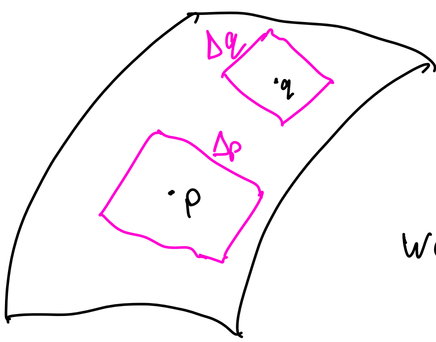
* Discussion tomorrow at 10 - 11
* Essay. for a maximum of 5% bonus (9-10)
* Both July 6 tutorial #1 and #2 are posted.



For $p \in M$, Chose a k-dim subspace $\Delta p \subseteq T_pM$

let $\Delta = \bigcup \Delta p \subseteq TM$

we call Δ a rank k distribution.

for $p \in M$, Pick a basis $\{X_1p, \dots, X_kp\}$ of Δp

This defines k vector fields $X_1, \dots, X_k$ defined as sections as follows

$$X_i : M \to TM$$
$$: p \mapsto X_ip$$

Remark: Any Choice of vector fields $X_1, \dots, X_k$ that are linearly independent at every point defines a rank k distribution Δ defined by $\Delta p = \{X_1p, \dots, X_kp\}$

Def: A rank k distribution Δ is smooth if $\exists\, X_1, \dots, X_k \in \cancel{\mathfrak{X}(M) \text{ s.t. } \Delta q = \operatorname{span}\{X_1q, \dots, X_kq\}}$

~~but then S^{2n} admits no smooth distribution.~~

A rank k distribution Δ is smooth if $\forall p \in M$, $\exists$ nbhd U of p and $X_1, \dots, X_k \in \mathfrak{X}(U)$ s.t. $\Delta_q = \text{span}\{X_{1q}, \dots, X_{kq}\}$ $\forall q \in U$.

Def: We say $X_1, \dots, X_k$ is a local frame of Δ near p.

Def: A section of Δ is a map $X : M \to \Delta \subseteq TM$ s.t. $\pi \circ X = Id$

If Δ is smooth, then we denote by $\Gamma(\Delta)$ the space of smooth sections of Δ $\left(\Gamma(\Delta) \subseteq \underbrace{\Gamma(M)}_{\mathfrak{X}(M)} \right)$

Question: Let Δ be a smooth rank k distribution.
For $p \in M$, does there exist a submanifold S containing p with property that:

$$T_q S = \Delta_q \quad \forall q \in S$$

Def: Such a submanifold is called an integral submanifold of Δ.

Def: A smooth rank k distribution Δ is integrable if for every $p \in M$, $\exists$ an integral submanifold of Δ containing p.

Ex 1: Let $k = 1$. The answer is always yes by the existence integral curve. (any rank 1 smooth distribution is integrable)

Let Δ be a smooth rank 1 distribution. Let $p \in M$.
Let X_1 be a local frame of Δ near p. ($\text{span}\{X_{1q}\} = \Delta_q$ $\forall q \in U$)
Then $\exists$ integral curve γ of X_1 starting at p.

Exc: The integral curve γ of X_1 defines an integral submanifold of Δ passing through p.

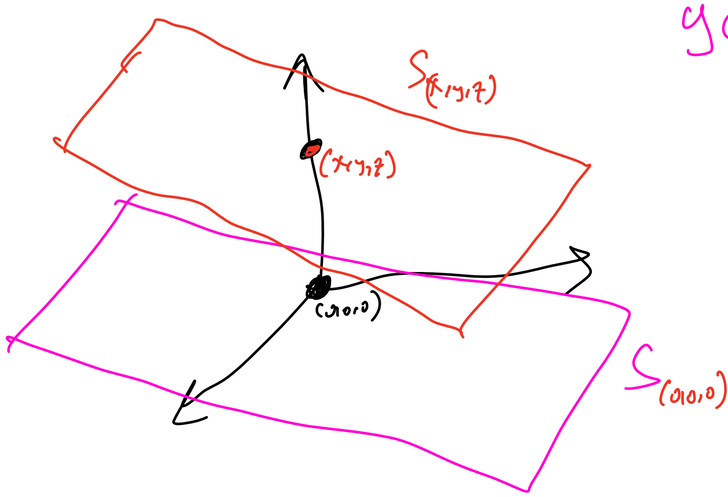
Ex 2 :

$$\left.\begin{aligned} X &= \frac{\partial}{\partial x} \\ Y &= \frac{\partial}{\partial y} \end{aligned}\right\} \in \mathfrak{X}(\mathbb{R}^3)$$

X and Y define a smooth rank 2 distribution Δ of $\mathbb{R}^3$.

Let $P = (0,0,0) \in \mathbb{R}^3$. Does there exist an integral surface of Δ containing P?

yes S = xy plane



(if rank of $\Delta = n = \dim(M)$
Then $\Delta = TM$)

Ex #2 :

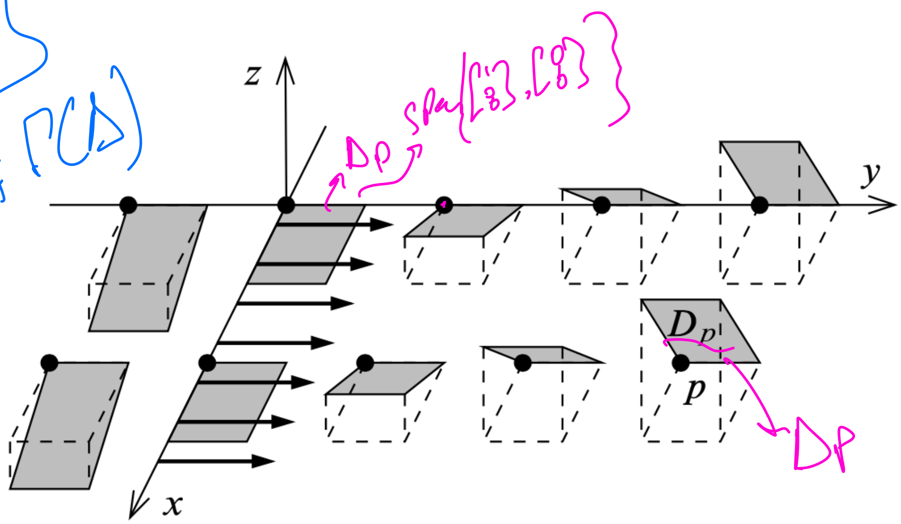
$$\left.\begin{aligned} X &= \frac{\partial}{\partial x} \\ Y &= \frac{\partial}{\partial y} + \cancel{x}\, y \frac{\partial}{\partial z} \end{aligned}\right\} \in \mathfrak{X}(\mathbb{R}^3)$$

X and Y define a smooth rank 2 distribution Δ of $\mathbb{R}^3$.

$$\Delta_{(x,y,z)} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} \right\}$$

Does there exist an integral surface passing through $(0,0,0)$?

$[X, Y] = \frac{\partial}{\partial z} \notin \Gamma(\Delta)$



No.

Let Δ be a smooth rank k distribution and fix a local frame $X_1, \dots, X_k$ of Δ near p.

Suppose Δ is integrable.

Let S be an integral submanifold of Δ containing p.

What does Assignment 4 Problem 2g say about $X_1, \dots, X_k$?

$$[X_i, X_j] \in \Gamma(\Delta)$$

or equivalently

$$[X_i, X_j] = \sum_{l=1}^{k} c_{ij}^{l} X_l \quad \text{for } c_{ij}^{l} \in C^\infty(U)$$

This is a necessary condition of Δ to be integrable.

In fact, it is also sufficient

<u>Def</u>: A smooth rank k distribution Δ is involutive if for every local frame $X_1, \dots, X_k$ of Δ,

$$[X_i, X_j] = \sum_{l=1}^{k} c_{ij}^{l} X_l \quad \text{for } c_{ij}^{l} \in C^\infty(U)$$

This is equivalent to saying that $\Gamma(\Delta)$ is a Lie subalgebra of $\Gamma(M)$.

<u>Meaning</u>: $\Gamma(\Delta)$ is a Lie algebra with the same Lie bracket defined on $\Gamma(M)$.

$$[\cdot,\cdot] : \Gamma(\Delta) \times \Gamma(\Delta) \to \Gamma(\Delta)$$

<u>Frobenius Theorem</u>: A smooth rank k distribution is integrable iff it's involutive.

($\Rightarrow$) by 2g.

($\Leftarrow$) Nontrivial.