(*) OH return to usual

Recall:

A 1-form ω is a section of T^*M

$$\omega: M \to T^*M$$
$$: p \mapsto (p, \omega_p), \text{ where } \omega_p \in T_p^*M$$

ω is smooth if it's smooth as a section.

Example #1: For $f \in C^\infty(M)$, we define the 1-form $df: M \to T^*M$

where $df_p \in T_p^*M$ defined by:

for $v \in T_pM$, $df_p(v) = v(f) \underset{\text{by Proposition}}{=} f_{*,p}(v) \ (\mathrm{Id})$

Proposition: The following diagram commutes.

$$\begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[dr, "f_{*,p}"'] & \mathbb{R} \arrow[d, "\phi"] \\ & T_{f(p)}\mathbb{R} \end{tikzcd}$$

where $\phi(c) = c \frac{d}{dx}\Big|_{x=f(p)}$

Under the identification $T_{f(p)}\mathbb{R} \cong \mathbb{R}$,

we have that $f_{*,p}$ "=" df_p

Both are called the differential of f at p.

Example #2: Let (U, ϕ) be a coordinate chart.

Then $x^i \in C^\infty(U)$ and so dx^i is a 1-form satisfying

$$dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \delta^i_j$$

$\Rightarrow$ $\{dx^1_p, \dots, dx^n_p\}$ is the dual basis of

$\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ called the coordinate dual basis.

Spoiler: Just like the coordinate vector fields $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ on U make a basis of $\mathfrak{X}(U)$ wrt module structure, we also have $\{dx^1, \dots, dx^n\}$ make a basis of the space of smooth 1-forms on U wrt the module structure.

Then for any $f \in C^\infty(U)$,

$$df_p = a_i \, dx^i_p \quad \text{for } a^i \in \mathbb{R}.$$

Apply $\frac{\partial}{\partial x^j}\Big|_p$ to both sides:

$$\frac{\partial f}{\partial x^j}\Big|_p = df_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = a_i \, dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = a_j$$

$$\Rightarrow \quad df = \frac{\partial f}{\partial x^i} dx^i \quad \text{on } U$$

In a different coordinate chart $(U, \psi = (y^1, \dots, y^n))$ then $df = \frac{\partial f}{\partial y^i} dy^i$

Note that,

$$\frac{\partial f}{\partial y^i}\, dy^i = \frac{\partial f}{\partial x^j} \underbrace{\left[\frac{\partial x^j}{\partial y^i} \frac{\partial y^i}{\partial x^k} \right]}_{\delta^j_k} dx^k$$

$$= \frac{\partial f}{\partial x^i}\, dx^i$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^i}$$

$$\boxed{\nabla f \overset{?}{=} \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}}$$

$$\frac{\partial f}{\partial x^i} \qquad \frac{\partial}{\partial x^i}$$

$$\boxed{df = \frac{\partial f}{\partial x^i}\, dx^i}$$

∇f is the vector field satisfying

$$\langle \nabla f_p , v \rangle = Df_p(v) \quad \forall v \in T_pM$$

Notice on $\mathbb{R}^n$, $\nabla f \neq \frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{\partial f}{\partial \phi}\frac{\partial}{\partial \phi}$

Recall:

Smoothness criterion lemma for V.F.

Let X be a section over TM. The following are equivalent:

1) $X : M \to TM$ is smooth
2) for any (U, ϕ) coordinate chart, $X = a^i \frac{\partial}{\partial x^i}$ on U where $a^i \in C^\infty(U)$

Smoothness criterion lemma for 1-forms :

Let $\omega : M \to T^*M$ be a 1-form. Then the following are equivalent

1) ω is smooth as a section
2) On any chart (U, ϕ), $\omega = a_i \, dx^i$ where $a_i \in C^\infty(U)$

(In Particular, dx^i are smooth 1 forms on U)

Exc

Recall what we did for vector fields ;

#1) We defined the action of a vector field on $C^\infty(M)$,
for $f \in C^\infty(M)$, $X(f) : M \to \mathbb{R}$
$: p \mapsto X(f)(p) := X_p(f)$

#2) Second Smoothness criterion :
A section $X : M \to TM$ is smooth iff $X(f) \in C^\infty(M)$
whenever $f \in C^\infty(M)$.
$[X : C^\infty(M) \to C^\infty(M)]$

#3) ⊛ Any vector field defines a derivation on $C^\infty(M)$,
$X : C^\infty(M) \to C^\infty(M)$ is a derivation

⊛ Every derivation on $C^\infty(M)$ coincides with the action of a unique smooth vector field on $C^\infty(M)$

#6)
$$\Gamma(M) \cong Der(C^\infty(M))$$

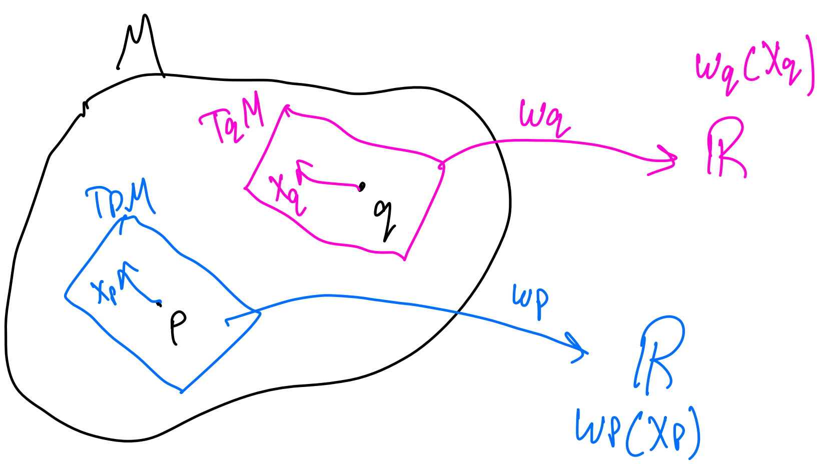
$$\left\{\begin{array}{l}\text{smooth sections}\\ X : M \to TM\end{array}\right\} \cong \left\{ \text{Derivations } X : C^\infty(M) \to C^\infty(M) \right\}$$

isomorphic wrt the module structure

We then denoted the space of smooth vector fields by $\mathfrak{X}(M) = \Gamma(M) = Der(C^\infty(M))$

We will do the same for 1-forms:

#7) Let ω be a 1-form. Let $X \in \mathfrak{X}(M)$



$$p \longmapsto \omega_p(X_p)$$

We define the action of 1-forms on $\mathfrak{X}(M)$,
For $X \in \mathfrak{X}(M)$, define $\omega(X) : M \longrightarrow \mathbb{R}$
$: p \longmapsto \omega_p(X_p)$

Proposition: Let ω be a 1-form. Then the action of ω on $\mathfrak{X}(M)$ is C^∞-linear

Proof: For $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$

Then
$$\omega(fX+Y)(p) = \omega_p\left(f(p)X_p + Y_p\right)$$
$$= f(p)\,\omega_p(X_p) + \omega_p(Y_p)$$
$$= \left[f\,\omega(X) + \omega(Y)\right](p)$$

#2) 2nd Smoothness criterion for 1-forms:

ω is C^∞ as a section over T^*M iff $\omega(X) \in C^\infty(M)$ whenever $X \in \mathfrak{X}(M)$.

Exc $\quad \left(\omega : \mathfrak{X}(M) \to C^\infty(M)\right)$

#3) (*) From above, we know that any smooth 1-form ω defines a C^∞-linear map $\omega : \mathfrak{X}(M) \to C^\infty(M)$.

Thm: Let $A : \mathfrak{X}(M) \to C^\infty(M)$ be a C^∞-linear map, Then $\exists!$ smooth 1-form ω s.t. the action of ω on $\mathfrak{X}(M)$ coincides with A.

Proof: The fact that A is C^∞-linear will imply that $A(X)(p)$ only depends on X_p.

proof

Then we can define $\omega_p \in T_p^*M$ by
for $v \in T_pM$, $\omega_p(v) := A(X)(p)$ where $X \in \mathfrak{X}(M)$ s.t. $X_p = v$

Let $X, Y \in \mathfrak{X}(M)$ s.t. $X_p = Y_p$.
WTS $A(X)(p) = A(Y)(p) \iff A(X-Y)(p) = 0$

It suffices to show that whenever $Z \in \mathfrak{X}(M)$, $Z_p = 0$
$$A(Z)(p) = 0$$

Ex
(Hint: bump functions)

Then the definition of ω_p is well defined and defines a 1-form
$$\omega : M \to T^*M$$
$$: p \mapsto (p, \omega_p)$$
and it satisfies for $X \in \mathfrak{X}(M)$, $\omega(X)(p) = \omega_p(X_p)$
$$= A(X)(p)$$

$$\Rightarrow \omega(X) = A(X)$$

Since $\omega(X) = A(X) \in C^\infty(M)$ for every $X \in \mathfrak{X}(M)$, ω is smooth

$\blacksquare$

#4)

$\omega: M \to T^*M \longmapsto \omega: \mathfrak{X}(M) \to C^\infty(M)$ is iso

(*) $\{$ smooth sections of $T^*M \}$ $\overset{\text{Id}}{=}$ (*) $\left\{ \omega: \mathfrak{X}(M) \to C^\infty(M) \;\middle|\; \omega \text{ is } C^\infty\text{-linear} \right\}$

modules over $C^\infty(M)$

for $\omega_1, \omega_2 \in$ (*), $f \in C^\infty(M)$,

define: $(\omega_1 + \omega_2)(p) := (p, \omega_{1p} + \omega_{2p})$

$(f\omega_1)(p) := (p, f(p)\omega_{1p})$

show $\omega_1 + \omega_2$, $f\omega \in$ (*)

for $\omega_1, \omega_2 \in$ (*), $f \in C^\infty(M)$

define

$(\omega_1 + \omega_2)(X) := \omega_1(X) + \omega_2(X)$

$(f\omega_1)(X) = f\omega_1(X)$

show $\omega_1 + \omega_2$, $f\omega_1 \in$ (*)

Show that it makes them modules over $C^\infty(M)$

Denote by $\Omega^1(M)$ the space of smooth 1-forms (either (*) or (*))

$\Omega^1(M)$ is a module over $C^\infty(M)$ & v.s. over $\mathbb{R}$

Post lecture Practice questions

1) Do the exercises above.

2) Let $f \in C^\infty(M)$ and let $(U, \phi = (x^1, \dots, x^n))$ and $(U, \psi = (y^1, \dots, y^n))$ be charts on M.

Define the vector field $X = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$ on U.
Write X wrt the basis $\left\{ \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \right\}$
and show it's not necessarily equal to $\sum_{i=1}^{n} \frac{\partial f}{\partial y^i} \frac{\partial}{\partial y^i}$.

3) Let $F: N \to M$ be a smooth map and consider charts $(U, \phi = (x^1, \dots, x^n))$ and $(V, \psi = (y^1, \dots, y^m))$ on N and M respectively.
Show that

$$F^* dy^i = \frac{\partial F^i}{\partial x^j} dx^j$$

where F^* is the pull back

4) What's wrong with this argument:

Let $A: \mathfrak{X}(M) \to C^\infty(M)$ be a C^∞-linear map and let $Z \in \mathfrak{X}(M)$ s.t. $Z_p = 0$.
Let (U, ϕ) be a chart near p, and so $Z = a^i \frac{\partial}{\partial x^i}$ where $a^i(p) = 0$.

Then $A(Z)(P) = A\left(a^i \frac{\partial}{\partial x^i}\right)(P)$

$= a^i(P)\, A\left(\frac{\partial}{\partial x^i}\right)(P)$ since A is C^∞-linear

$= 0$

5) If $w \in \Omega^1(M)$ and $U \subseteq M$ is open, then $w|_U \in \Omega^1(U)$ where $w|_U$ is defined by

$$w|_U : U \longrightarrow T^*U$$
$$: P \longmapsto (P, w_P)$$

6) Show that $\{$ smooth sections $w : M \to T^*M \}$ and

$\{ w : \mathfrak{X}(M) \to C^\infty(M) \mid w \text{ is } C^\infty\text{-linear} \}$ are modules over $C^\infty(M)$

and $\Phi : (w : M \to TM^*) \longrightarrow (w : \mathfrak{X}(M) \to C^\infty(M))$

is an module isomorphism.

7) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x,y,z) = (x^3 e^{yz}, \sin x)$

Let $g \in C^\infty(\mathbb{R}^2)$ be the function $g(u,v) = uv$

Compute dg, $F^*(dg)$, F^*g, and $d(F^*g)$

Verify that $F^*(dg) = d(F^*g)$

8)

let $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$

which is an integral curve of $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

Let $\beta = dx - dy \in \Omega^1(\mathbb{R}^2)$

Compute $\beta(X)$, $\gamma^*\beta$, $\gamma^*\beta\left(\frac{d}{dt}\right)$, $\gamma^*(\beta(X))$

Verify that $\gamma^*\beta\left(\frac{d}{dt}\right) = \gamma^*(\beta(X))$