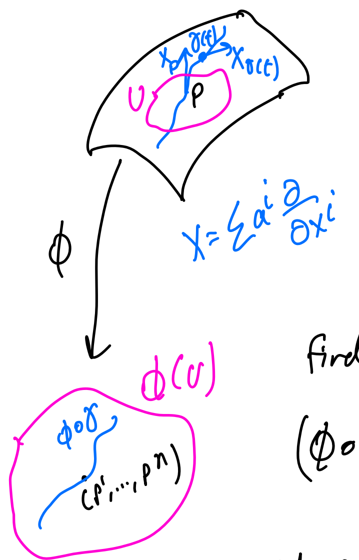
1) Office hours from July 6 – July 16 will change to ∴ (announcement.
2) Assignment 4 is due by the end of Sunday

<u>Review</u>: Let $p \in M$ and let $X \in \mathfrak{X}(M)$

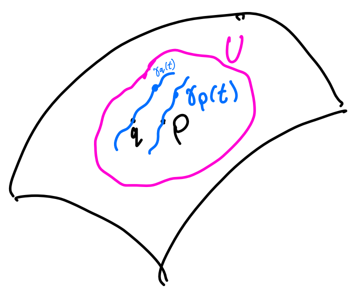


$\gamma : (a,b) \to M$ is an integral curve of X starting at p if :
$$\gamma'(t) = X_{\gamma(t)}$$
$$\gamma(0) = p$$

find $\phi \circ \gamma$ satisfying:
$$(\phi \circ \gamma)'(t) = \begin{bmatrix} \phi \circ a^1 \\ \vdots \\ \phi \circ a^n \end{bmatrix} (\phi \circ \gamma(t))$$
$$(\phi \circ \gamma)(0) = (p^1, \dots, p^n)$$

Thanks to existence & uniqueness Thm of ODEs:

<u>Proposition</u>: for any coordinate open set U, any $p \in U$, and any $X \in \mathfrak{X}(M)$ $\exists!$ maximal integral curve of $X|_U$ starting at p.



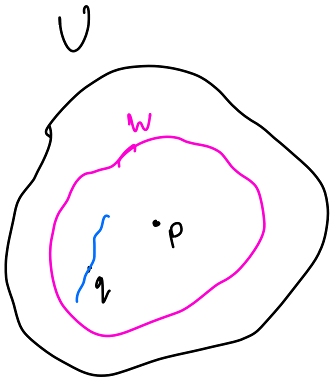
Thanks to "smooth dependence of solutions on the initial point" Theorem of ODEs,

<u>Proposition</u>: Let U be a coordinate open set and $p \in U$. Let $X \in \mathfrak{X}(M)$

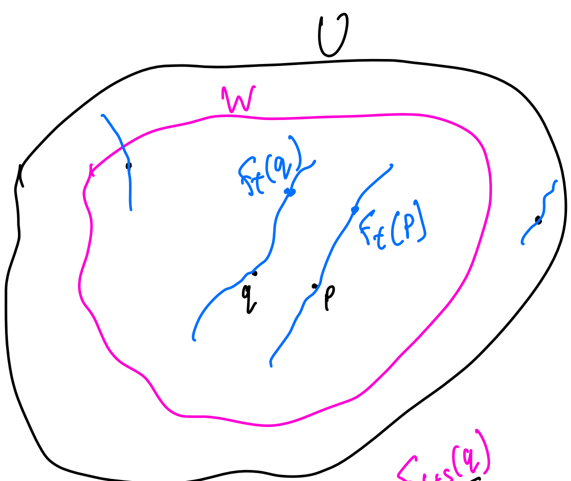
$\exists \varepsilon > 0$ and nhbd $W \subseteq U$ of p and a C^∞ map

$F : (-\varepsilon, \varepsilon) \times W \to U$ s.t.

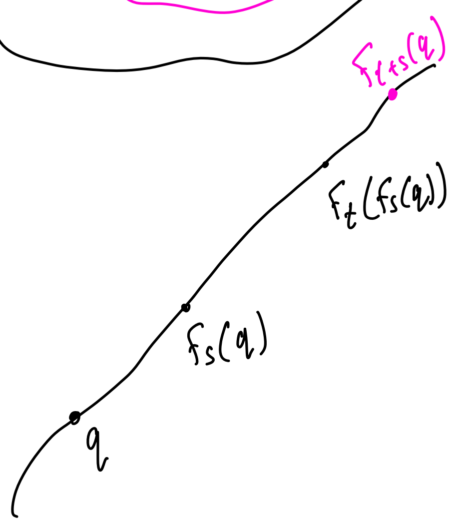
$$\left.\begin{aligned} \frac{\partial}{\partial t} F(t,q) &= X_{F(t,q)} \\ F(0,q) &= q \end{aligned}\right\}$$



Fix $q \in W$, Then $t \mapsto F_t(q) := F(t,q)$ is an integral curve of X starting at q.



fix $t \in (-\varepsilon, \varepsilon)$,
as I change the initial point $F_t(q)$ changes smoothly since $q \mapsto F_t(q)$ is C^∞



Let $q \in W$

Suppose $s, t \in (-\varepsilon, \varepsilon)$

s.t. $F_{t+s}(q)$ and $F_t(F_s(q))$ is defined.

$t+s \in (-\varepsilon, \varepsilon)$

$(t, F_s(q)) \in (-\varepsilon, \varepsilon) \times W$

$(F_s(q) \in W)$

Claim: The curves $\gamma_1 : t \mapsto F_t(F_s(q))$
and $\gamma_2 : t \mapsto F_{t+s}(q)$
are both integral curves of X starting at $F_s(q)$
and so by uniqueness, they are equal.

$$\gamma_1'(t) = \frac{\partial}{\partial t}\left(F_t(F_s(q))\right) = \frac{\partial}{\partial t} F(t, F_s(q)) = X_{F_t(F_s(q))}$$

$$\gamma_1(0) = F_0(F_s(q)) = F(0, F_s(q)) = F_s(q)$$

$$\gamma_2'(t) = \frac{\partial}{\partial t} F_{t+s}(q) = \frac{\partial}{\partial t} F(t+s, q) = \frac{\partial F}{\partial t}(t+s, q) \cdot \cancel{\frac{\partial(t+s)}{\partial t}} = X_{F_{t+s}(q)}$$

$$\gamma_2(0) = F_{0+s}(q) = F_s(q)$$

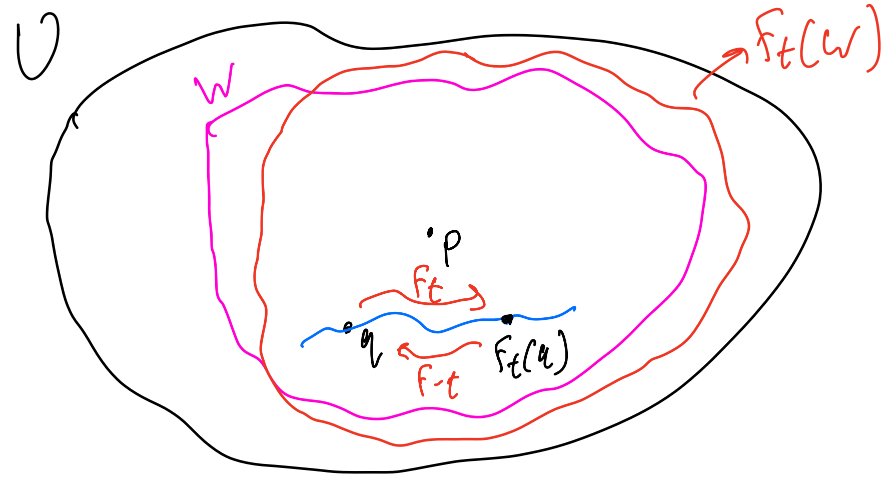
Lemma: $F_t(F_s(q)) = F_{t+s}(q)$

whenever both sides make sense.

$\left((t, F_s(q)),\ (t+s, q) \in (-\varepsilon, \varepsilon) \times W \right)$

In particular, for $t \in (-\varepsilon, \varepsilon)$, $q \in W$

$$\left.\begin{aligned} F_{-t} \circ F_t &= \mathrm{Id}_W \\ F_t \circ F_{-t} &= \mathrm{Id}_W \end{aligned}\right\} \quad F_t^{-1} = F_{-t} \text{ where } F_t^{-1} : F_t(W) \to W$$



Def: f is called the local flow near p generated by vectorfield X.

The map $t \mapsto f_t(q)$ is called the flow line of the local flow, so each flow line is an integral curve of X.

If f is defined on $\mathbb{R} \times M$, then it's called a global flow. We have shown that any $X \in \mathfrak{X}(M)$ admits a local flow near any point $p \in M$, but not necessarily a global flow.

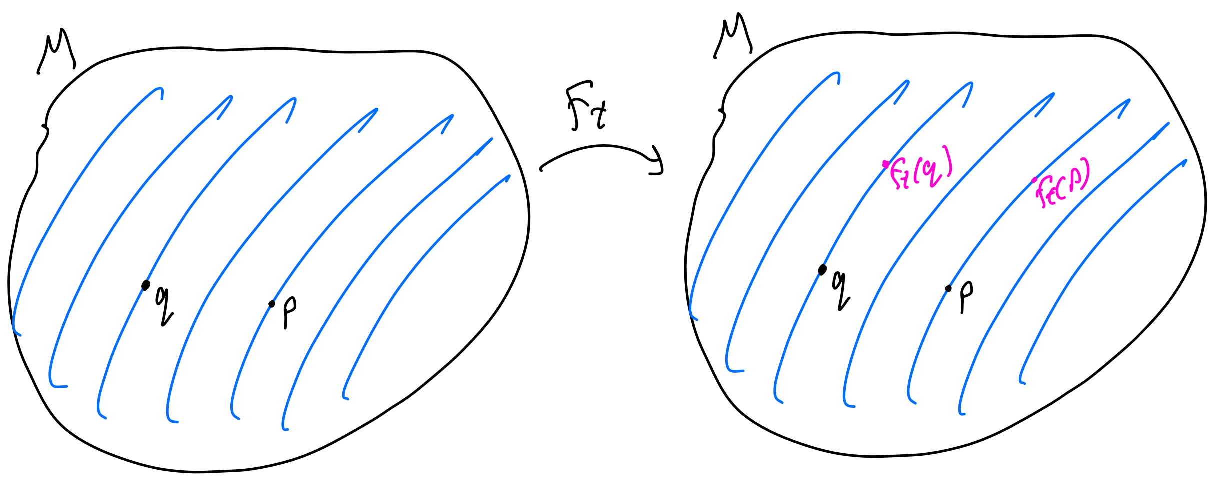
A vector field that admits a global flow is called a complete vector field.

Example: $X := x^2 \dfrac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$

$\gamma(t) = \left(\frac{1}{1-t}, 0\right)$ is the integral curve of X starting at $(1,0)$. γ cannot be extended to all $\mathbb{R}$. So X is not complete.

If F is a global flow, then $\forall t \in \mathbb{R}$ $F_t^{-1} = F_{-t}$

let $t \in \mathbb{R}$, $F_t : M \to M$ is a diffeomorphism.



Note that $F_0 = Id$

Example: $X := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

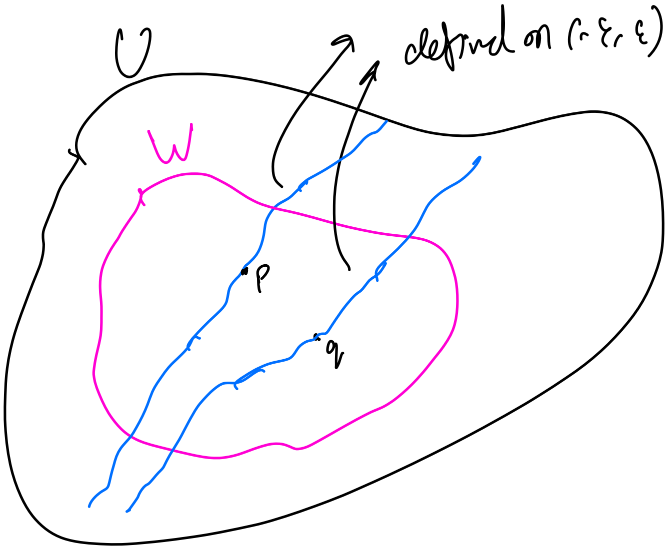
X admits a global flow
$F : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

So X is complete

$$f_t : \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is a diffeomorphism.



$\exists$ maximal integral curve passing through every point in U.

let $D^{(q)}$:= maximal interval in which the maximal integral curve starting at p is defined $\subseteq \mathbb{R}$

$$\text{let } D = \{ (t, q) : q \in U,\ t \in D^{(q)} \} \subseteq \mathbb{R} \times U$$

Then the maximal flow $F : D \to U$ is defined by

$$F(t, q) = \gamma_q(t)$$

maximal integral curve starting at q.

from what we did earlier:

exc

1) F is C^∞ on D

(2) $\forall q \in U$, $\exists \varepsilon > 0$ s.t. $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^{(q)}$

however, this might not be true:

false $\exists \varepsilon > 0$ s.t. $\forall q \in U$, $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^{(q)}$

We Just Proved:

<u>Proposition</u>: Let $X \in \mathfrak{X}(U)$. $\exists!$ maximal flow $F : \mathcal{D} \to U$.

All we have is that on a coordinate set U, for any $X \in \mathfrak{X}(U)$:

#1) $\exists!$ C^∞ maximal integral curve starting at any point in U.

#2) $\exists!$ maximal flow.

<u>Fundamental Theorem of flow</u>: Let $X \in \mathfrak{X}(M)$

#1) for any $p \in M$, $\exists!$ maximal integral curve starting at p.

#2) $\exists!$ C^∞ maximal flow generated by X.

Post - Lecture Practice Questions

1) Do all exercises above.
2) do Problem 14.6, 14.7, 14.8
3) find an example of $X \in \mathfrak{X}(U)$ s.t. $\nexists\, \varepsilon > 0$ satisfying $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^{(q)}$ $\forall q \in U$. (but $\exists \varepsilon > 0$ and $\exists W \subseteq U$ s.t. $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^{(q)}$ $\forall q \in W$).
4) Let $F: \mathbb{R} \times M \to M$ be a C^∞ map satisfying
   1) $F(0, q) = q \quad \forall q \in W$
   2) $F(t+s, q) = F_t(F_s(q))$ whenever the right side make sense.

   Show that F is the local flow generated by a unique $X \in \mathfrak{X}(M)$.
5) let $\gamma : (a,b) \to M$ be an integral curve of X starting at p.
   Let $\lambda \in \mathbb{R} \setminus \{0\}$ Define $\gamma_1(t) = \gamma(\lambda t)$
   and $\gamma_2(t) = \gamma(t+\lambda)$

   Show that γ_1 is an integral curve of λX starting at p and is defined on $(\frac{a}{\lambda}, \frac{b}{\lambda})$
   and γ_2 is an integral curve of X starting at $\gamma(\lambda)$ and is defined on $(a-\lambda, b-\lambda)$
6) Every compactly supported vector field is complete.
7) Compute the flow of $x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$
8) Let $S \subseteq M$ be a submanifold and let $X \in \mathfrak{X}(M)$ be tangent to S. Is it true that any integral curve passing through $p \in S$ stays in S?

9) let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(t, x) = x + tx.$$

Is F the flow of a vector field?