{t=0} f_t^*\eta\right)$$

$= \dots$

2) $\mathcal{L}_X d = d\mathcal{L}_X$

Recall $f_t^*(\omega) = \omega + t\mathcal{L}_X\omega + o(t)$ $\leftarrow \in \Omega^k(M)$

$f_t^*(d\omega) = d(f_t^*\omega) = \cancel{d\omega} + t\, d\mathcal{L}_X\omega + o(t)$

$\downarrow$

$\cancel{d\omega} + t\mathcal{L}_X d\omega + o(t)$

$\Rightarrow \quad t\mathcal{L}_X d\omega = t\, d\mathcal{L}_X\omega + o(t)$

$\Rightarrow \quad \mathcal{L}_X d\omega = d\mathcal{L}_X\omega$

3) $\mathcal{L}_X\left(\omega(X_1,\dots,X_k)\right)$

(*) global formula for $\mathcal{L}_X$ on forms that is useful computations

$= \mathcal{L}_X\omega(X_1,\dots,X_k) + \sum_{i=1}^{k}\omega(X_1,\dots,\mathcal{L}_XX_i,\dots,X_k)$

$f_t^*(\omega) = \omega + t\mathcal{L}_X\omega + o(t)$

$\mathcal{L}_X\omega(X_1,\dots,X_k)$
$= X(\omega(X_1,\dots,X_n)) - \sum \omega(X_1,\dots[X,X_i]\dots X_k)$

$f_t^*(\omega)(X_1,\dots,X_k) = \omega(X_1,\dots,X_k) + t\mathcal{L}_X\omega(X_1,\dots,X_k) + o(t)$

$\downarrow$

$\omega(f_{t*}X_1,\dots,f_{t*}X_k)\circ f_t$

$f_t = f_{-(-t)}$

$f_{-t}\left(Y_{f_t(p)}\right) = Y_p + t\mathcal{L}_XY + o(t)$

$\hookrightarrow X_1 - t\,\mathcal{L}_X X_1 + o(t)$

$$\omega\left(X_1 - t\mathcal{L}_X X_1 + o(t), \dots, X_n - t\mathcal{L}_X X_n + o(t)\right) \circ F_t = \omega(X_1, \dots, X_n) + t\,\mathcal{L}_X \omega(X_1, \dots, X_n) + o(t)$$

$$\downarrow$$

$$\omega(X_1, \dots, X_n) \circ F_t = \cancel{\omega(X_1, \dots, X_n)} + t\left\{\mathcal{L}_X \omega(X_1, \dots, X_n) + \sum_{i=1}^{n} \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_n)\right\} + o(t)$$

$$\cancel{\omega(X_1, \dots, X_n)} + t\,\mathcal{L}_X(\omega(X_1, \dots, X_n)) + o(t)$$

$$\Longrightarrow \quad \mathcal{L}_X\left(\omega(X_1, \dots, X_n)\right) = \mathcal{L}_X \omega(X_1, \dots, X_n) + \sum_{i=1}^{n} \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_n)$$

4) Cartan's Magic Formula:

$$\mathcal{L}_X = \iota_X d + d\iota_X \qquad !!!$$

Proof: Lemma: $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ is the unique $\mathbb{R}$-linear map satisfying

1) for $f \in \Omega^0(M)$, $\mathcal{L}_X(f) = X(f)$
2) $\mathcal{L}_X d = d\mathcal{L}_X$
3) $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$

Proof of lemma:

Let $D: \Omega^k(M) \to \Omega^k(M)$ be an $\mathbb{R}$-linear map satisfying the above Properties.

let $\omega \in \Omega^k(M)$, let (U, ϕ) be a chart.

Since D satisfies (3), D is a local operator ($D\omega|_U$ only depends on $\omega|_U$)

So on U, $D\omega|_U = D|_U(\omega|_U)$

$$= D(\omega|_U)$$

$$= D(a_I dx^I)$$

$$= Da_I \wedge dx^I + a_I \wedge D(dx^I)$$

$$= X(a_I) dx^I + a_I \wedge \underbrace{D(dx^{i_1} \wedge \dots \wedge dx^{i_k})}_{\sum_{l=1}^{k} dx^{i_1} \wedge \dots \wedge \underline{D dx^{i_l}} \wedge \dots \wedge dx^{i_k}}$$

$$= X(a_I) dx^I + a_I \sum_{l=1}^{k} dx^{i_1} \wedge \dots \wedge db_{i_l} \wedge \dots \wedge dx^{i_k}$$

where $\omega = a_I dx^I$, $X = b^i \frac{\partial}{\partial x^i}$

$$= \mathcal{L}_X \omega$$

$$\Rightarrow \mathcal{L}_X = d$$

It is then sufficient to show that

$$i_X d + d i_X : \Omega^k(M) \to \Omega^k(M)$$

satisfies the same properties.

$\mathbb{R}$-linear map, ✓

$$(i_X d + d i_X)(f) = i_X df = df(X) = X(f) \checkmark$$

$$d(i_X d + d i_X) = (i_X d + d i_X) d \checkmark$$

$$(i_X d + d i_X)(\omega \wedge \eta) \underset{\text{etc}}{=} \cdots = (i_X d + d i_X)\omega \wedge \eta + \omega \wedge (i_X d + d i_X)\eta$$

so done by uniqueness

$$\mathcal{L}_X = i_X d + d i_X$$

5) $\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}$

let $\omega \in \Omega^1(M)$,

$$\mathcal{L}_X(i_Y \omega) - i_Y(\mathcal{L}_X(\omega))$$
$$= \mathcal{L}_X(\omega(Y)) - \mathcal{L}_X \omega(Y)$$

$$= w([X,Y]) \qquad \text{by } ③$$

$$= L_{[X,Y]}\, w$$

(you can also use ④ to prove it)

6) $\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}$

(you can also use ④ to prove it)

for $w \in \Omega^k(M)$,

$$(\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X)\, w\, (X_1, \cdots, X_k)$$

$$= \mathcal{L}_X \left(\mathcal{L}_Y w(\cdots) \right) - \sum \mathcal{L}_Y w(\cdots, [X, X_i], \cdots) \quad (*)$$

$$- \mathcal{L}_Y \left(\mathcal{L}_X w(\cdots) \right) + \sum \mathcal{L}_X w(\cdots, [Y, X_i], \cdots) \quad (\#)$$

$$= \mathcal{L}_X \left(\mathcal{L}_Y (w(\cdots)) - \sum w(\cdots [Y, X_i], \cdots) \right) - (*)$$

$$- \mathcal{L}_Y \left(\mathcal{L}_X (w(\cdots)) + \sum w(\cdots [X, X_i], \cdots) \right) + (\#)$$

$$= [X,Y](w(\cdots)) - \sum\sum w(\cdots [X, X_j], \cdots, [Y, X_i], \cdots)$$

$$+ \sum\sum w(\cdots [Y, X_j], \cdots, [X, X_i] \cdots)$$

$$= [X,Y](w(\cdots)) - \sum w(\cdots,[X,[Y,X_i]],\cdots)$$
$$+ \sum w(\cdots,[Y,[X,X_i]],\cdots)$$

$$= [X,Y](w(\cdots)) + w(\cdots,[[X,Y],X_i],\cdots,)$$

$$= \mathcal{L}_{[X,Y]} w (\cdots)$$

Alternative Proof of global intrinsic formula for d.

Let $w \in \Omega^1(M)$, let $X, Y \in \mathfrak{X}(M)$

Then $dw(X,Y) = \iota_Y \iota_X dw$

$$= \iota_Y (\mathcal{L}_X w) - \iota_Y d\iota_X w$$

$$= \mathcal{L}_X \iota_Y w - \iota_{[X,Y]} w - \iota_Y d\iota_X w$$

$$= \mathcal{L}_X(w(Y)) - w([X,Y]) - \underbrace{\iota_Y d(w(X))}_{Y(w(X))}$$

$$= X(w(Y)) - Y(w(X)) - w([X,Y])$$

Similarly, you could use the properties to prove: $w \in \Omega^n(M)$

$$d\omega(y_0, \dots, y_k) = \sum_{i=0}^{k} (-1)^{i-1} y_i\left(\omega(y_0, \dots, \hat{y}_i, \dots, y_k\right)$$

$$+ \sum_{0 \le i \le j \le k} (-1)^{i+j} \omega([y_i, y_j], y_0, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_k)$$

(do it by induction)

Summary

We introduced 3 operators on $\Omega^*(M)$:

$d : \Omega^k(M) \to \Omega^{k+1}(M)$ antiderivation of degree 1

$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ antiderivation of degree -1

$\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ derivation.

They interact with $\wedge$:

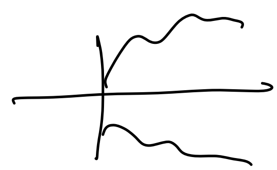
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$
$$\iota_X(\omega \wedge \eta) = \iota_X \omega \wedge \eta + (-1)^k \omega \wedge \iota_X \eta$$
$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta$$

They interact with each other in the following way:

$$\left.\begin{aligned}
&d^2 = \iota_x^2 = 0 \\
&\mathcal{L}_x \mathcal{L}_y - \mathcal{L}_y \mathcal{L}_x = \mathcal{L}_{[x,y]} \\
&\iota_x \iota_y + \iota_y \iota_x = 0 \\
&d \mathcal{L}_x - \mathcal{L}_x d = 0 \\
&\mathcal{L}_x \iota_y - \iota_y \mathcal{L}_x = \iota_{[x,y]} \\
&d \iota_x + \iota_x d = \mathcal{L}_x
\end{aligned}\right\} \text{The Cartan Calculus.}$$

Cartan's magic formula.

Orientation



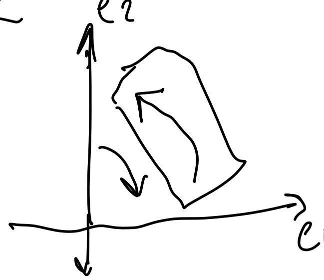
<u>Orientation on Vectorspaces:</u>

on $\mathbb{R}$

$\longrightarrow$ $\qquad \int_{[a,b]} f(x)\,dx := \int_a^b f(x)\,dx$

$\longleftarrow$ $\qquad \int_{[a,b]} f(x)\,dx := \int_b^a f(x)\,dx$

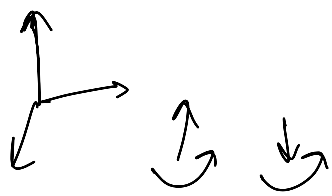
on $\mathbb{R}^2$



Notice: $A(R)$ satisfies $v_1 \times v_2 = A(R) e_3$

Notice: Choosing an orientation on $\mathbb{R}^2$ is equivalent to choosing an ordered basis.
We call the orientation wrt ~~the~~ $\{e_1, e_2\}$ the "positive" orientation.

on $\mathbb{R}^3$



We call the orientation wrt $\{e_1, e_2, e_3\}$ the "positive" orientation.

Notice that the orientation specified by an ordered basis $\{v_1, v_2, v_3\}$ is the same as the positive orientation iff $\det\{v_1\ v_2\ v_3\} > 0$

Generalization:

Let V be a vectorspace

(Recall that an orientation on $\mathbb{R}^n$ can be specified by fixing an ordered basis)

Let $\alpha = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_n\}$ be ordered basis of V

We define an equivalence relation:

α and β specify the same orientation if

$A_\alpha^\beta = \left[[v_1]_\beta \cdots [v_n]_\beta \right]$ has positive determinant.

A_α^β satisfies

* $v_i = w_j {A_\alpha^\beta}^j_i$
* $A_\alpha^\beta : [v]_\alpha \longmapsto [v]_\beta$
* $A_\alpha^\beta = {A_\beta^\alpha}^{-1}$

The equivalence relation partitions the space of all ordered basis into two equivalence classes.

Each class is called an orientation on V.

Fix $\gamma \in \Lambda^n(V^*)$. Let α, β be two ordered basis for V,

Then $\gamma(v_1, \dots, v_n) = \det(A_\beta^\alpha)\, \gamma(w_1, \dots, w_n)$

$\gamma(A_\beta^\alpha w_1, \dots, A_\beta^\alpha w_n)$

$\Rightarrow$ $\gamma(v_1,\dots,v_n)$ has the same sign as $\gamma(w_1,\dots,w_n)$

$(*)$ iff $\det(A^\alpha_\beta)>0$

iff $[\alpha]=[\beta]$ (α,β specify the same orientation)

We say γ specifies the orientation $[\{v_1,\dots,v_n\}]$ if $\gamma(v_1,\dots,v_n)>0$

By $(*)$, This independent of the choice of ordered basis in $[\{v_1,\dots,v_n\}]$

So $\gamma\in\Lambda^n(V^*)$ specifies an orientation on V.

This defines an equivalence relation on $\Lambda^n(V^*)\setminus\{0\}$:

$\gamma\sim\gamma'$ if they both specify the same orientation

or equivalently, $\gamma=a\gamma'$ where $a>0$

Which partitions $\Lambda^n(V^*)\setminus\{0\}$ into 2 equivalence classes, each associated to an orientation on V.

Orientation on a Manifold

next lecture.

Post-lecture Practice Questions

1) do the exercises above.

2) On $\mathbb{R}^3$, let $\alpha = x^2 dy - y\,dx$,

$$\beta = y\, dx \wedge dz$$

$$X = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

Calculate the following forms:

a) $\alpha \wedge \beta$

b) $\iota_X \beta$

c) $d\beta$

d) $\iota_X(\alpha \wedge \beta)$

e) $\mathcal{L}_X(\alpha \wedge \beta)$

d) verify Cartan's Magic formula for α and β.

3) Suppose $X, Y \in \mathfrak{X}(M)$ make a smooth rank-2 involutive distribution.

Let $\omega \in \Omega^1(M)$ s.t. $\omega(X) = \omega(Y) = 0$.

Show $d\omega(X, Y) = 0$.

(use the global intrinsic formula)

4) Show that if $\mathcal{L}_{\frac{\partial}{\partial x^1}} \omega = 0$ $(\omega \in \Omega^k(\mathbb{R}^n))$,

then coefficients of ω do not depend on x^1.

5) Problem 20.4.

6) Show $F_{t*}(Y_p) = Y_{F_t(p)} - t\, \mathcal{L}_X Y\big|_{F_t(p)} + o(t)$

(Albert's question)

7) Show that for any $\omega \in \Omega^1(M)$ and any $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$,

if $\omega(Y) = 0$, then $\mathcal{L}_X \omega(Y) = -\omega([X, Y])$

So if $\omega(Y) = 0$ and $[X, Y] = 0$, $\mathcal{L}_X \omega(Y) = 0$

8) On $\mathbb{R}^n$, show $\mathcal{L}_{\frac{\partial}{\partial x^i}} \omega = \mathcal{L}_{\frac{\partial}{\partial x^i}} (\omega_J \, dx^J)$

$$= \frac{\partial \omega_J}{\partial x^i} dx^J$$

$$\mathcal{L}_{\frac{\partial}{\partial x^i}} Y = \mathcal{L}_{\frac{\partial}{\partial x^i}} \left(Y^j \frac{\partial}{\partial x^j} \right)$$

$$= \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

(similar ☺)