* Assignment 5 solutions are up
* Assignment 7 will be posted soon.

Cartan Calculus:

1) The exterior derivative d
2) The interior multiplication ι
3) The Lie derivative $\mathcal{L}$

how they interact together.

Recall: We defined the exterior derivative $d : \Omega^*(M) \to \Omega^*(M)$, which satisfies, and is completely determined by, the following properties:

1) d is an antiderivation of degree 1 on $\Omega^*(M)$,

$$\left(\begin{array}{ll} * & d : \Omega^k(M) \to \Omega^{k+1}(M) \text{ and is } \mathbb{R}\text{-linear} \\ * & d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad \text{for } \omega \in \Omega^k(M) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \eta \in \Omega^\ell(M) \end{array} \right)$$

2) $d : \Omega^0(M) \to \Omega^1(M)$ is the differential.
3) $d^2 = 0$

Interior multiplication

We first define it on a vector space V.
Let $\beta \in \Lambda^k(V^*)$ for $k \geq 2$. Fix $v \in V$.

Define $\iota_v(\beta) \in \Lambda^{k-1}(V^*)$ by

$$\iota_v(\beta)(v_1, \cdots, v_{k-1}) = \beta(v, v_1, \cdots, v_{k-1}) \quad \text{for } v_1, \cdots, v_{k-1} \in V$$

$\iota_v(\beta)$ is called the interior multiplication or contraction of β with v.

If $k=1$, $\iota_v(\beta) := \beta(v)$
If $k=0$, $\iota_v(\beta) := 0$

$\iota : V \times \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*)$ is linear wrt both arguments. (Exc)

Proposition: Some properties of ι_v (fix $v \in V$)

1) If $\alpha^1, \cdots, \alpha^k \in V^* = \Lambda^1(V^*)$
Then $\iota_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v)\, \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k$

$$\alpha^1 \wedge \cdots \wedge \alpha^k (v, v_1, \cdots, v_{k-1}) = \det \begin{bmatrix} \alpha^1(v) & \alpha^1(v_1) & \cdots & \alpha^1(v_{k-1}) \\ \alpha^2(v) & \vdots & \cdots & \vdots \\ \vdots & & & \\ \alpha^k(v) & \alpha^k(v_1) & & \alpha^k(v_{k-1}) \end{bmatrix}$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v)\, \underline{\alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k (v_1, \cdots, v_{k-1})}$$

2) $\iota_v \circ \iota_v = 0$

$\iota_v \circ \iota_v(\beta)(v_1, \dots, v_{k-1}) = \beta(v, v, v_1, \dots, v_{k-1}) = 0$

3) for $\beta \in \Lambda^k(V^*)$ and $\gamma \in \Lambda^l(V^*)$,

$$\iota_v(\beta \wedge \gamma) = \iota_v(\beta) \wedge \gamma + (-1)^k \beta \wedge \iota_v \gamma$$

Hint: Use (1) (Exe.)

So $\iota_v : \Lambda^*(V^*) \longrightarrow \Lambda^*(V^*)$ is an anti derivation of degree -1 on the graded algebra $\Lambda^*(V^*)$ whose square 0.

We define interior multiplication on manifolds:

Fix $X \in \mathfrak{X}(M)$. Let $\omega \in \Omega^k(M)$

Define $\iota_X(\omega)$ as the $k-1$ form given by:

$$(\iota_X \omega)_p = \iota_{X_p} \omega_p$$

(Recall $d\omega$ could not be defined first on vector spaces and then pointwise on manifolds) for $X_1, \dots, X_{k-1} \in \mathfrak{X}(M)$, $\iota_X \omega(X_1, \dots, X_{k-1})$

$$= \omega(X, X_1, \dots, X_{k-1}) \in C^\infty(M)$$

$\Rightarrow \iota_X \omega \in \Omega^{k-1}(M)$

If $k=1$, $\iota_X \omega = \omega(X)$
If $k=0$, $\iota_X \omega = 0$

So $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ satisfies

1) $\mathbb{R}$-linear
2) $\iota_X(\omega \wedge \eta) = \iota_X \omega \wedge \eta + (-1)^k \omega \wedge \iota_X(\eta)$
for $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$
3) $\iota_X^2 = 0$
$\hookrightarrow \iota_X \circ \iota_X$

So ι_X is an antiderivation on the graded algebra $\Omega^*(M)$ of degree -1 s.t. $\iota_X^2 = 0$

Also $\iota : \mathfrak{X}(M) \times \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ is C^∞-linear wrt both arguments.

translates to $(\iota_X \omega)_p$ depends only on X_p and ω_p

<u>Proposition</u>: For $X, Y \in \mathfrak{X}(M)$,

$$\iota_X \circ \iota_Y + \iota_Y \circ \iota_X = 0$$

<u>Proof</u>: $(\iota_X \circ \iota_Y + \iota_Y \circ \iota_X)\,\omega(X_1, \dots, X_{k-2})$

$$= \omega(X, Y, X_1, \dots, X_{k-2}) + \omega(Y, X, X_1, \dots, X_{k-2})$$

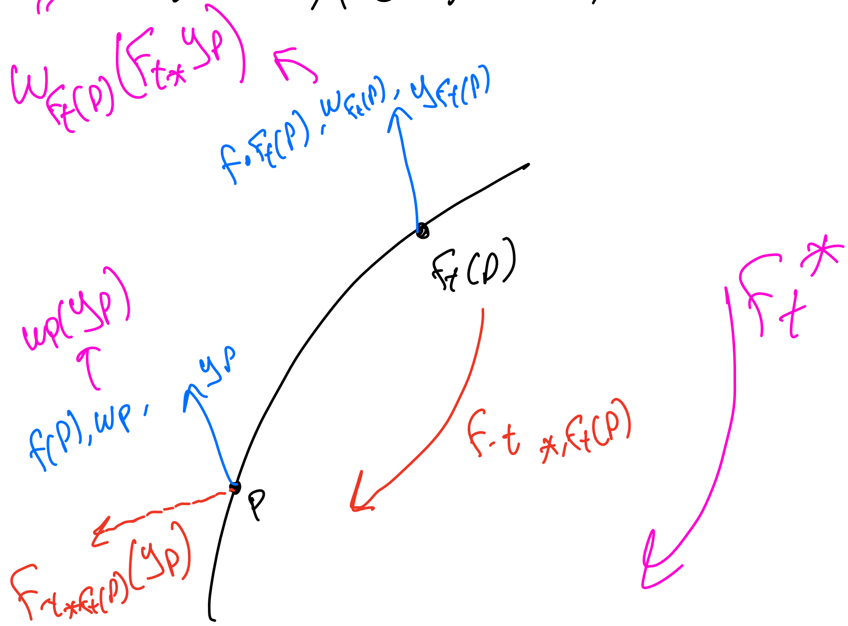
$$= 0$$

∎

Lie Derivative of k-form

$F_t^* \omega_{F_t(p)}(Y_p) = \omega_{F_t(p)}(F_{t*} Y_p)$

Let $X \in \mathfrak{X}(M)$, let F be the flow of X.



Recall we already defined $\mathcal{L}_X$ on $\Omega^0(M)$:

let $f \in \Omega^0(M)$,

$$\circledast \quad X(f) = \mathcal{L}_X f\Big|_p := \lim_{t\to 0} \frac{\overbrace{f\circ F_t(p)}^{F_t^*(f)} - f(p)}{t} = \frac{d}{dt}\Big|_{t=0} \overbrace{f\circ F_t(p)}^{F_t^*(f)}$$

Recall we defined $\mathcal{L}_X$ on $\mathfrak{X}(M)$,

Let $Y \in \mathfrak{X}(M)$,

$$\mathcal{L}_X Y\Big|_p := \lim_{t\to 0} \frac{F_{-t_*,F_t(p)}(Y_{F_t(p)}) - Y_p}{t} = \frac{d}{dt}\Big|_{t=0} \underbrace{F_{-t_*,F_t(p)}(Y_{F_t(p)})}_{\text{curve on } T_pM}$$

Similarly, we define on $\mathcal{L}_X$ on $\Omega^k(M)$,

For $\omega \in \Omega^k(M)$,

$$\mathcal{L}_X \omega\Big|_p = \lim_{t\to 0} \frac{F_t^*(\omega_{F_t(p)}) - \omega_p}{t}$$

$$= \frac{d}{dt}\Big|_{t=0} F_t^*(\omega_{F_t(p)}) \in \Lambda^k(T_p^*M)$$

$\searrow$ cure on $\Lambda^k(T_p^* M)$

If $\omega \in \Omega^0(M)$,

it agrees with the definition above (*)

Proposition: The limit always exists & also

$\mathcal{L}_X \omega \in \Omega^k(M)$ whenever $\omega \in \Omega^k(M)$.

In coordinates:

$$\mathcal{L}_X \omega|_p \left(\frac{\partial}{\partial x^I}\Big|_p\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f_t^* \omega_{f_t(p)} \left(\frac{\partial}{\partial x^I}\Big|_p\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f_t^* \left(b_J \circ f_t(p)\, dx^J_{f_t(p)}\right) \frac{\partial}{\partial x^I}\Big|_p$$

$$= \frac{d}{dt}\Big|_{t=0} b_J \circ f_t(p)\, d f_{t\,p}^{j_1} \wedge \cdots \wedge d f_{t\,p}^{j_k} \left(\frac{\partial}{\partial x^I}\Big|_p\right)$$

$$= \cdots = ?$$

etc