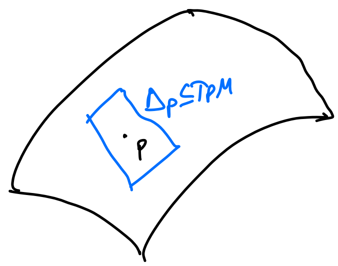
(*) Assignment 5 is posted & is due July 23

Let Δ be a smooth rank k distribution



It is smooth in the sense that

$\forall p \in U, \exists X_1, \cdots, X_k \in \mathfrak{X}(U)$ s.t.

$\Delta_q = \text{span}\{X_{1q}, \cdots, X_{kq}\} \quad \forall q \in U.$

(i.e. $\exists$ local frame of Δ near p) $\uparrow$

For $p \in M$, is there a submanifold S containing p s.t.

$$T_qS = \Delta_q \quad \forall q \in S\,?$$

<u>Frobenius Thm</u> A smooth rank k distribution Δ is integrable iff it's involutive.

($\Rightarrow$) by 2g. If S is an integral submanifold containing p and $X_1, \cdots, X_k$ is a local frame of Δ near p on U

Then $X_1, \cdots, X_k$ are tangent to S and "span TS"

$(T_qS = \text{span}\{X_{1q}, \cdots, X_{kq}\} \ \forall q \in S \cap U)$

And so $[X_i, X_j] = \sum_{l=1}^{k} c_{ij}^{l} X_l$ where $c_{ij}^{l} \in C^\infty(U)$

$\Rightarrow \Delta$ is involutive

($\Leftarrow$) Suppose Δ is involutive

Outline of the proof:

Step 1: Let $X_1, \dots, X_k$ be a local frame of Δ on U near p.

So we know $[X_i, X_j] = \sum_{l=1}^{k} c_{ij}^{l} X_l$ where $c_{ij}^{l} \in C^\infty(U)$

Then $\exists$ nhbd $\tilde{U} \subseteq U$ of p and another local frame $Y_1, \dots, Y_k$ of Δ s.t. $[Y_i, Y_j] = 0$

(question on Assignment 6)

we will show that $\exists$

Step 2: We define an embedding $A : (t_1, \dots, t_k) \longmapsto A(t_1, \dots, t_k) \in M$ with the property that
1) $A(0, \dots, 0) = p$
2) $\dfrac{\partial A}{\partial t^i} = Y_i$

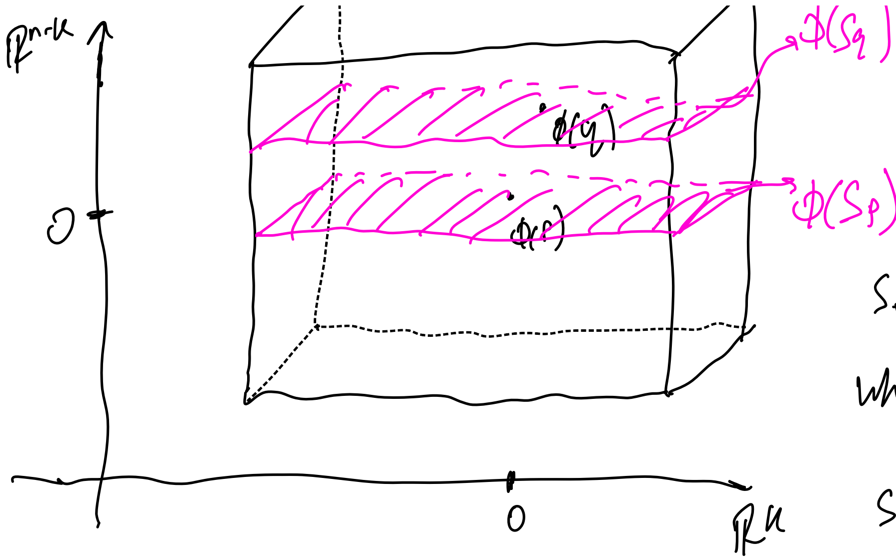
Because A is an embedding, the image of A is a submanifold with the property that the tangent space of the submanifold is spanned by Y_i & so it's an integral submanifold containing p.

Step 3: We will show $\exists$ a chart $(\underset{\subseteq \tilde{U}}{\tilde{\tilde{U}}}, \phi = (x^1, \dots, x^n))$ near p s.t.

$$\frac{\partial}{\partial x^i} = Y_i$$

If so, then

suppose $\phi(\tilde{\tilde{U}})$ is a cube



let $\phi(p) = (p^1, \dots, p^n)$

$S_p := \{x^{k+1} = p^{k+1}, \dots, x^n = p^n\}$

which is a submanifold containing p

So $\underset{Y_1}{\frac{\partial}{\partial x^1}}, \dots, \underset{Y_k}{\frac{\partial}{\partial x^k}}$ make a smooth frame for S_p

$\Rightarrow S_p$ is an integral submanifold of Δ containing p.

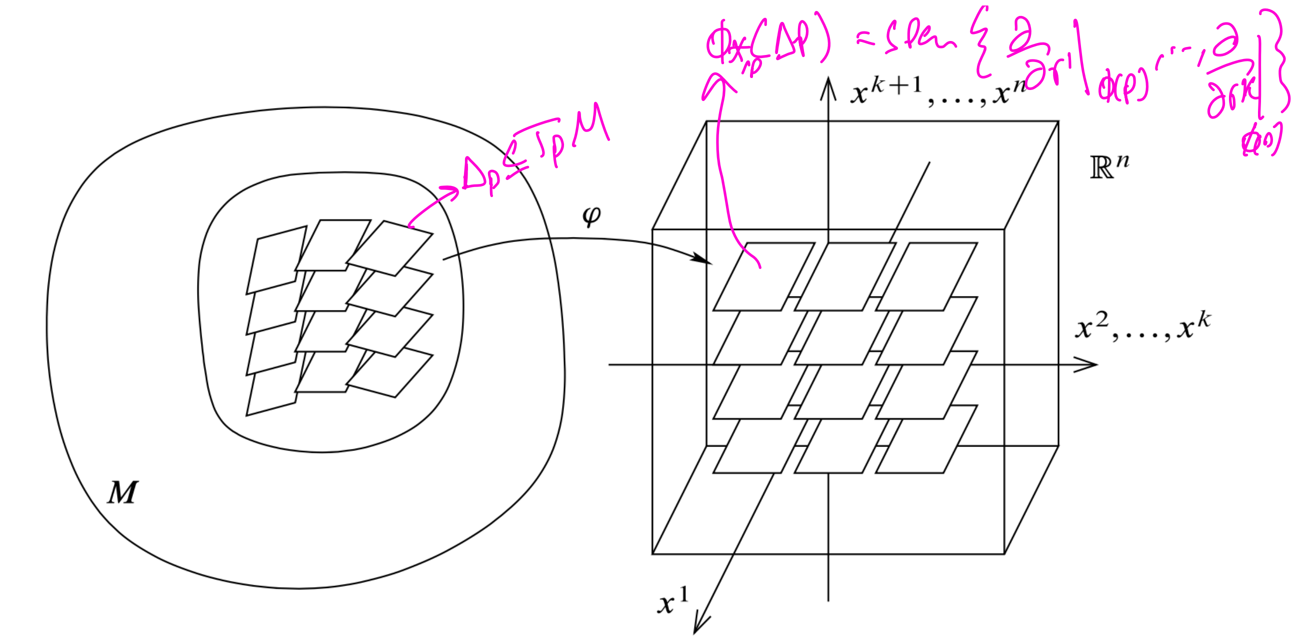
In fact, for any $q \in \tilde{U}$ with $\phi(q) = (q^1, \dots, q^n)$

$S_q := \{x^{k+1} = q^{k+1}, \dots, x^n = q^n\}$ is an integral submanifold of Δ containing q.

We say that the coordinate chart $(\tilde{U}, \phi)$ is flat for Δ

<u>Def</u>: D is completely integrable if $\exists$ flat chart near every point.

Completely integrable $\Rightarrow$ integrable



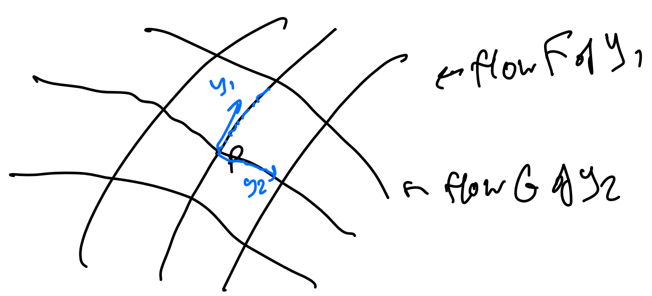
Frobenius Thm (strengthened)

completely integrable $\Rightarrow$ integrable $\Rightarrow$ involutive $\Rightarrow$ completely integrable

proof:

Step2: Let $Y_1, \dots, Y_n$ be a local frame of Δ on U near p s.t. $[Y_i, Y_j] = 0$ for $1 \le i, j \le k$

For simplicity, suppose $k = 2$



Let $\varepsilon > 0$ and $\tilde{U} \subseteq U$ be a nbhd of p s.t.

$$F, G : (-\varepsilon, \varepsilon) \times \tilde{U} \longrightarrow M$$

Define
$$A : (-\varepsilon, \varepsilon)^2 \longrightarrow M$$
$$: (t, s) \longmapsto F_t \circ G_s(p) = G_s \circ F_t(p)$$

Fix (t_0, s_0), define $\left.\frac{\partial A}{\partial t}\right|_{(t_0,s_0)} := A_{*,(t_0,s_0)}\left(\left.\frac{\partial}{\partial t}\right|_{(t_0,s_0)}\right)$

$\left\{\left.\frac{\partial}{\partial t}\right|_{(t_0,s_0)}, \left.\frac{\partial}{\partial s}\right|_{(t_0,s_0)}\right\}$ is basis for $T_{(t_0,s_0)}\mathbb{R}^2$

or equivalently, it's the velocity vector of the curve
$$t \longmapsto A(t, s_0) \quad \text{at } t = t_0.$$

$$\left.\frac{\partial A}{\partial t}\right|_{(t_0,s_0)} = \left.\frac{d}{dt}\right|_{t=t_0} F_t \circ G_{s_0}(p) = Y_{1\,A(t_0,s_0)}$$

similarly
$$\left.\frac{\partial A}{\partial s}\right|_{(t_0,s_0)} = \left.\frac{d}{ds}\right|_{s=s_0} G_s \circ F_{t_0}(p) = Y_{2\,A(t_0,s_0)}$$

$:= A_{*,(t_0,s_0)}\left(\left.\frac{\partial}{\partial s}\right|_{(t_0,s_0)}\right)$

Since Y_1 and Y_2 are linearly independent, A is an immersion and so A is locally an embedding. So $\exists\, \varepsilon' < \varepsilon$ s.t. $A(-\varepsilon', \varepsilon')^2$ is a submanifold of M containing p with the property that $T_q S = \operatorname{span}\left\{\frac{\partial A}{\partial t}, \frac{\partial A}{\partial s}\right\} = \operatorname{span}\{Y_{1q}, Y_{2q}\}$

& so S is an integral submanifold of Δ containing p.

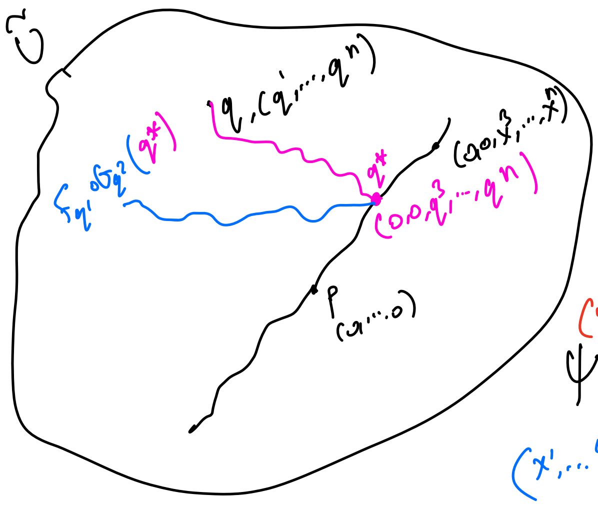
Step 3: Recall that Y_1, Y_2 is a local commuting frame of Δ near P on U.

WTS $\exists$ chart near P s.t. $\frac{\partial}{\partial x^i} = Y_i$ for $i = 1, 2$.

Let $(\tilde{U}, \phi)$ be a chart near P s.t. $\phi(P) = 0$

$$\left.\frac{\partial}{\partial x^1}\right|_P = Y_{1P}$$

$$\left.\frac{\partial}{\partial x^2}\right|_P = Y_{2P}$$



Define $\psi : \cancel{\tilde{U}}\ V \subseteq \mathbb{R}^n \to \cancel{\mathbb{R}^n}\ \tilde{U}$ by

$$\psi(\cancel{q}\ (y^1,\dots,y^n)) = F_{\cancel{x^1}\, y^1} \circ G_{\cancel{x^2}\, y^2}\left(\phi^{-1}(0,0,\cancel{x^3}\, y^3,\dots,\cancel{x^n}\, y^n)\right) \quad (*)$$

$(x^1, \dots, x^n)$

~~$\psi_{*,P}\left(\left.\frac{\partial}{\partial x^1}\right|_P\right) = \left.\frac{d}{dx^1}\right|_{x^1=0} F_{x^1} \circ G_0 \underbrace{\left(\phi^{-1}(0,0,0,\dots,0)\right)}_{P}$~~

~~$= Y_{1P} = \left.\frac{\partial}{\partial x^1}\right|_P$~~

~~$\psi_{*,P}\left(\left.\frac{\partial}{\partial x^2}\right|_P\right) = \left.\frac{d}{dx^2}\right|_{x^2=0} G_{x^2} \circ F_0\left(\phi^{-1}(0,\dots,0)\right)$~~

~~$= Y_{2P} = \left.\frac{\partial}{\partial x^2}\right|_P$~~

~~similarly $\psi_{*,P}\left(\left.\frac{\partial}{\partial x^i}\right|_P\right) = \left.\frac{\partial}{\partial x^i}\right|_P$ for $i > 2$~~

Then By the inverse function Theorem, Ψ is a local diffeomorphism on a nbd $\tilde{\tilde{U}} \subseteq \tilde{U}$ of p and so defines a chart near p.

<u>WTS</u> Wrt the chart Ψ, the first two coordinate vector fields are Y_1 and Y_2.

for $q \in \tilde{\tilde{U}}$,

$$\frac{d}{dx^1}\Big|_{x^1=q^1} f_{x^1} \circ G_{q^2}\left(\Phi^{-1}(0,0,q^3,\cdots,q^n)\right)$$

fill Those gaps. Exc.

$$= Y_{1_{f_{q^1}\circ G_{q^2}(\Phi^{-1}(0,0,q^3,\cdots,q^n))}} = Y_{1q}$$

Similarly $$\frac{d}{dx^2}\Big|_{x^2=q^2} G_{x^2} \circ f_{q^1}\left(\Phi^{-1}(0,0,q^3,\cdots,q^n)\right)$$

$$= Y_{2q}$$

$\left\{\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n}\right\}$ is the coordinate basis wrt to Φ.

$\left\{Y_1, Y_2, \frac{\partial}{\partial x^3},\cdots,\frac{\partial}{\partial x^n}\right\}$ is the Coordinate basis wrt to Ψ.

$\Rightarrow$ $(\tilde{\tilde{U}}, \Psi)$ is flat for Δ near p

(after shrinking $\tilde{\tilde{U}}$ so that $\Psi(\tilde{\tilde{U}})$ is a cube).

Cotangent Bundle

Let M be a manifold and let $p \in M$.

Def $T_p^*M := \{ f : T_pM \to \mathbb{R} \mid f \text{ is linear} \}$ ← dual vectors ← covectors

Recall from linear Algebra, T_p^*M is also an n-dimensional vectorspace.

Let $\{v_1, \dots, v_n\}$ be a basis of T_pM.

Define $\theta^i \in T_p^*M$ by $\theta^i(v_j) = \delta^i_j$.

Then $\{\theta^1, \dots, \theta^n\}$ is a basis T_p^*M called dual basis of $\{v_1, \dots, v_n\}$

Define a covector field as a choice of covector in T_p^*M at each point p. $\quad$ (a map $p \mapsto \theta \in T_p^*M$)

Introducing Einstein notation:

#1) Vectors: v_i , Coefficient of vector: a^i

Covector: θ^i , Coefficient of covector: b_i

Fact

~~Idea~~ : If index i appears twice, one up and the other is down, and you are summing over the index i, then the object will be basis independent.

Example : let $v \in T_pM$ and $\theta \in T_p^*M$.
So $\underbrace{v = \sum_{i=1}^{n} a^i v_i}$ and $\underbrace{\theta = \sum_{i=1}^{n} b_i \theta^i}$

basis independent

#1) $\underbrace{\sum_{i=1}^{n} a^i b_i}_{\text{basis indefendent}}$

$\theta(v)$

#2) $\sum_{i=1}^{n} a^i \theta^i \in T_p^*M$ #3) $\sum_{i=1}^{n} b_i v_i \in T_pM$

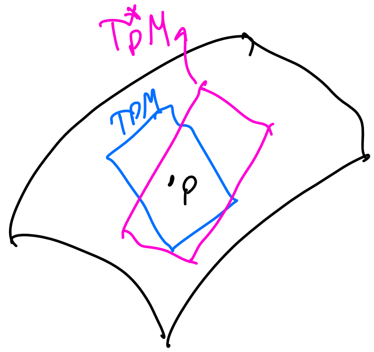
basis defendent

#2) If an index appears twice, one up and the other down, then the $\sum$ is dropped and assumed.

Why is $(x^1, \cdots, x^n)$ written with the index up?

$v \in T_pM$, $v = a^i v_i$ where $a^i = v([x^i])$

We want to make a choice of a covector at every point $p \in M$.



Def: define the cotangent bundle $T^*M := \bigcup_{p\in M} T_p^*M$

$$= \left\{ (p,\theta) : p\in M, \ \theta\in T_p^*M \right\}$$

we have the natural map $\pi : T^*M \to M$

$$: (p,\theta_p) \mapsto p \qquad \text{where } \theta_p\in T_p^*M$$

We define a topology on T^*M in the following way:

Let (U,ϕ) be a chart on M and define the bijective map

$$\tilde{\phi} : T^*U \to \phi(U)\times\mathbb{R}^n$$

$$: (p,\theta_p) \longmapsto (x^1,\dots,x^n, c_1,\dots,c_n) \quad \text{s.t.}$$

$$\theta = c_i\, dx^i{}_p$$

where $\{dx^1{}_p, \dots, dx^n{}_p\}$ is the dual basis of

$\longrightarrow \left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$

(called the dual basis)

Choose the unique topology on T^*U that makes $\tilde{\phi}$ a homeomorphism. (this topology is independent of the coordinate map ϕ)

Then define the following topology on T^*M:

$$\mathcal{T} = \left\{ A \subseteq T^*M : \begin{array}{l} A \cap T^*U \text{ is open in } T^*U \\ \text{for any chart } (U, \phi) \end{array} \right\}$$

show $\mathcal{T}$ is a topology on T^*M (Exc)

Also, the collection of charts $\{(T^*U, \tilde{\phi})\}$ is a C^∞ atlas on T^*M making it a $2n$ dimensional smooth manifold.

Define a section ω of T^*M as a map $\omega : M \to T^*M$ satisfying $\pi \circ \omega = Id_M$.

This is called a 1-form. (differential 1-form)

Def: A smooth 1-form ω is a C^∞ section of T^*M.

Why study dual spaces & 1-forms?

Examples of 1-forms:

Let $f \in C^\infty(M)$, define $df : M \to T^*M$ by

$$df(p) = (p, \underbrace{df_p}_{\in T_p^*M})$$

$$\left.\begin{aligned} df_p &: T_pM \to \mathbb{R} \\ &: v \mapsto v([f]) \end{aligned}\right\} \text{Check linear}$$

Recall that $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R}$
also describes how fast f changes.

Proposition: Let $f \in C^\infty(M)$. Then for $p \in M$, and $X_p \in T_pM$

$$f_{*,p}(X_p) = df_p(X_p) \left.\frac{d}{dx}\right|_{f(p)}$$

Proof: Let $Id : \mathbb{R} \to \mathbb{R}$. Let $a \in \mathbb{R}$ s.t.

$$f_{*,p}(X_p) = a \left.\frac{d}{dx}\right|_{f(p)}$$

$$\begin{aligned} a = a\left.\frac{d}{dx}\right|_{f(p)}(Id) &= f_{*,p}(X_p)(Id) \\ &= X_p(Id \circ f) \\ &= X_p(f) \\ &= df_p(X_p) \end{aligned}$$

since we identify $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ via

$$a \left.\frac{d}{dx}\right|_{f(p)} \longleftrightarrow a$$

We also call df the differential of f.

Post-lecture Practice Questions.

- Solve exercises above.
- For the proof of step 3, let $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ be the coordinate basis wrt ϕ and $\left\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right\}$ be the coordinate basis wrt ψ. Find the relation between them.
- If $f \in C^\infty(M)$, is df a <u>smooth</u> 1-form
- let $f \in C^\infty(\mathbb{R})$. how does df relate with the standard derivative f' from first year Calculus?
- on $\mathbb{R}^3$, define dx, dy, dz to be the differential of the projection maps. Show that at each point p, $\{dx_p, dy_p, dz_p\}$ is the dual basis of $\left\{\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p, \left.\frac{\partial}{\partial z}\right|_p\right\}$

- $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2+y^2$
  Write df interms of the standard dual basis $\{dx, dy, dz\}$

- Show that for any $f \in C^\infty(S^2)$, $df_p = 0$ for some $p \in S^2$.