- feedback form
- OH and prereq form
- Assignment 1 posted on friday

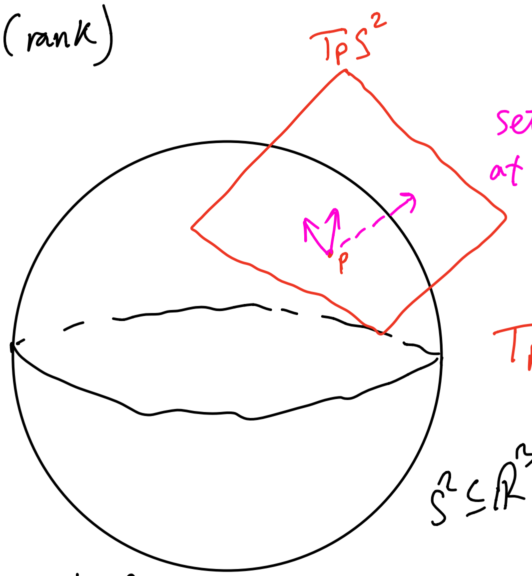
Point set topology $\longrightarrow$ Topological Manifold $\longrightarrow$ Smooth manifold $\longrightarrow$ Riemannian manifold

K-dim Manifolds in $\mathbb{R}^n$

$M \subseteq \mathbb{R}^n$ is a "K-dim manifold in $\mathbb{R}^n$" if

$\forall p \in M$, $\exists U$ neighbhd of p and $\exists V \subseteq \mathbb{R}^K$ that is open and a function $f: V \to U$ satisfying

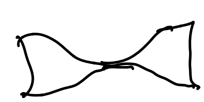
1) f is C^∞
2) f is a homeomorphism
3) Df is 1-1 (rank)



set of all vectors starting at p $= \mathbb{R}^3_p$
$= T_p \mathbb{R}^3$

$T_pS^2 = Df(\mathbb{R}^2)$
$= \text{Image}(Df)$
$\uparrow$
dim 2

Since Df is 1-1, the tangent space is welldefined and is of the same dimension



Remark: We want k-dim manifolds in $\mathbb{R}^n$ to satisfy the more general def. that we are trying to come up with.

Point Set Topology

Let M be a set.

Need the notion of continuity, compact, neighbourhood, connected, ...

$\mathbb{R}^n$ — Cont, compact, Connect

What structure do we need on M so that we can talk about these notions?

Topological structure

First Notice: Continuity in $\mathbb{R}^n$ could be formulated only using of open sets

Let $\tau \subseteq P(M)$ satisfying:

1) $\emptyset, M \in \tau$, 2) τ is closed under unions and finite intersections

If so, we say (M, τ) is a topological space with topology τ.

Remark: Read Appendix A (mandatory)

Topological Properties:

1) closed: $A \subseteq M$ is closed if A^c is open

2) Compact: ...

3) Connectedness: ...

4) continuous: $f: M \to M$ is cont if $f^{-1}(A)$ is open whenever A is open,

5) Bounded? Not a topology property. It depends on the fact $\exists d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $: (p, q) \mapsto \|p-q\|$

6) differentiable function? Not a topological property.
It depends on V.S of $\mathbb{R}^n$. We don't have a notion of addition & subtraction

Remark: A property is topological iff it stays invariant under homeomorphism.
(topological invariant)

$$A \xleftrightarrow{\text{homeomorphism}} B$$

Subspace Topology.

If $(M, \mathcal{T})$ is a topological space, then $A \subseteq M$ inherits a topology from the ambient space.

(Characterized by: Unique topology s.t.
$f: N \to A$ is cont iff
$i \circ f: N \to M$ is cont)

where $i: A \to M$
$: p \mapsto p$

let $\tau_A = \{ A \cap U : U \in \tau \}$

τ_A is called the subspace topology

exc: verify τ_A is a topology on A

Bases

Def: Let (M, τ) be a topological space. $\mathcal{B} \subseteq \tau$ is called a basis for τ if for every $U \in \tau$ and $p \in U$, $\exists B \in \mathcal{B}$ s.t. $p \in B \subseteq U$

equivalently

$\mathcal{B}$ is a basis for τ if every $U \in \tau$ is a union of elements in $\mathcal{B}$.

Ex: $\mathbb{R}^n$. $\mathcal{B} = \{ B_r(p) : p \in \mathbb{R}^n, r \in \mathbb{R}^+ \}$

is a basis for the regular topology.

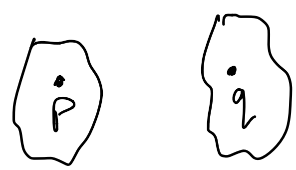
In fact, we can find a countable basis

$$\mathcal{B}^* = \{ B_r(p) : p \in \mathbb{Q}^n, r \in \mathbb{Q}^+ \}$$

Def: $(M, \mathcal{T})$ is second countable if it admits a countable basis.

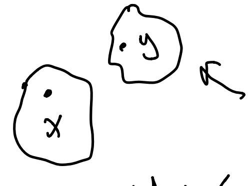
Proposition: A subspace A of a second countable space is also second countable.

Def: $(M, \mathcal{T})$ is a Hausdorff space if $\forall p, q \in M$, $p \neq q$, $\exists$ disjoint open sets U and V s.t. $p \in U$ and $q \in V$



Proposition: Any subspace of a Hausdorff space is Hausdorff

Proposition: $\{x\}$ are closed in Hausdorff spaces.



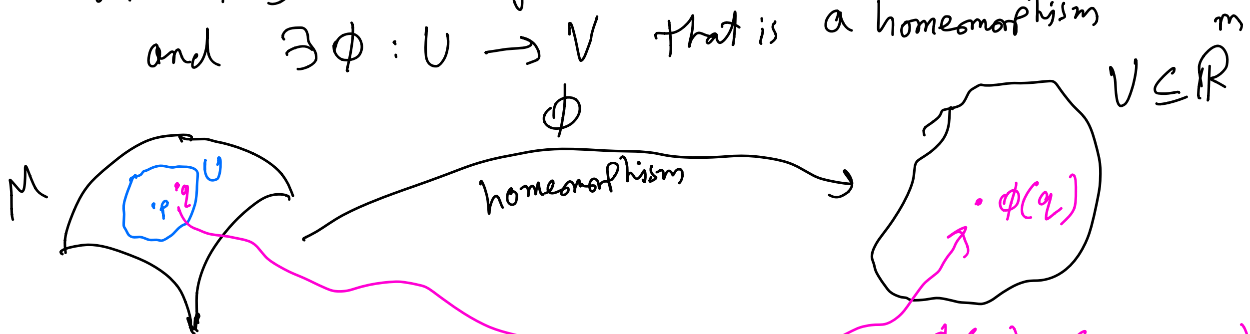
$\Rightarrow M \setminus \{x\}$ is open

Back to Manifolds: Topological Manifolds

Let $(M, \mathcal{T})$ be a topological space
Suppose M is Hausdorff and second-countable

<u>Technical reasons</u>:
1) Partition of unity
2) Compact exhaustion
3) Existence of a Riemannian metric
4) Embedding theorems.

<u>Def</u>: $(M, \mathcal{T})$ is locally Euclidean of dim m if $\forall p \in M$, $\exists U$ neighbd of p and $\exists V \subseteq \mathbb{R}^m$ open and $\exists \phi : U \to V$ that is a homeomorphism



$$\phi(q) = (x_1, \dots, x_m)$$
$$= (x_1(q), \dots, x_m(q))$$

$$x_i : U \to \mathbb{R} \quad \underbrace{}_{\text{coordinates of } q \in U}$$

IMPORTANT TERMINOLOGY

(U, ϕ) is called a coordinate chart

U is called a coordinate neighbhd of P (coordinate set)

ϕ is called a coordinate map (coordinate system)

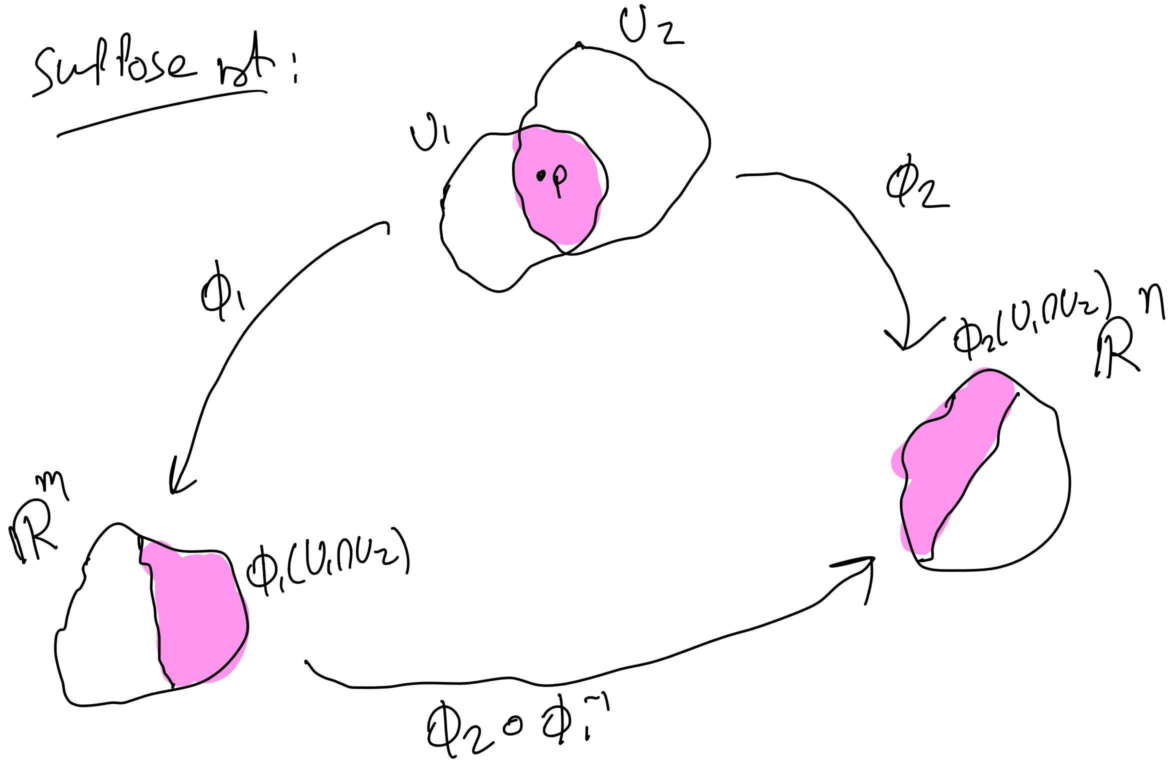
(equivalent to M is locally homeomorphic to $\mathbb{R}^m$)

$\boxed{\text{Def of Topological Manifold}}$

M is a topological manifold of dim m if it's a Hausdorff second countable topological space that is locally Euclidean of dim m.

Is the dimension well defined?

Suppose st:



$$\cancel{\Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1) \to \Phi_2(U_2)}$$

$$\Phi_2 \circ \Phi_1^{-1} : \overset{\subseteq \mathbb{R}^m}{\Phi_1(U_1 \cap U_2)} \longrightarrow \overset{\subseteq \mathbb{R}^n}{\Phi_2(U_1 \cap U_2)}$$

homeomorphism

Theorem: Invariance of dimension.

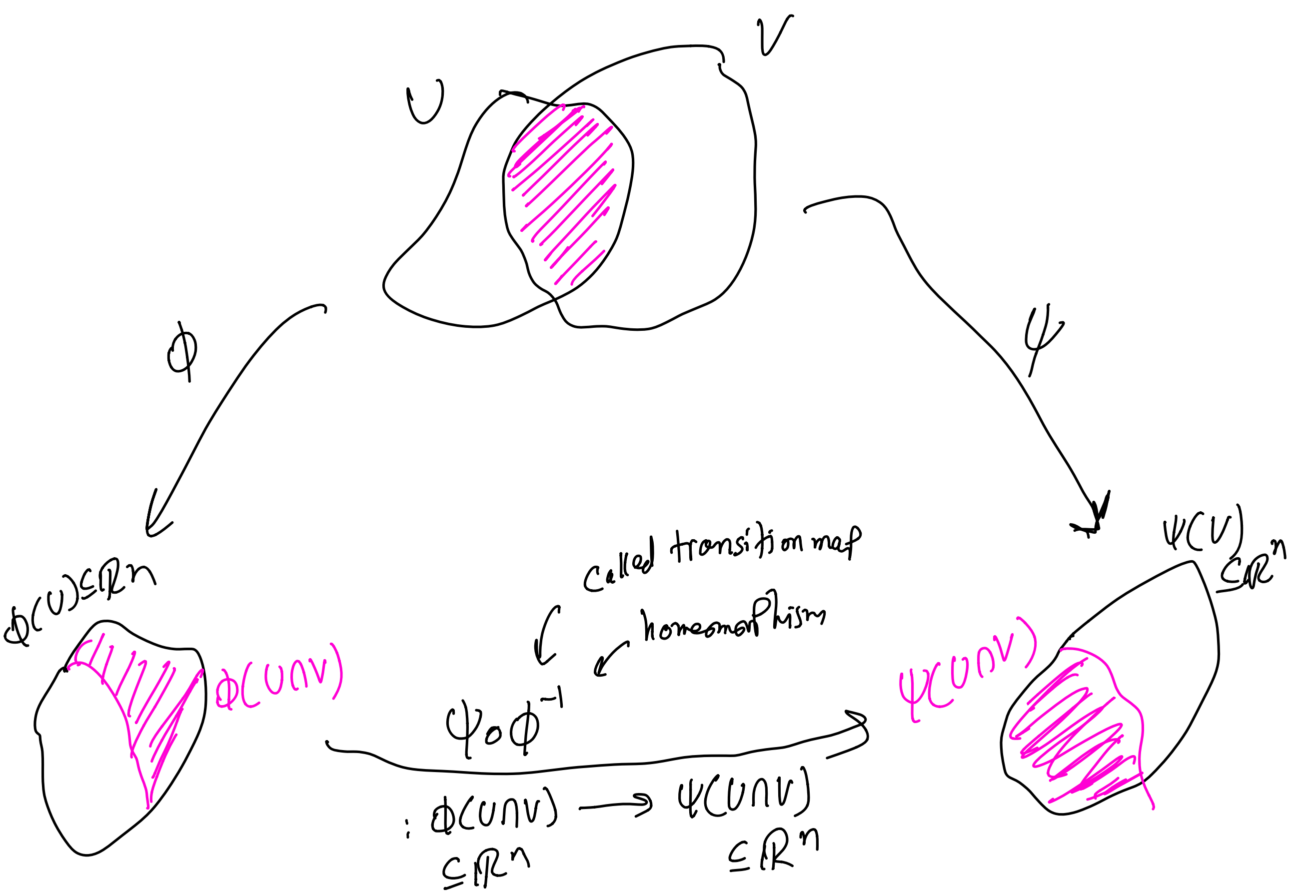
If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open sets that are homeomorphic, then $m = n$

Needs the tools of Algebraic Topology
(beyond the course)

Definition of smooth Manifold

Let M be a topological manifold of dim n.

Consider two coordinate charts
(U, ϕ) and (V, ψ)



Def: 2 charts (U, ϕ) and (V, ψ) are C^∞ compatible if the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

$$\left(\left.\begin{aligned} \psi \circ \phi^{-1} &: \phi(U \cap V) \to \psi(U \cap V) \\ \phi \circ \psi^{-1} &: \psi(U \cap V) \to \phi(U \cap V) \end{aligned}\right\} C^\infty \right)$$

Def: A C^∞ atlas is a collection of charts

$\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ covering the manifold

and are pairwise C^∞ compatible.

precisely:
1) $\bigcup_\alpha U_\alpha = M$
2) (U_α, ϕ_α) and (U_β, ϕ_β) are C^∞ compatible $\forall \alpha, \beta$

Question: Is C^∞ compatible an equivalence relation?

Reflexive: ✓

Symmetric : ✓

Transitive :

Suppose (U_1, ϕ_1) is compatible with (U_2, ϕ_2)
(U_2, ϕ_2) is compatible with (U_3, ϕ_3)

question: is (U_1, ϕ_1) C^∞ compatible with (U_3, ϕ_3)
more precisely:

is $\phi_3 \circ \phi_1^{-1} : \phi_1(U_{13}) \to \phi_3(U_{13})$
a diffeomorphism? $(U_{13} := U_1 \cap U_3)$

Akira's Answer:

where is this true? it's only true on $\phi_1(U_{123})$

$$\phi_3 \circ \phi_1^{-1} = \underbrace{(\phi_3 \circ \phi_2^{-1})}_{\text{diffeom}} \circ \underbrace{(\phi_2 \circ \phi_1^{-1})}_{\text{diffeo}}$$

$\Rightarrow$ $\phi_3 \circ \phi_1^{-1}$ is a diffeomorphism.

all we can say $\phi_3 \circ \phi_1^{-1} : \phi_1(U_{123}) \to \phi_3(U_{123})$
is a diffeom.

Not transitive.

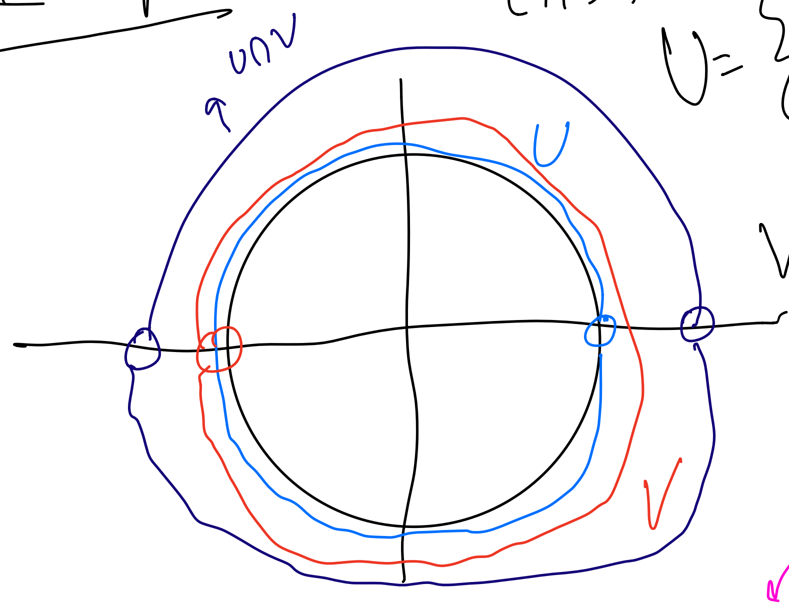
Example:

$(-1,0) \in U$ $\quad$ $(-1,0) \notin V$

$(1,0) \notin U$ $\quad$ $(-1,0) \in V$

$U = \{(\cos\theta, \sin\theta) : \theta \in (0, 2\pi)\}$

$V = \{(\cos\theta, \sin\theta) : \theta \in (-\pi, \pi)\}$



$\theta \in (0, 2\pi)$

$$\phi : U \to (0, 2\pi), \quad (\cos\theta, \sin\theta) \mapsto \theta$$

$$\psi : V \to (-\pi, \pi), \quad (\cos\theta, \sin\theta) \mapsto \theta$$

$\theta \in (-\pi, \pi)$

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$

exc: show it's a diffeomorphism

This makes $\{(U, \phi), (V, \psi)\}$ a C^∞ atlas for S^1.

$$\boxed{\text{Def of a smooth manifold}}$$

A smooth manifold of dim n is a topological manifold of dim n equipped with a C^∞ atlas.

The C^∞ atlas gives the topological manifold a smooth structure making it a smooth manifold.