$\textcircled{*}$ Assignment 4 solutions

Exterior derivative

We defined $\quad d: \Omega^0(M) \longrightarrow \Omega^1(M) \quad$ (differential)

$$: f \longmapsto df : \mathfrak{X}(M) \to C^\infty(M)$$
$$: X \mapsto X(f)$$

We wish to extend this definition to $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

<u>Thm</u>: $\exists!$ collection of $\mathbb{R}$-linear maps, called the exterior derivative, $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ satisfying:

1) for $f \in \Omega^0(M)$, df is the differential of f
2) for $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$,
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$
3) $d^2 = 0 \qquad$ ~~$\Rightarrow d = 0$ by Albert (Albert's Theorem).~~

allow for Stokes Theorem
$$\int_M d\omega = \int_{\partial M} \omega$$

Some intuition of d : (Assignment 7)

1) for $\omega \in \Omega^k(\mathbb{R}^n)$,

$d\omega_p(v_1, \cdots, v_{k+1})$ = $k+1$ coefficient in the Taylor expansion of $F(t)$

defined $F(t) := \int_{\partial V(t)} \omega$ where $V(t)$ is the Parallelopiped with vertices $p, p+tv_1, p+tv_2, \cdots$

2) It shows up in Frobenius thm.

Δ is completely integrable iff $\left\{ \begin{array}{l} \omega \in \Omega^k(M) \\ \text{s.t. } \omega(X_1, \cdots, X_k) = 0 \end{array} \right|$ whenever $X_1, \cdots, X_k \in \Gamma(\Delta) \Big\}$

is closed under d

3) Generalizes notions in Vector Calculus, such as gradient, divergence, curl, etc...

4) $\exists!$ collection of linear maps $d : \Omega^k(M) \to \Omega^k(M)$ satisfying

1) for a function f, df is the differential of f.

2) $\forall f \in C^\infty(M)$, $\omega \in \Omega^k(M)$,
$d(f\omega) = df \wedge \omega + f d\omega$

3) for any $f : M \to N$,
$f^* \circ d = d \circ f^*$

Def: An antiderivation on a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D : A \to A$

satisfying $D(\omega \cdot \tau) = D(\omega) \cdot \tau + (-1)^k \omega \cdot D(\tau)$

for $\omega \in A^k$, $\tau \in A^l$.

The antiderivation is of degree m if $\deg(D\omega) = \deg(\omega) + m$ $\forall \omega \in A^k$

Recall that $\Omega^*(M)$ is the graded algebra of differential forms on M.

We defined $d : \Omega^0(M) \to \Omega^1(M)$ (the differential)
$: f \longmapsto df : X \mapsto X(f)$

In local coordinates, $df = \frac{\partial f}{\partial x^i} dx^i$

coordinate independent

Alternative definition of df:

Let $f \in C^\infty(M)$. Define df to be the 1-form s.t. on any chart, $df = \frac{\partial f}{\partial x^i} dx^i$ on U. $*$

This is well defined since if (U, ϕ) and (V, ψ) are charts,

$$\frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial y^i} dy^i \quad \text{on } U \cap V.$$

so $*$ defines a global 1-form on M.

Let's define exterior derivative on charts first:

<u>Motivation</u>:

We say $w \in \Omega^1(M)$ is exact if $\omega = df$ for some $f \in \Omega^0(M)$.

In assignment 6, we find a necessary condition:

If ω is exact, Then $\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} = 0$

on every chart.

So $\left\{ \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0 \quad \text{on every chart} \right\}$ $(*)$

We say w is closed if it satisfies $(*)$.

<u>Proposition</u>: w is closed iff $X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = 0$ $\forall X, Y \in \mathfrak{X}(M)$.

$\Rightarrow$ That the property $(*)$ is coordinate independent.
(just let $X = \frac{\partial}{\partial y^i}$, $Y = \frac{\partial}{\partial y^j}$. Then leftside is $\frac{\partial \omega_j}{\partial y^i} - \frac{\partial \omega_i}{\partial y^j}$)

<u>Remark</u>: $\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}$ is <u>not</u> coordinate independent.

Exc

Since $\frac{\partial w_j}{\partial x^i} - \frac{\partial w_i}{\partial x^j}$ is antisymmetric, it can be interpreted as the ij^{th} component of a 2-form on U.

On U, $$d\omega := \sum_{i<j} \left(\frac{\partial w_j}{\partial x^i} - \frac{\partial w_i}{\partial x^j}\right) dx^i \wedge dx^j$$
$$= \sum_{i,j} \left(\frac{\partial w_j}{\partial x^i} dx^i\right) \wedge dx^j$$
$$= dw_j \wedge dx^j$$

Does this define a global 2-form on M?

<u>Lemma</u>: $dw_j \wedge dx^j$ is a coordinate independent expression:

<u>Proof</u>:

<u>Method #1</u>: by brute force.

let $(U, \phi = (x^1, \cdots, x^n))$ and $(V, \psi = (y^1, \cdots, y^n))$ be 2 charts.

Then on $U \cap V$, $\omega = a_i\, dx^i = b_i\, dy^i$

$$da_j \wedge dx^j = - - - - = d b_j \wedge dy^j$$

$$\left(\frac{\partial a_j}{\partial y^l} dy^l\right) \wedge \left(\frac{\partial x^j}{\partial y^m} dy^m\right) - - -$$

Exc

Method #2: Let $\eta : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$

$: (X, Y) \longmapsto X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$

It's Easy to see that η is C^∞-multilinear and alternating

and so it defines a smooth 2-form $\eta \in \Omega^2(M)$. Exc

Notice that on U, $\eta = \sum_{i<j} \eta\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) dx^i \wedge dx^j$

$$= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j$$

$$= d\omega_j \wedge dx^j$$

$$= d\omega$$

Btw: for any k-form $\alpha \in \Omega^k(M)$,

$\alpha = \sum_I \underbrace{\alpha_I}_{\alpha\left(\frac{\partial}{\partial x^I}\right)} dx^I$

On every Chart, $\eta = d\omega_j \wedge dx^j \implies$ The expression is coordinate independent.

and so $d\omega$ is well defined global 2-form and $d\omega = \eta$

$\rightarrow$ gives us a global intrinsic formula for $d : \Omega^1(M) \to \Omega^2(M)$

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

$\implies \omega$ is closed iff $d\omega = 0 \implies d^2(f) = d(df) = 0$

$\implies d^2 = 0$

Recall $X \in \mathfrak{X}(M)$, let (U, ϕ) be a chart,

$$X = a^i \frac{\partial}{\partial x^i} = X(x^i)\frac{\partial}{\partial x^i}$$

$\hookrightarrow a^i = X(x^i)$

Similarly $\omega = a_I \, dx^I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right) dx^{i_1} \wedge \dots \wedge dx^{i_k}$

Assignment 6, 5a

Method #3 (sneaky) (Could be easily applied to $\Omega^k(M)$)

We defined $d\omega := da_j \wedge dx^j$ on U.

We find some properties of $d : \Omega^1(U) \to \Omega^2(U)$.

and then we show that $\exists!$ operator $d : \Omega^1(U) \to \Omega^2(U)$

satisfying those properties.

$\Rightarrow \exists!$ 2-form $d\omega$ satisfying those properties

Then defining $d\omega$ on charts to be $d\omega := da_i \wedge dx^i$

defines a global 2-form $d\omega$ on M.

$\blacksquare$

Def: Let $\omega \in \Omega^k(M)$. We define $d\omega \in \Omega^{k+1}(M)$ in the following way.

On a chart (U, ϕ), where $\omega = a_I \, dx^I$, $\quad I = \{i_1, \dots, i_k\}$

$d\omega := da_I \wedge dx^I$ on U. $\quad \sum_{1 \le i_1 < i_2 \dots < i_k \le n} d(a_{(i_1, \dots, i_k)}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

If $k=0$, d is the differential on $\Omega^0(M)$ defined earlier.

Lemma: $d\omega$ is well defined.

Proof: Fix a chart (U,ϕ). We define $d\omega$ on U to be

$$d\omega = da_I \wedge dx^I$$

Then $d: \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$ satisfies the following properties:

1) d is an antiderivation of degree on $\Omega^*(U)$.
d is $\mathbb{R}$-linear ✓ and

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

$$d(a_I b_J\, dx^I \wedge dx^J) = d(a_I b_J\, dx^{IJ})$$

$\{i_1, \ldots, i_k, j_1, \ldots, j_\ell\}$

$$= d(a_I b_J) \wedge dx^{IJ}$$

$$= (da_I\, b_J + a_I\, db_J) \wedge dx^I \wedge dx^J$$

$$= da_I \wedge dx^I \wedge (b_J dx^J) + a_I\, db_J \wedge dx^I \wedge dx^J$$

$$= d\omega \wedge \eta + (-1)^k a_I\, dx^I \wedge (db_J \wedge dx^J)$$

$$= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

2) for $f \in \Omega^0(M)$, df is the differential of f (by definition).

3) $d^2 = 0$

$$d(d\omega) = d(da_I \wedge dx^I)$$
$$= \cancel{d^2 a_I}^{\,0} \wedge dx^I - da_I \wedge \underbrace{d(dx^I)}$$

$$d(dx^{i_1} \wedge \dots \wedge dx^{i_k})$$
$$= \sum_{j=1}^{k} (-1)^{j-1} dx^{i_1} \wedge \dots \wedge \underbrace{d^2 x^{i_j}}_{\text{vanishes}} \wedge \dots \wedge dx^{i_k}$$
$$= 0$$

We claim that if $D : \Omega^k(U) \to \Omega^{k+1}(U)$ also satisfies the above properties, then $D = d$

For $\omega \in \Omega^k(U)$,

$$D\omega = D(a_I\, dx^I) = D(a_I \wedge dx^I)$$
$$= Da_I \wedge dx^I + \cancel{a_I\, D(dx^I)}$$
$$= da_I \wedge dx^I + a_I \Big(\sum_{j=1}^{k} (-1)^{j-1} dx^{i_1} \wedge \dots \wedge \underbrace{Ddx^{i_j}}_{D^2 x^{i_j} = 0} \wedge \dots \wedge dx^{i_k} \Big)$$
$$= da_I \wedge dx^I$$
$$= d\omega$$

$$\Rightarrow D = d$$

And so defining $d\omega = da_I \wedge dx^I$ on charts defines a global $k+1$ form $d\omega \in \Omega^{k+1}(M)$

And so this defines $d : \Omega^k(M) \to \Omega^{k+1}(M)$

<u>Corollary</u>: $\exists !$ collection of $\mathbb{R}$-linear maps
$d : \Omega^k(M) \to \Omega^{k+1}(M)$
satisfying:

1) d is an antiderivation of degree 1
2) $d : \Omega^0(M) \to \Omega^1(M)$ is the differential.
3) $d^2 = 0$

<u>Proof</u>: Existence ✓

Uniqueness ✓ ?

$\leadsto$ $D : \Omega^k(U) \to \Omega^{k+1}(U)$ satisfies ① ② ③

Let $D : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the above.

for $\omega \in \Omega^k(M)$, and (U, ϕ) be a chart

nonsense

$$\cancel{D(\omega|_U)} = d\omega|_U \quad \Rightarrow \underline{D\omega = d\omega} \text{ on } U$$

$D\omega|_U$

$$(D - d)(\omega|_U) = 0 \qquad \Rightarrow D\omega = d\omega$$

<u>Lemma</u>: D is a local operator.

(for any open $U \subseteq M$, $\omega \in \Omega^k(M)$,
$D\omega|_U$ only depends on $\omega|_U$)

(any operator that is an antiderivation is a local operator.

<u>Ex</u>: $\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

Since $\frac{df}{dx}\Big|_U$ only depends on $f|_U$

we can talk about $\underbrace{\frac{d}{dx}\Big|_U}_{\frac{d}{dx}} : C^\infty(U) \to C^\infty(U)$

<u>Proof</u>: It suffices to show that if $\omega \in \Omega^k(M)$ s.t. $\omega|_U = 0$,
then $D\omega|_U = 0$ (for any U)

Let $U \subseteq M$ be open. Let ρ be a bump function s.t. $\operatorname{supp}(\rho) \subseteq U$,
let $p \in U$ $\rho(p) = 1$

Then $\rho\omega \equiv 0$ and so $0 = D(\rho\omega)$
$= D(\rho) \wedge \omega + \rho\, D\omega$

by evaluating at p, $0 = D\omega_p$

$\Rightarrow D\omega|_U = 0$

∎

Since D is a local operator, we can define

$$D|_U : \Omega^k(U) \to \Omega^{k+1}(U)$$
$$: \omega \longmapsto D|_U(\omega) \cancel{:= D(\tilde{\omega})|_U}$$ ~~for any $\tilde{\omega} \in \Omega^k(M)$ that is an extension of ω.~~

small mistake. where is i ∈

Fix

$D|_U(\omega)$ defined by :

for $p \in U$, $D|_U(\omega)_p = D\tilde{\omega}_p$ for any $\tilde{\omega} \in \Omega^k(M)$ that is an extension of ω near p.

This is well defined since D is a local operator.

Verify $D|_U : \Omega^k(U) \to \Omega^{k+1}(U)$ satisfies ①②③

and so by uniqueness, $D|_U = d$ on U (if U is a chart)

$\Rightarrow$ for any $\omega \in \Omega^k(M)$ and any chart (U, ϕ),

$$d\omega|_U = D|_U(\omega|_U) = D\omega|_U$$

$$\Rightarrow d\omega|_U = D\omega|_U$$
$$\Rightarrow d\omega = D\omega$$
$$\Rightarrow d = D$$

Thm: A fourth property:

Let $f: N \to M$ be smooth maps

$$f^* \circ d = d \circ f^* \quad \text{on } \Omega^k(M)$$

We found a global intrinsic formula for

$$d: \Omega^1(M) \to \Omega^2(M)$$

for $\omega \in \Omega^1(M)$,

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Thm: Global intrinsic formula for the exterior derivative $d: \Omega^k(M) \to \Omega^{k+1}(M)$ $(k \geq 1)$.

Let $\omega \in \Omega^k(M)$. Then $d\omega$ satisfies: for $k=0$ $df(X) = X(f)$

for $X_0, \ldots, X_k \in \mathfrak{X}(M)$,

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^{i-1} X_i\left(\omega(X_0, \ldots, \hat{X_i}, \ldots, X_k)\right)$$

$$- \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)$$

Proof: Step 1: The right side: $\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \to C^\infty(M)$ is C^∞-multilinear and alternating

Step 2: ① Define $D: \Omega^k(M) \to \Omega^{k+1}(M)$

$: \omega \mapsto \text{RHS}$

$\to$ and show it satisfies properties and by uniqueness, we are done.

② Let $\omega \in \Omega^k(M)$, let (U, ϕ) be a chart.

It suffices to show:

$$D\omega\left(\frac{\partial}{\partial x^J}\right) = d\omega\left(\frac{\partial}{\partial x^J}\right) \quad \text{for any multiindex}$$

$J = \{j_1, \dots, j_{k+1}\}$ in ↗ order.

Exc

Post-lecture Practice Questions

1) do exercises above.

2) Problem 19.7

3) Compute $F^*(d\omega)$ and $d(F^*\omega)$:

a) $M = \mathbb{R}^2$, $N = \mathbb{R}^3$,

$F(x,y) = \big((\cos y + 2)\cos x, (\cos y + 2)\sin x, \sin y \big)$

$\omega = y \, dz \wedge dx$

b) $M = \{ (a,b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1 \}$

$N = \mathbb{R}^3 \setminus \{0\}$

$F(u,v) = (u, v, \sqrt{1-u^2-v^2})$

$\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}} \left(x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy \right)$

4) Show that any antiderivation on $\Omega^*(M)$ is a local operator.

5) Show that if $D : \Omega^k(M) \to \Omega^{k+1}(M)$ is a local operator, and $U \subseteq M$ is open,

$$D|_U (\omega|_U) = D\omega|_U \quad \forall \omega \in \Omega^k(U)$$