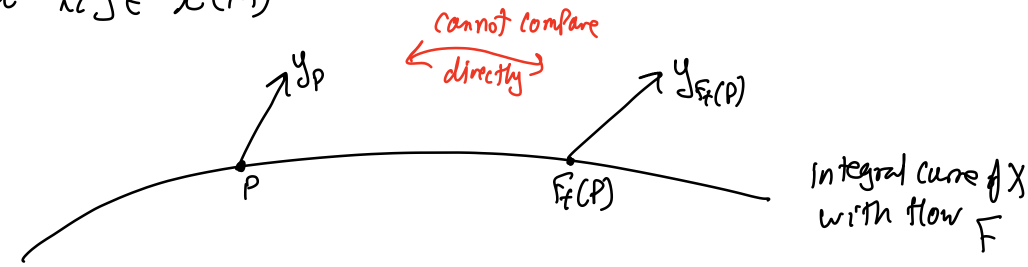
1) OH

2) Edited Assignment 4
Problem 2c, 4d

3) Tutorial (2 videos)

We want a notion of how fast a vector field changes in the direction of another vector field.

Let $X, Y \in \mathfrak{X}(M)$



Fix a coordinate system $(U, \phi = (x^1, \dots, x^n))$

Then $Y_{F_t(p)} = \sum_{i=1}^{n} c^i(t) \left.\frac{\partial}{\partial x^i}\right|_{F_t(p)}$, $\left[Y_{F_t(p)}\right] = \begin{bmatrix} c^1(t) \\ \vdots \\ c^n(t) \end{bmatrix}$

We can define $D_X Y|_p := \sum_{i=1}^{n} c^{i\,\prime}(0) \left.\frac{\partial}{\partial x^i}\right|_p$

(check that this is coordinate dependent exc) on $\mathbb{R}^n$, we have a prefered global coordinate system and so all $T_p\mathbb{R}^n$ are identified with $\mathbb{R}^n$

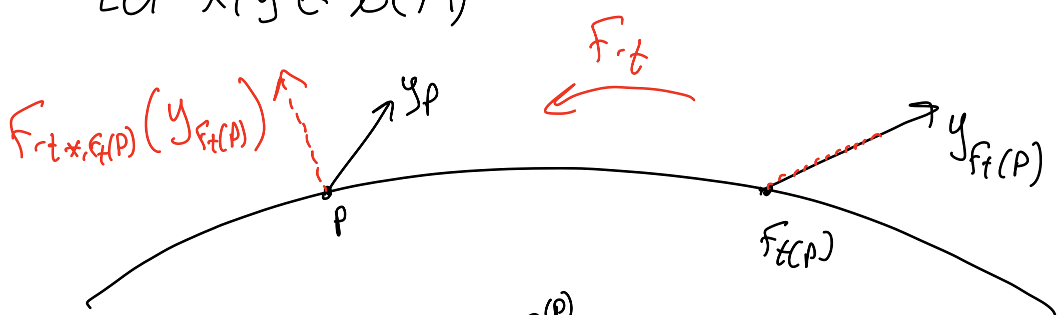
$$T_p\mathbb{R}^n \text{ "="} \mathbb{R}^n$$ ← old def of Tangent space

$$\sum v^i \frac{\partial}{\partial x^i}\Big|_p \text{ "="} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

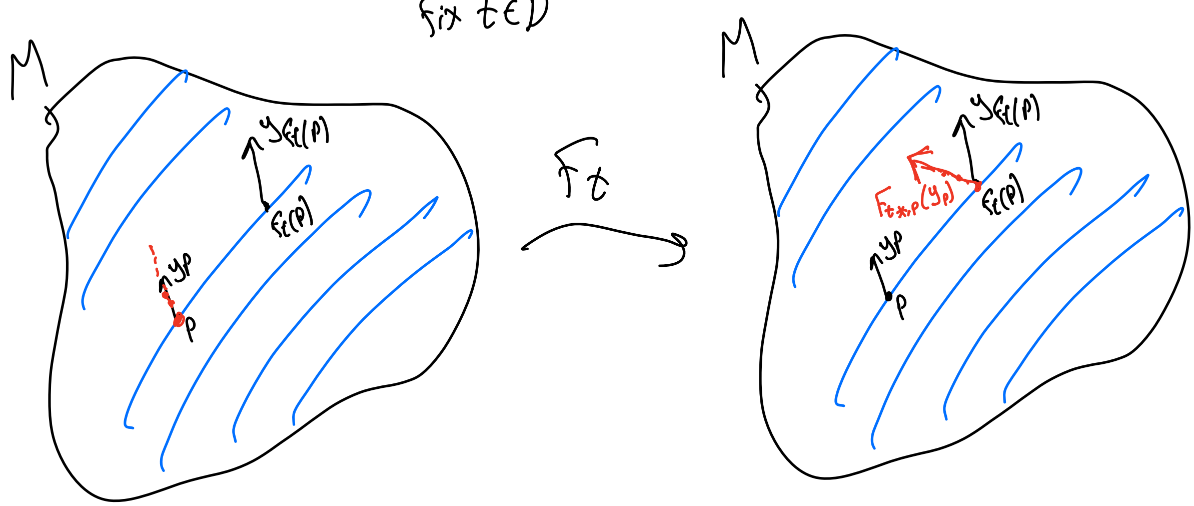
So we can compare directly $y_{F_t(p)}$ with y_p and we can define

$$D_X y\big|_p = \lim_{t\to 0} \frac{y_{F_t(p)} - y_p}{t} = \sum c^{i\,\prime}(0) \frac{\partial}{\partial x^i}\Big|_p$$

Let $X, y \in \mathfrak{X}(M)$



Fix $t \in D^{(p)}$



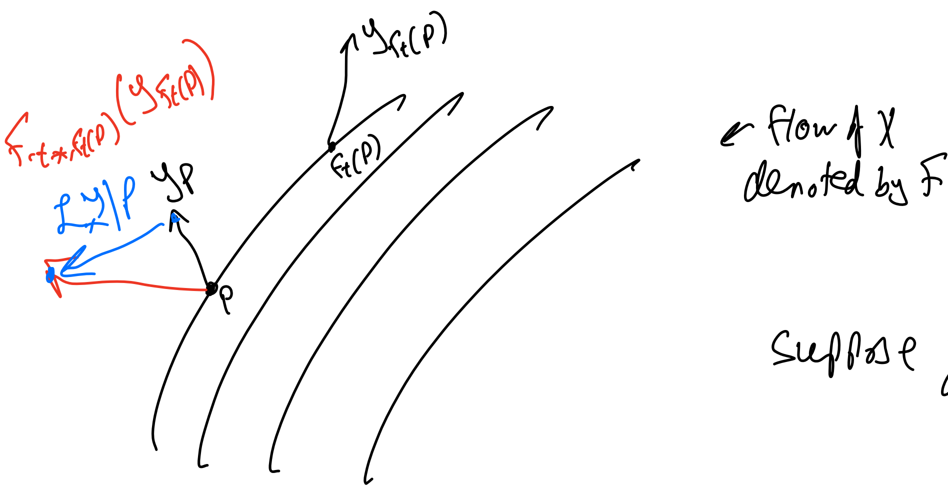
Def: The Lie derivative of Y in the direction of X at p is

$$\mathcal{L}_X Y\Big|_p := \lim_{t\to 0} \frac{F_{-t*,F_t(p)}\left(Y_{F_t(p)}\right) - Y_p}{t}$$

Proposition: 1) limit always exist (excercise)
2) $\mathcal{L}_X Y \in \mathfrak{X}(M)$

What does $\mathcal{L}_X Y \equiv 0$ mean

Let $X, Y \in \mathfrak{X}(M)$



suppose $\mathcal{L}_X Y \equiv 0$

Let $H(t) := F_{-t*,F_t(p)}\left(Y_{F_t(p)}\right)$

$\left(H : \mathcal{D}^{(p)} \longrightarrow T_pM \right)$

note

$$H'(0) = \mathcal{L}_X y|_p = 0$$

$$H'(t_0) = \frac{d}{dt}\Big|_{t=t_0} F_{-t+t_0-t_0\,*,F_{t-t_0+t_0}(p)}\left(y_{F_{t-t_0+t_0}(p)}\right)$$

let $q := F_{t_0}(p)$

$$= \frac{d}{dt}\Big|_{t=t_0} \widetilde{F_{-t_0 *,q}} \circ F_{-t+t_0 *, F_{t-t_0}(q)}\left(y_{F_{t-t_0}(q)}\right)$$

$$= F_{-t_0 *,q} \frac{d}{dt}\Big|_{s=0} F_{-s *,F_s(q)}\left(y_{F_s(q)}\right)$$

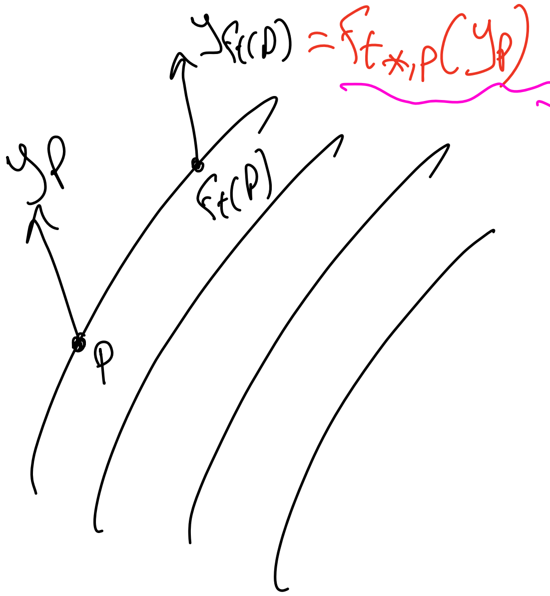
$$= F_{-t_0 *,q}\left(\mathcal{L}_X y\Big|_{F_{t_0}(p)}\right) = 0$$

$\Rightarrow$ H is constant.

$\Rightarrow$ $F_{-t *,F_t(p)}\left(y_{F_t(p)}\right) = y_p \quad \forall t \in D^{(p)}$

$\Rightarrow$ Apply $F_{t *,p}$ to both sides: $\left[\left(F_{t *,p}\right)^{-1} = F_{-t *F_t(p)}\right]$

$$\boxed{y_{F_t(p)} = F_{t *,p}\left(y_p\right)}$$



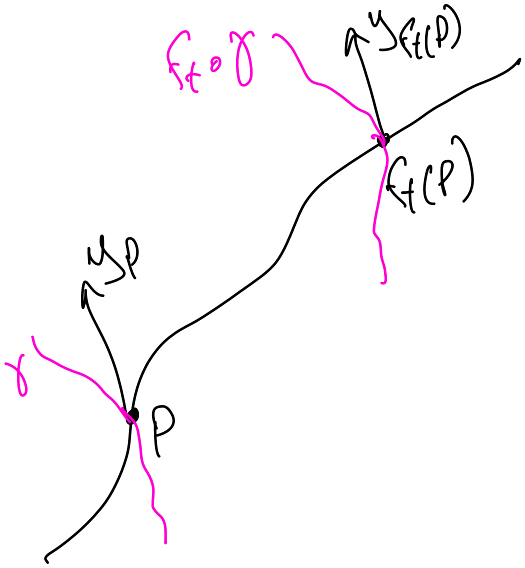
$\rightsquigarrow$ only depends on X and Y_p

Y does not change along the flow of X

<u>Def</u>: We say that Y is invariant under the flow of X

if $\quad f_{t*,p}(Y_p) = Y_{f_t(p)} \quad \forall p \in M, \ \forall t \in D^{(p)}$

$$\boxed{\mathcal{L}_X Y \equiv 0 \implies Y \text{ is invariant under the flow of } X}$$



$$\begin{aligned} f_t \circ \gamma'(0) &= f_{t*,p}(\gamma'(0)) \\ &= f_{t*,p}(Y_p) \\ &= Y_{f_t(p)} \end{aligned}$$

Choose γ to be the unique integral curve of $\mathcal{Y}$ starting at p.

Then if $\mathcal{Y}$ is invariant under the flow of X,

$$\underline{F_t \circ \gamma'(s)} = F_{t*,\gamma(s)}(\gamma'(s))$$
$$= F_{t*,\gamma(s)}(\mathcal{Y}_{\gamma(s)})$$
$$= \underline{\mathcal{Y}_{F_t(\gamma(s))}}$$

$\Rightarrow F_t \circ \gamma$ is an integral curve of starting at $F_t(p)$

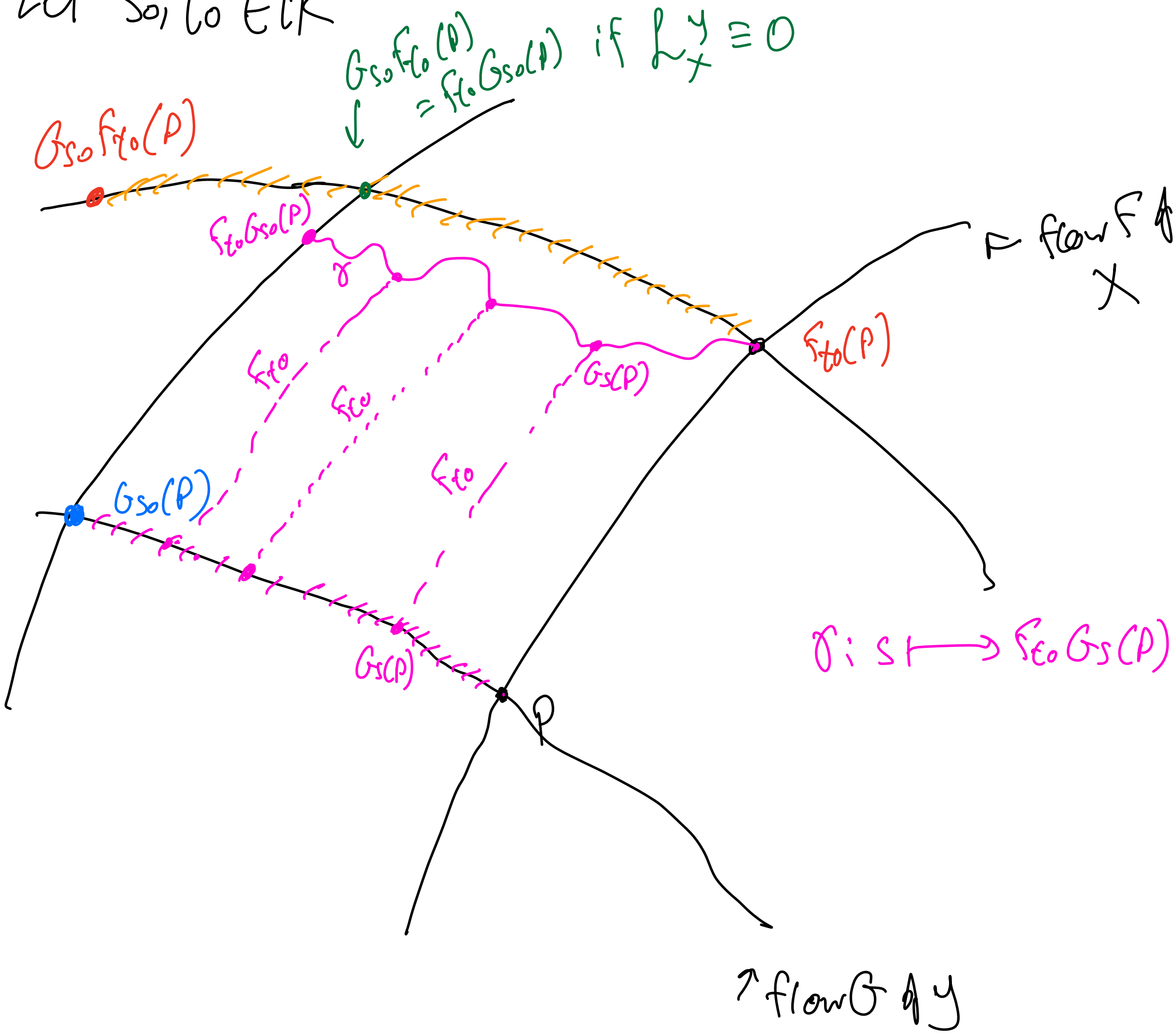
$\Rightarrow F_t$ sends integral curves of $\mathcal{Y}$ to integral curves of $\mathcal{Y}$!!

$\mathcal{Y}$ is invariant under the flow of X	$\Rightarrow$	F_t sends integral curves of $\mathcal{Y}$ to integral curves of $\mathcal{Y}$

Let $X, \mathcal{Y} \in \mathfrak{X}(M)$ with flows F and G

Assume $X, \mathcal{Y}$ are complete

Let $s_0, t_0 \in \mathbb{R}$



Suppose $\mathcal{L}_X Y \equiv 0$. Since $s \longmapsto G_s(P)$ is an integral curve of Y starting at P.

Then $\gamma(s) = F_{t_0}(G_s(P))$ is an integral curve starting at $F_{t_0}(P)$. (since F_{t_0} sends integral curves of Y to integral curves of Y)

Also, $G_s F_{t_0}(P)$ is also an integral curve of Y starting at $F_{t_0}(P)$

so by uniqueness $G_s F_{t_0}(p) = F_{t_0} G_s(p)$

In particular $G_{s_0} \circ F_{t_0}(p) = F_{t_0} \circ G_{s_0}(p)$

$\Longrightarrow \quad G_s \circ F_t = F_t \circ G_s \qquad \forall t, s \in \mathbb{R}$

<u>Def</u>: We say the flows of X and Y commute if $F_t \circ G_s = G_s \circ F_t \quad \forall s, t \in \mathbb{R}$

F_t sends integral curves of Y to integral curves of Y $\Longrightarrow$ flows of X and Y commute

<u>Theorem</u>: Let $X, Y \in \mathfrak{X}(M)$ be complete. Then the following are equivalent:

1) $\mathcal{L}_X Y \equiv 0$

2) Y is invariant under the flow of X

3) Integral curves of Y $\xrightarrow{F_t}$ integral curves of Y

4) The flows of commute $(F_t \circ G_s = G_s \circ F_t)$

5) (because it is symmetric), $\mathcal{L}_Y X \equiv 0$

Proof: we only need to prove $4 \Rightarrow 1$

$$\frac{\partial}{\partial s}\Big|_{s=0}\left[G_s \circ f_t(p) = \underline{f_t \circ G_s(p)} \right]$$

$$Y_{f_t(p)} = f_{t*,p}(Y_p)$$

$$\Rightarrow \underbrace{f_{-t*, f_t(p)}\left(Y_{f_t(p)}\right)}_{H(t)} = Y_p$$

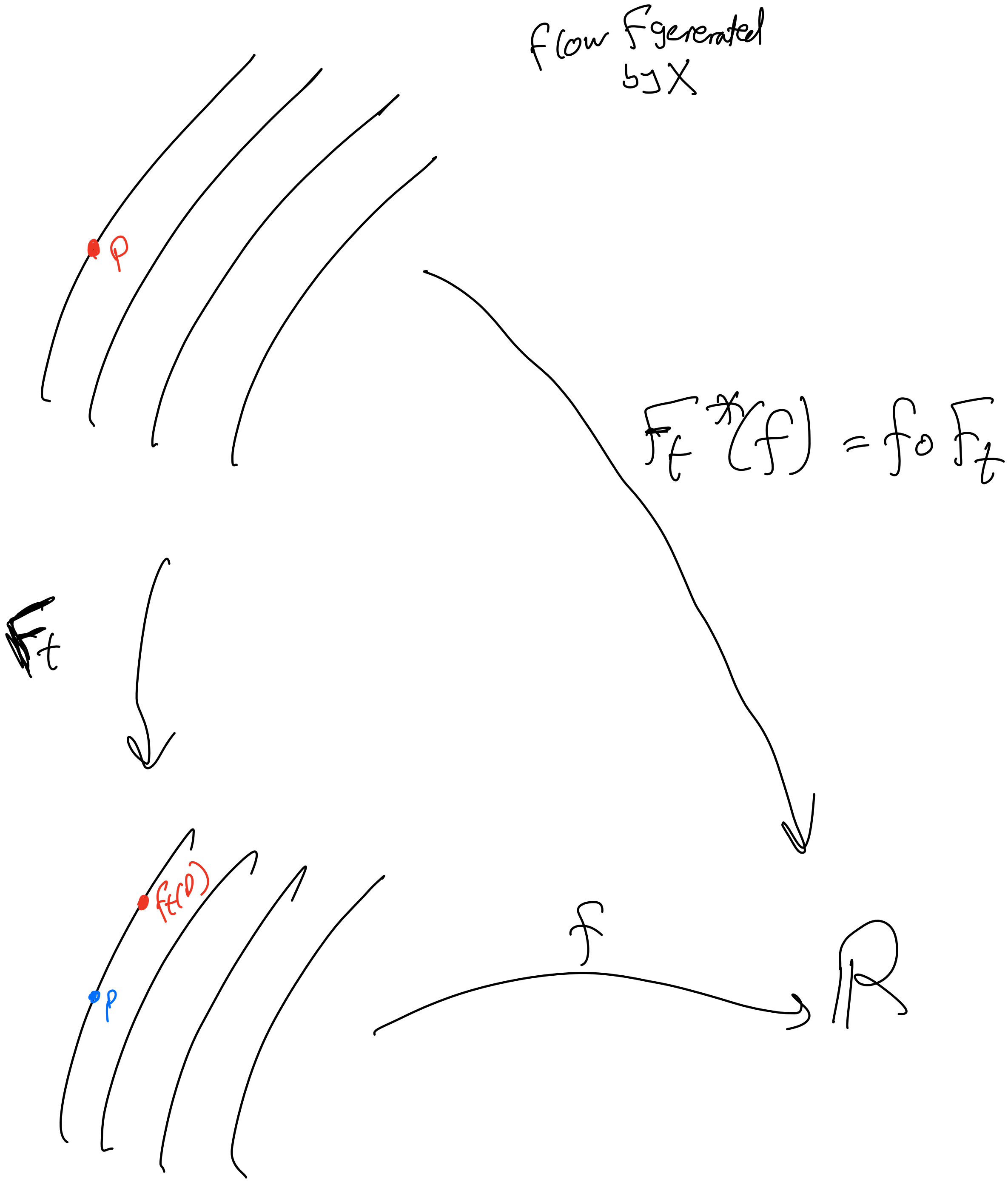
$$\Rightarrow H'(0) = 0 \Rightarrow \mathcal{L}_X Y\big|_p = 0$$

$\blacksquare$

Remark: There is a version of this for non complete V.F.

Taylor Series def

Let $f \in C^\infty(M)$
Let $X \in \mathfrak{X}(M)$



We can talk about

exc

$$\lim_{t \to 0} \frac{f \circ F_t(p) - f(p)}{t} = \frac{d}{dt}\Big|_{t=0} f \circ F_t(p) = X(f)\Big|_p$$

We extend the definition of Lie derivatives on functions:

$$\text{for } f \in C^\infty(M), \quad \mathcal{L}_X f = \lim_{t \to 0} \frac{F_t^*(f)(P) - f(P)}{t}$$
$$= X(f)\big|_P$$

We can equivalently define it as the first order term in the Taylor expansion of $t \mapsto f \circ F_t(P)$

$$\text{so } f \circ F_t(P) = f(P) + t\,\mathcal{L}_X f + o(t)$$
$$= f(P) + t\,X(f)|_P + o(t)$$

Similarly, we can define Lie derivative of Y in the direction of X using Taylor series:

$$\text{Let } H : \mathbb{R} \to T_PM$$
$$: t \mapsto F_{-t*, F_t(P)}\left(Y_{F_t(P)}\right)$$

$$\text{so } H'(0) = \mathcal{L}_X Y|_P$$

or equivalently $\mathcal{L}_X Y|_P$ is the unique vector in T_PM satisfying:

$$H(t) = H(0) + t\,H'(0) + o(t)$$

$$F_{-t*,F_t(p)}\left(Y_{F_t(p)}\right) = Y_p + t\,\mathcal{L}_X Y|_p + o(t)$$

Apply $F_{t*,p}$ to both sides:

$$Y_{F_t(p)} = \underbrace{F_{t*,p}(Y_p)}_{\hookleftarrow} + t\,F_{t*,p}\left(\mathcal{L}_X Y|_p\right) + o(t)$$

Let $f \in C^\infty(M)$

$$\left[Y_{F_t(p)} - F_{t*,p}(Y_p)\right](f) = t\,F_{t*,p}\left(\mathcal{L}_X Y|_p\right)(f) + o(t)$$

$$Y(f)\circ F_t(p) - Y_p(\underbrace{f\circ F_t}) = t\,\mathcal{L}_X Y|_p\left(f\circ F_t(p)\right) + o(t)$$

$Y(f)|_p + t\,XY(f)|_p + o(t)$

$f + tX(f) + o(t)$

$f + tX(f) + o(t)$

$$\cancel{Y(f)|_p} + t\,XY(f)|_p - \cancel{Y_p(f)} - t\,Y_p(X(f)) =$$

$$t\,\mathcal{L}_X Y|_p(f) + \left\{t^2\,\mathcal{L}_X Y|_p(X(f)) + o(t)\right\}$$

$o(t)$

$$t\left[XY - YX\right](f)|_p = t\,\mathcal{L}_X Y|_p(f) + o(t)$$

divide by t and take $t \to 0$

$$[XY - YX](f)|_p = \mathcal{L}_X Y|_p(f)$$

Theorem: $\mathcal{L}_X Y = [X, Y]$ ← Lie Bracket defined in Assignment 2.

Lie Bracket & Lie Algebra

Recall if $X, Y \in \mathfrak{X}(M)$

$$XY - YX =: [X, Y] \in \mathfrak{X}(M)$$

Def: We say X and Y commute if $[X, Y] = 0$

$$XY(f) = YX(f)$$
$$\forall f \in C^\infty(M)$$

Lie Bracket has the following Properties

i) bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

$\forall a, b \in \mathbb{R}$ $\quad [Z, aX + bY] = a[Z, X] + b[Z, Y]$

($a = b = 1$, distributivity)

($b = 0$, homogeneity)

ii) $[X, Y] = -[Y, X]$ $\left(\Rightarrow \mathcal{L}_X Y = -\mathcal{L}_Y X \right)$

iii) Jacobi identity: $\sum_{\text{cyclical}} [X, [Y, Z]] = 0$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

$[X, [Y, Z]]$
$= [[X, Y], Z]$

Def: A Lie algebra over $\mathbb{R}$ is a vector space V over $\mathbb{R}$ together with a product $[\cdot, \cdot] : V \times V \to V$ called the bracket satisfying i, ii, iii.

Recall: An algebra over $\mathbb{R}$ is a vector space V over $\mathbb{R}$ together with a product $\cdot : V \times V \to V$ making V a ring satisfying homogeneity condition;

$$a(X \cdot Y) = aX \cdot Y = X \cdot aY$$

Is a Lie Algebra an algebra?

No: Lie bracket is not necessarily associative

However, any algebra admits a Lie bracket

$[X, Y] := X \cdot Y - Y \cdot X$ which satisfies i, ii, iii making it a Lie algebra.

<u>Example</u>: $\mathfrak{X}(M)$ together with the Lie bracket is a Lie algebra.

1) V.S. over $\mathbb{R}$
2) Module over C^∞
3) Lie algebra over $\mathbb{R}$

(Not an algebra)

<u>Def</u>: A derivation on a Lie algebra V is a map

$D: V \to V$ that is linear (wrt the v.s. structure so $\mathbb{R}$-linear)

$$D(aX + bY) = aD(X) + bD(Y)$$
$$\forall a, b \in \mathbb{R},\ X, Y \in V$$

and satisfies the Leibniz rule wrt the bracket:

$$D[X, Y] = [DX, Y] + [X, DY]$$

<u>Example</u>: Let $X \in \mathfrak{X}(M)$. Then
$\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a derivation

on the Lie algebra $\mathfrak{X}(M)$.

exc (Jacobi identity)

Proposition: Lie derivative satisfies:

i) $\mathcal{L}_Y X = -\mathcal{L}_X Y$

ii) $\mathcal{L}_X [Y,Z] = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$

since $\mathcal{L}_X$ is a derivation on $\mathfrak{X}(M)$.

iii) $\mathcal{L}_{[X,Y]} Z = \mathcal{L}_X \mathcal{L}_Y Z - \mathcal{L}_Y \mathcal{L}_X Z$

iv) let $g \in C^\infty(M)$

$$\mathcal{L}_X (gY) = \mathcal{L}_X g \, Y + g \, \mathcal{L}_X Y$$
$$= X(g) \, Y + g \, \mathcal{L}_X Y$$

Example: Let $(U, \phi = (x^1, \dots, x^n))$ be a chart

Then $\dfrac{\partial^2 f}{\partial x^i \partial x^j} = \dfrac{\partial^2 f}{\partial x^j \partial x^i}$, $\forall f \in C^\infty(U)$ by Clairut's Thm.

$$\Rightarrow \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$$

So coordinate vector fields commute

Question #1: Is this sufficient

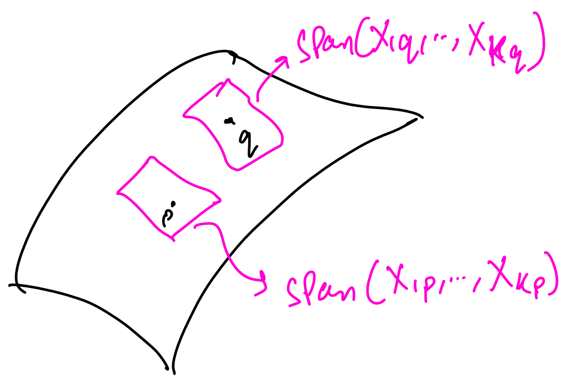
Let $X_1, \cdots, X_k$ be a k-frame
(k smooth vector fields s.t. $X_{1p}, \cdots, X_{k,p}$ are linearly independent $\forall p \in M$)
s.t. $[X_i, X_j] = 0$ for $i,j = 1, \cdots, k$

Does that imply that they are coordinate vector fields?

For $p \in M$, does $\exists (U, \phi)$ near p s.t.

$$\frac{\partial}{\partial x^i} = X_i \quad \text{for } i = 1, \cdots, k$$

Question #2 Let $X_1, \cdots, X_k$ be a k-frame.



i.e. suppose we fix a smooth choice of k-dim subspace of T_pM at every point.

does $\exists$ k-dim submanifold S s.t. $\forall p \in S$, $T_pS = \text{span}(X_{1p}, \cdots X_{kp})$?

(we call this an integral submanifold of $X_1, \cdots, X_k$)

We have completely answered this question for $k=1$.
(by fundamental theorem flows)

Frobenius Thm: Let $X_1, \cdots, X_k$ be any k-frame satisfying "some property", $\exists$ integral submanifold of $X_1, \cdots, X_k$ passing through any $P \in M$.

(iff)

Fubini's Theorem is something else

Post-lecture Practice questions :

1) Do the exercises above.

2) Is every derivation on $\mathfrak{X}(M)$ of the form $\mathcal{L}_X$ for some $X \in \mathfrak{X}(M)$.

3) Let $X = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$

$Y = g \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$ for $g \in C^\infty(\mathbb{R}^2)$

a) Draw and compute the flow of X and Y for $g(x,y) = x$

Then show that the flows don't commute.

Then find an integral curve of Y that doesn't get mapped to an integral flow of Y by F_t.

b) find a condition on g so that $\mathcal{L}_X Y \equiv 0$