Summary of last lecture:

let $\omega : M \to T^*M$ be a 1-form. The following are equivalent.

1) ω is C^∞ as a section.
2) for any chart (U, ϕ), $\omega = a_i dx^i$ where $a_i \in C^\infty(U)$
3) By its action on $\mathfrak{X}(M)$, $\omega(X) \in C^\infty(M)$ whenever $X \in \mathfrak{X}(M)$

$$(\omega : \mathfrak{X}(M) \to C^\infty(M))$$

modules over $C^\infty(M)$ (V.S. over $\mathbb{R}$)

Also, $\Phi : \left\{ \begin{array}{c} \text{smooth sections} \\ \omega : M \to T^*M \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \omega : \mathfrak{X}(M) \to C^\infty(M) \\ \omega \text{ is } C^\infty\text{-linear} \end{array} \right\}$

$$\omega : M \to T^*M \longmapsto \omega : \mathfrak{X}(M) \to C^\infty(M)$$
$$: X \longmapsto (\omega(X) : p \mapsto \omega_p(X_p))$$

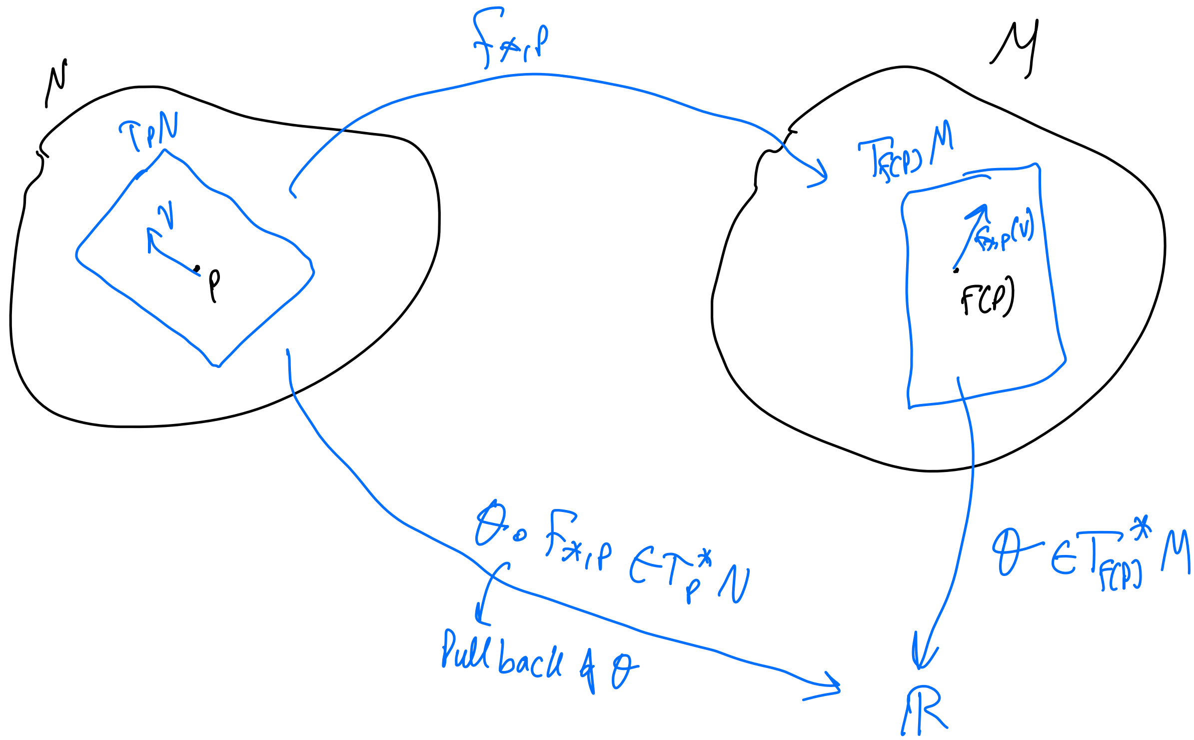
is an isomorphism on modules

We denote the space of C^∞ 1-forms by $\Omega^1(M)$

Pull back of 1-forms

Let $F : N \to M$ be a C^∞ map.

Recall $F_{*,p} : T_pN \longrightarrow T_{F(p)} M$



$$F^{*,p} : T_{F(p)}^* M \longrightarrow T_p^* N$$
$$: \theta \longmapsto \theta \circ F_{*,p}$$

Recall that vector fields cannot always be pushed forward
However, 1-forms can always be pulled back

Let ω be a 1-form on M
Then the pull back of ω is a 1-form on N defined by

$$F^*\omega : N \longrightarrow T^*N$$
$$: p \longmapsto (F^*\omega)_p := F^{*,p}(\omega_{F(p)})$$
$$= \omega_{F(p)} \circ F_{*,p}$$

$$\begin{tikzcd} \mathfrak{X}(N) \arrow[r, dashed, "?"] \arrow[rr, bend right, "F^*\omega"'] & \mathfrak{X}(M) \arrow[r, "\omega"] & \text{space of function} \end{tikzcd}$$

by its action on $\mathfrak{X}(N)$, $F^*\omega$ satisfies:

for $X \in \mathfrak{X}(N)$, $F^*\omega(X)(p) = (F^*\omega)_p(X_p)$

$$= \omega_{F(p)} \circ F_{*,p}(X_p)$$

If F is a diffeomorphism, then $F^*\omega(X) = \omega(F_*(X))$

push forward vector field

So we can write: $F^*\omega = \omega \circ F_*$

<u>Proposition</u>: 1) $F^*(g\omega) = F^*(g)\, F^*\omega$

$$= g \circ F\, F^*\omega$$

2) If $g \in C^\infty(M)$, $F^*(dg)$

$$= d(F^*g)$$

Proof:

$$dg(F_{*,p}(v)) = F_{*,p}(v)(g) = v(g\circ F) = d(g\circ F)(v)$$

$$\parallel \qquad\qquad\qquad = d(F^*g)(v)$$

$$F^*(dg)(v) \qquad\qquad = d(g\circ F)$$

3) If $\omega \in \Omega^1(M)$, then $\omega \in \Omega^1(N)$

and $F^*: \Omega^1(M) \to \Omega^1(N)$

and is $\mathbb{R}$-linear

(linear wrt the v.s. of $\Omega^1(M)$ and $\Omega^1(N)$)

Exc: Prove it.

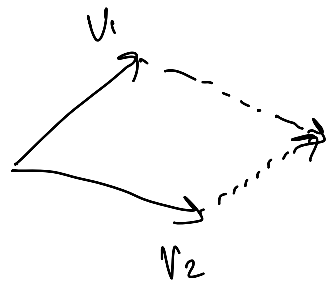
Alternating tensors

Let V be a vectorspace

Def: a k-tensor on V is a map

$$T: \underbrace{V\times\cdots\times V}_{k} \longrightarrow \mathbb{R} \quad \text{that is multilinear.}$$

We are interested in tensors that gives us a notion of a signed k-dimensional volume meter.



make sense of the signed 2-dim volume of $\{tv_1 + sv_2 \mid t,s \in [0,1]\} =: P_{v_1,v_2} \subseteq V$

<u>Example</u>: If $V = \mathbb{R}^3$, an example of signed 2-dim volume meter

$$: \underset{\in \mathbb{R}^3 \times \mathbb{R}^3}{(v_1, v_2)} \longmapsto \operatorname{sgn} |v_1 \times v_2| = \det[v_1\ v_2\ v_1\times v_2]$$

$$\hookrightarrow \frac{\det[v_1\ v_2\ v_1\times v_2]}{|\det[v_1\ v_2\ v_1\times v_2]|}$$

What should a signed k-dim volume meter satisfy?

1) $f : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$

$: (v_1, \dots, v_k) \longmapsto$ the signed k-dim volume of $P_{v_1,\dots,v_k}$

2) f is multilinear

3) $f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$
(alternating)

Any signed k-dim volume meter should satisfy 1–3.
Any such map f gives us a notion of a signed k-dim volume meter.

Def: An alternating k-tensor (k-covector) is a k-tensor on V that is alternating.

Proposition: Let f be a k-tensor. The following are equivalent:

1) f is alternating
2) $f(v_1, \dots, v_k) = 0$ whenever $v_i = v_j$ for $i \neq j$
3) $f(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k$ are linearly dependent
4) for any $\sigma \in S_k$,

$$\sigma f(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$
$$= \operatorname{sgn}(\sigma) f(v_1, \dots, v_k)$$

Example: $\theta^1, \theta^2 \in V^*$

define $\theta^1 \wedge \theta^2 : (v_1, v_2) \mapsto \theta_1(v_1)\theta_2(v_2) - \theta_1(v_2)\theta_2(v_1)$

Then $\theta_1 \wedge \theta_2$ is an alternating 2-tensor

Denote by $T_k(V)$ the v.s. of k-tensors
" $A_k(V)$ the v.s. of alternating k-tensor
" $S_k(V)$ the v.s. of symmetric k-tensors

Define the projection operator :

$$\mathrm{Sym} : T_k(V) \longrightarrow S_k(V)$$
$$: f \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \sigma f$$

Then $\mathrm{Sym}(f) = f$ iff f is symmetric

Similarly
$$\mathrm{Alt} : T_k(V) \longrightarrow A_k(V)$$
$$: f \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \sigma f$$

then $\mathrm{Alt}(f) = f$ iff f is alternating

We define $\otimes : T_k(V) \times T_\ell(V) \longrightarrow T_{k+\ell}(V)$

for $f \in T_k(V)$, $g \in T_\ell(V)$, $f \otimes g \in T_{k+\ell}(V)$

defined by
$$f \otimes g(v_1, \dots, v_k, w_1, \dots, w_\ell) = f(v_1, \dots, v_k)\, g(w_1, \dots, w_\ell)$$

(associative : $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, so we can just write $f \otimes g \otimes h$)

If $f \in A_k(V)$, $g \in A_\ell(V)$, then is $f \otimes g \in A_{k+\ell}(V)$?

We define $f \wedge g \in A_{k+\ell}(V)$ by

$$f \wedge g = \frac{(k+l)!}{k!\,l!} \operatorname{Alt}(f \otimes g) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}\sigma \; \sigma(f \otimes g)$$

$(*)$

Note that when $\underset{c}{f} \in A_0(V) := \mathbb{R}$, $c \wedge g := cg$)

Properties of $\wedge : A_k(V) \times A_l(V) \longrightarrow A_{k+l}(V)$)

#1) Bilinear

#2) Let $\{e_1, \dots, e_n\}$ be a basis for V and $\{\alpha^1, \dots, \alpha^n\}$ be the dual basis of $V^* = T_1(V) = A_1(V)$

$(\alpha^i(e_j) = \delta^i_j)$

Define for any $I \subseteq \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ s.t. $1 \leq i_1 < \dots < i_k \leq n$ the unique alternating k-tensor α^I satisfying:

$$\alpha^I(e_J) = \delta^I_J \qquad \text{for any multindex } J \text{ in } \uparrow \text{order.}$$

Then $\{\alpha^I : \text{multindex } I \text{ in } \uparrow \text{order}\}$ makes a basis for $A_k(V)$

and so $\dim(A_k(V)) = \binom{n}{k}$

#3) $\alpha^I \wedge \alpha^J = \alpha^{IJ}$

Thanks to the mysterious constant *

#4) Associativity:

$$(f \wedge g) \wedge h = f \wedge (g \wedge h)$$

so we can write $f \wedge g \wedge h$.

Hint:

$$(\alpha^I \wedge \alpha^J) \wedge \alpha^K = \alpha^{IJ} \wedge \alpha^K = \alpha^{IJK}$$
$$= \alpha^I \wedge \alpha^{JK}$$
$$= \alpha^I \wedge (\alpha^J \wedge \alpha^K)$$

$$\Rightarrow \boxed{\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}}$$

#5) Anticommutativity, $f \wedge g = (-1)^{lk} g \wedge f$

Hint: on basis elements: $\alpha^I \wedge \alpha^J = \alpha^{IJ}$

$$= \operatorname{sgn}(\sigma)\, \alpha^{JI}$$
$$= (-1)^{lk}\, \alpha^{JI}$$
$$= (-1)^{lk}\, \alpha^J \wedge \alpha^I$$

$i_1, \dots, i_k, j_1, \dots, j_l$

(σ sends $IJ \to JI$)

#6) For any $\theta^1, \dots, \theta^k \in V^*$ and any $v_1, \dots, v_k \in V$,

$$\theta^1 \wedge \cdots \wedge \theta^k (v_1, \dots, v_k) = \det\left(\theta^i(v_j)\right)$$

#7) The wedge product is the unique bilinear, associative, and anticommutative product $A_k(V) \times A_l(V) \to A_{k+l}(V)$ satisfying $\alpha^I = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$.

!!

We denote $A_k(V)$ by $\Lambda^k(V^*)$

$$= \text{span}\{\alpha^{i_1} \wedge \dots \wedge \alpha^{i_k} \mid 1 \le i_1 < \dots < i_k \le n\}$$

Define $\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$

<u>Def</u>: An algebra A over $\mathbb{R}$ is graded if it can be written as a direct sum $A = \bigoplus_{k=0}^{\infty} A^k$

where A^k are V.S. over $\mathbb{R}$ s.t. the multiplication map sends $A^k \times A^l \longrightarrow A^{k+l}$

A graded algebra is anticommutative if it satisfies $ab = (-1)^{lk} ba \quad \forall a \in A^k, b \in A^l$.

Then the wedge product make $\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$ an associative anticommutative graded algebra of dim 2^n.

Differential forms

Let M be a smooth manifold & let $p \in M$ and (U, ϕ) be a chart near p.

Then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is the coordinate basis for T_pM

and $\left\{ dx^1_p, \dots, dx^n_p \right\}$ is the coordinate dual basis

So $\left\{ dx^{i_1}_p \wedge \dots \wedge dx^{i_k}_p \;\middle|\; 1 \le i_1 < \dots < i_k \le n \right\}$ is a basis of

$\Lambda^k(T_p^* M)$.

Define $\Lambda^k(T^*M) := \bigcup_{p \in M} \Lambda^k(T_p^* M)$

The bundle of alternating k-tensors

which comes with the map $\pi : \Lambda^k(T^*M) \longrightarrow M$

$: (p, \omega_p) \longmapsto p$, $\omega_p \in \Lambda^k(T_p^* M)$

Excercise: Equip $\Lambda^k(T^*M)$ with a topology and a smooth structure (just like for TM and T^*M)

for a chart (U, ϕ), define

$$\tilde{\phi} : \Lambda^k(T^*U) \longrightarrow \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$
$$: (p, \omega_p) \longmapsto (\phi(p), \{c_I(p, \omega_p)\}_I)$$

$$\text{s.t. } \omega_p = \sum_I c_I(p, \omega_p) \underbrace{dx^I_p}_{dx^{i_1}_p \wedge \cdots \wedge dx^{i_k}_p}$$

<u>Def</u>: A section of $\Lambda^k(T^*M)$ is a map

$$\omega : M \longrightarrow \Lambda^k(T^*M)$$
$$p \longmapsto (p, \omega_p) \quad , \omega_p \in \Lambda^k(T^*_pM)$$

so $\pi \circ \omega = Id_M$

Sections of $\Lambda^k(T^*M)$ are called differential k-forms

<u>Example</u>: Let (U, ϕ) be a chart

define $dx^I : U \to \Lambda^k(T^*U)$

$$: p \longmapsto (p, dx^I_p = dx^{i_1}_p \wedge \cdots \wedge dx^{i_k}_p)$$

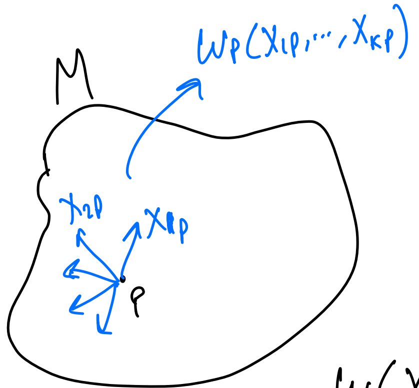
so dx^I is a differential k-form on U.

We extend the wedge product: Let ω and η be a k-form and a l-form on M, we define the $k+l$ form

$$\omega \wedge \eta : M \longrightarrow \Lambda^{k+l}(T^*M)$$
$$: p \longmapsto (p, (\omega \wedge \eta)_p := \omega_p \wedge \eta_p)$$

And so since $dx^I_p = dx^{i_1}_p \wedge \cdots \wedge dx^{i_k}_p \quad \forall p \in U$

then $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$



We define the action of k-forms on $\mathfrak{X}(M)$:

For $X_1, \cdots, X_k \in \mathfrak{X}(M)$, we define the function

$$\omega(X_1, \cdots, X_k) : M \longrightarrow \mathbb{R}$$
$$: p \longmapsto \omega_p(X_{1p}, \cdots, X_{kp})$$

The action of k-forms on $\mathfrak{X}(M)$ satisfies:

for $g \in C^\infty(M)$, $X_1, \cdots, X_k \in \mathfrak{X}(M)$,

$$\omega(X_1, \cdots, gX_i, \cdots, X_k) = g\, \omega(X_1, \cdots, X_k)$$

so ω is C^∞ multilinear and alternating!

$$\omega(X_1, \cdots, X_i, \cdots, X_j, \cdots, X_k) = -\omega(X_1, \cdots, X_j, \cdots, X_i, \cdots, X_k)$$

Def: A differential k-form is C^∞ if it's smooth as a section.

Proposition: Let ω be a k-form. The following are equivalent.

1) ω is C^∞ as a section.
2) on any chart, $\omega = c_I dx^I$ on U where $c_I \in C^\infty(U)$

3) By it's action on $\mathfrak{X}(M)$,

$\omega(X_1, \dots, X_k) \in C^\infty(M)$ whenever $X_1, \dots, X_k \in \mathfrak{X}(M)$

$$\left(\text{so } \omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M) \right)$$

modules over $C^\infty(M)$ (V.S. over $\mathbb{R}$)

Also,

$(*)$

$$\Phi : \left\{ \begin{array}{l} \text{smooth sections} \\ \text{over } \Lambda^k(T^*M) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M) \\ \omega \text{ is } C^\infty\text{-multilinear} \\ \text{and alternating} \end{array} \right\}$$

$$: \omega : M \to \Lambda^k(T^*M) \longmapsto \omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

is an isomorphism of modules

$(*)$

In particular, every C^∞-multilinear map and alternating $A : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ coincides with the action of a unique smooth differential k-form

We denote by $\Omega^k(M)$ the space of smooth differential k-forms on M. This is a module over $C^\infty(M)$ and V.S. over $\mathbb{R}$.

Example,

$f^1, \dots, f^k \in C^\infty(M)$

Then $df^1, \dots, df^k \in \Omega^1(M)$

On a chart (U, ϕ), $df^1 \wedge \dots \wedge df^k = c_I \, dx^I$

Apply $\frac{\partial}{\partial x^I}\Big|_p \Rightarrow$

$$df^1_p \wedge \dots \wedge df^k_p \left(\frac{\partial}{\partial x^I}\Big|_p \right) = c_I(p)$$

$$\det\left(df^i_p \left(\frac{\partial}{\partial x^{i_j}}\Big|_p \right) \right)$$

$$= \det\left(\frac{\partial f^i}{\partial x^{i_j}}\Big|_p \right)$$

$$=: \frac{\partial(f^1, \dots, f^k)}{\partial(x^{i_1}, \dots, x^{i_k})} \in C^\infty(U)$$

$\Rightarrow df^1 \wedge \dots \wedge df^k \in \Omega^k(M)$

(smooth)

If $k=n$, $df^1 \wedge \cdots \wedge df^n = \frac{\partial(f^1,\ldots,f^n)}{\partial(x^1,\ldots,x^n)} dx^1 \wedge \cdots \wedge dx^n$

In fact, if $(df^1 \wedge \cdots \wedge df^n)_p \neq 0$, Then $\left.\frac{\partial(f^1,\ldots,f^n)}{\partial(x^1,\ldots,x^n)}\right|_p \neq 0$

$\Rightarrow (f^1,\ldots,f^n)$ make a coordinate map near p.

In general, for $w \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$

On a chart, $w \wedge \eta = (c_I dx^I) \wedge (b_J dx^J)$

$$= c_I b_J dx^I \wedge dx^J$$

$$= \underbrace{c_I b_J}_{\in C^\infty(U)} dx^{IJ}$$

$\Rightarrow w \wedge \eta \in \Omega^{k+l}(M)$

$\Rightarrow \wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$

Remark : $\Omega^0(M) = \left\{ w : p \underset{\text{smooth}}{\longmapsto} (p, w_p) \quad \in \Lambda^0(T_p^*M) =: \mathbb{R} \right\}$

$= C^\infty(M)$

and $f \in \Omega^0(M)$, $\omega \in \Omega^k(M)$,

$f \wedge \omega := f\omega$

Define the vectorspace $\Omega^*(M)$ of C^∞ differential forms on M to be the direct sum

$$\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$$

V.S. over $\mathbb{R}$

modules over $C^\infty(M)$

With the wedge product, $\Omega^*(M)$ becomes an associative anticommutative graded algebra.

Post-lecture Practice questions

1) Fill in all the gaps and do all exercises above.

2) Verify that the properties of wedge product on $\Lambda^k(T_p^*M)$ stay true on $\Omega^k(M)$.

3) Let $\omega \in \Omega^k(M)$ and let $U \subseteq M$ be an open set. Show that $\omega|_U : U \to \Lambda^k(T^*U)$

$: p \mapsto (p, \omega(p))$ is in $\Omega^k(U)$

and $\omega|_U : \mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) \longrightarrow C^\infty(U)$ satisfying

for $X_1, \cdots, X_k \in \mathfrak{X}(U)$, $\omega|_U (X_1, \cdots, X_k) = \omega(\tilde{X}_1, \cdots, \tilde{X}_k)\Big|_U$ for any extensions $\tilde{X}_1, \cdots, \tilde{X}_k \in \mathfrak{X}(M)$.

4) If (U, ϕ) is a chart and $\omega \in \Omega^k(U)$, then
$\omega = c_I \, dx^I$ where $c_I = \omega|_U \left(\frac{\partial}{\partial x^I}\right)$.

we sometime drop the "$|_U$" since it's implied.

5) Describe an element $\Omega^*(M)$. Can you describe how it acts on $\mathfrak{X}(M)$?
Is $\Omega^*(M)$ a module over $C^\infty(M)$?