* Assignment 6 is posted
* mistake in Proof of Frobenius Thm lecture
* mistake in the action of differential k-form on $\mathfrak{X}(M)$

Any k-form $\omega \in \Omega^k(M)$ defines a map $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ which is C^∞-multilinear $\boxed{\text{and alternating}}$

Any C^∞ multilinear map $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ that is alternating defines a smooth k-form.

* Practice with k-covectors, differential k-forms on $\mathbb{R}^n$ by doing Problems in Section 3 and 4.

<u>Summary of the last lecture</u> :

* We defined a k-form $\omega : M \to \Lambda^k(T^*M)$ as a section over $\Lambda^k(T^*M)$

$$: p \mapsto (p, \underbrace{\omega_p}_{\in \Lambda^k(T_p^*M)})$$

* We defined smoothness in 3 equivalent ways

We defined the space of smooth k-forms

$$\left\{ \begin{array}{l} \omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M) \\ \omega \text{ is } C^\infty\text{-multilinear} \\ \quad \text{and alternating} \end{array} \right\} = \Omega^k(M) = \left\{ \text{smooth sections of } \Lambda^k(T^*M) \right\}$$

⊛ $\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$

making $\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ an associative and anticommutative graded algebra

⊛ On a Chart, $\left\{ dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k} \;\middle|\; I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\} \right\}$ in ↗ order

makes a basis of $\Omega^k(U)$ wrt the module structure.

Pull back of k-forms

Let $F : N \to M$ be C^∞ map

⊛ We can pull back of 0-forms :

for $f \in \Omega^0(M) = C^\infty(M)$, $F^*(f) = f \circ F \in \Omega^0(N) = C^\infty(N)$

so $F^* : \Omega^0(M) \to \Omega^0(N)$

⊛ We can pull back covectors :

$F^{*,p} : T^*_{F(p)} M \to T^*_p N$

$: \theta \to \theta \circ F_{*,p}$

and so we can pull back 1-forms :

for $w \in \Omega^1(M)$, define $F^* w : N \longrightarrow T^* N$

$: p \longmapsto (p, F^{*,p}(w_{F(p)}) = w_{F(p)} \circ F_{*,p})$

Furthermore, $F^* : \Omega^1(M) \to \Omega^1(N)$.

⊛ Similarly we define pull back of k-covectors (alternating k tensor)

$$F^{*,p} : \Lambda^k(T^*_{F(p)}M) \longrightarrow \Lambda^k(T^*_pN)$$
$$: \theta \longmapsto F^{*,p}(\theta) : (v_1, \dots, v_k) \mapsto \theta(F_{*,p}v_1, \dots, F_{*,p}v_k)$$

which has the property: for $\theta_1 \in \Lambda^k(T^*_{F(p)}M)$, $\theta_2 \in \Lambda^l(T^*_{F(p)}M)$,

$$F^{*,p}(\theta_1 \wedge \theta_2) = F^{*,p}\theta_1 \wedge F^{*,p}\theta_2 \in \Lambda^{k+l}(T^*_pN)$$

So we can pull back k-forms:

for $\omega \in \Omega^k(M)$, define $F^*\omega : N \longrightarrow \Lambda^k(T^*N)$
$$: p \mapsto (p, F^{*,p}(\omega_{F(p)}))$$

which has the property: for $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$,

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

On a coordinate chart:

$$
\begin{aligned}
F^*(\omega) &= F^*(a_I \, dx^I) \\
&= F^*(a_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&= F^*a_I \wedge F^*dx^{i_1} \wedge \cdots \wedge F^*dx^{i_k} \\
&= \underbrace{a_I \circ F}_{\in C^\infty(U)} \; \underbrace{\underbrace{d(x^{i_1} \circ F)}_{\in \Omega^1(U)} \wedge \cdots \wedge \underbrace{d(x^{i_k} \circ F)}_{\in \Omega^1(U)}}_{\Omega^1(U)} \Bigg\} = \frac{\partial(f^{i_1}, \dots, f^{i_k})}{\partial(x^{j_1}, \dots, x^{j_k})} dx^{j_1} \wedge \cdots \wedge dx^{j_k}
\end{aligned}
$$

smooth

$\Rightarrow \quad F^*(\omega) \in \Omega^k(N)$

and $\quad F^* : \Omega^k(M) \to \Omega^k(N)$

Properties of Pullback :

1) F^* is $\mathbb{R}$-linear

(for any $a \in \mathbb{R}$, k-forms η and ω , $F^*(a\omega+\eta) = aF^*\omega + F^*\eta$)

2) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$

3) If $\omega \in \Omega^k(M)$, $F^*\omega \in \Omega^k(N)$

Exterior derivative

↳ Generalization of

$d : \Omega^0(M) \longrightarrow \Omega^1(M)$

We wish to extend this definition to $\Omega^k(M)$

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

Proposition : $\exists!$ extension of d (exterior derivative)

($\exists!$ collection of linear maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$)

s.t. The following properties hold :

1) for $f \in \Omega^0(M)$, df is the differential of f defined earlier.

2) $\forall f \in C^\infty(M)$,

$$d(f\omega) = df \wedge \omega + f\, d\omega$$

3) $\underbrace{\text{for any } F: N \to M, \quad F^* \circ d = d \circ F^*}_{d^2 = 0}$

The objects of study in calculus are functions

1) $f: \mathbb{R} \to \mathbb{R}$, smooth $\omega = f\,dx \in \Omega^1(\mathbb{R})$

2) $\underbrace{\frac{d}{dx} : f \mapsto f'}$ $\quad d: f \mapsto df$

3) $\int_a^b : f \mapsto \int_a^b f \quad \int_{[a,b]} : f\,dx \mapsto \int_{[a,b]} f\,dx$

4) FTC: $\int_a^b f' = f(b) - f(a)$

$$\int_{[a,b]} df = \int_{\partial [a,b]} f$$

The objects of study in the theory of manifolds are differential forms

1) $\omega \in \Omega^k(M)$

2) $d : \Omega^k(M) \to \Omega^{k+1}(M)$

3) $\int_M : \omega \mapsto \int_M \omega$

4) Stokes Theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

$$\Phi : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^n(\mathbb{R}^n)$$

$$: f \longmapsto f\, dx^1 \wedge \cdots \wedge dx^n$$

Post Lecture Practice Questions

1) do Problems in section 3 and 4

2) Let $A : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$ be C^∞-multilinear that is alternating.

Define $\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R}$

$: \omega_p(v_1, \dots, v_k) \longmapsto A(X_1, \dots, X_k)(p)$ for any $X_i \in \mathfrak{X}(M)$ s.t. $X_i|_p = v_i$ for $i = 1, \dots, k$.

Show that ω_p is well defined (independent of the extensions X_i)

This defines a k-form ω s.t. by its action on $\mathfrak{X}(M)$:

$$\omega(X_1, \dots, X_k) = A(X_1, \dots, X_k) \quad \forall X_1, \dots, X_k \in \mathfrak{X}(M)$$

Since $A(X_1, \dots, X_k) \in C^\infty(M)$, ω is smooth.

This shows every C^∞-multilinear alternating map is a smooth k-form.

3) Let $F : N \to M$ be a smooth map and $(U, \phi = (x^1, \dots, x^n))$, $(V, \psi = (y^1, \dots, y^m))$ be charts on N and M s.t. $F(U) \subseteq V$

Show

$\Omega^n(U) = \{ f\, dx^1 \wedge \cdots \wedge dx^n \mid f \in C^\infty(U) \}$ and

$\Omega^m(V) \{ g\, dy^1 \wedge \cdots \wedge dy^m \mid g \in C^\infty(V) \}$

Find $f \in C^\infty(U)$ s.t. $F^*(dy^1 \wedge \cdots \wedge dy^m)$

$= f\, dx^1 \wedge \cdots \wedge dx^n$

4) solve Problem 19.10, 19.12

we already did this