1) Assignment 3
2) Test on June 17

Review:

$$TM := \bigcup_{p\in M} T_pM = \{ (p,v) : v \in T_pM \}$$

Let $(U, \phi = (x^1, \dots, x^n))$ be a chart near $p \in M$.

We defined $\tilde{\phi} : \underset{\bigcup_{p\in U} T_pM}{\overset{\shortparallel}{TU}} \to \phi(U) \times \mathbb{R}^n$

$: (p,v) \mapsto (x^1, \dots, x^n, c^1, \dots, c^n)$ ← coefficients of $v$ wrt the basis $\{ \frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p \}$

$\left(v = \sum_{i=1}^n c^i \frac{\partial}{\partial x^i}\big|_p\right)$

We equipped $TU$ with the topology that makes $\tilde{\phi}$ a homeomorphism.

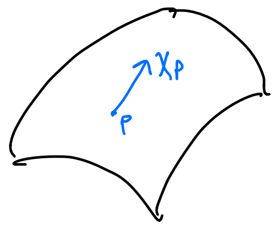
If $(U, \psi = (y^1, \dots, y^n))$ is another chart, then

$\tilde{\psi} \circ \tilde{\phi}^{-1} : (x^1, \dots, x^n, c^1, \dots, c^n) \longmapsto (y^1, \dots, y^n, b^1, \dots, b^n)$ is a homeomorphism (diffeomorphism)

$\Longrightarrow$ The topology on $TU$ is independent of the coordinate map.

We assert $A \subseteq TM$ is open in $TM$ if $A \cap TU$ is open in $TU$ for every coordinate set $U \subseteq M$. (This topology on $TM$ is Hausdorff and second countable)

Then $(TU, \tilde{\phi})$ is a chart on $TM$ and if $\{ (U_\alpha, \phi_\alpha) : \alpha \in A \}$ is a $C^\infty$ atlas on $M$, $\{ (TU_\alpha, \tilde{\phi}_\alpha) : \alpha \in A \}$ is a $C^\infty$ atlas on $TM$.




A smooth vector field $X$ is a smooth section over $TM$.

$$X : M \longrightarrow TM \qquad (\pi \circ X = Id_M)$$
$$p \longmapsto X(p) = (p, \underbrace{X_p}_{\in T_pM})$$

Recall that $\Gamma(M)$ is a vector space over $\mathbb{R}$ and a module over $C^\infty(M)$

<u>Ex</u>: Let $(U, \phi = (x^1, \dots, x^n))$ be a chart. Then $\frac{\partial}{\partial x^i} \in \Gamma(U)$

$$\frac{\partial}{\partial x^i} : p \longmapsto \left(p, \left.\frac{\partial}{\partial x^i}\right|_p\right)$$

in coordinates, $: (x^1, \dots, x^n) \longmapsto (x^1, \dots, x^n, 0, \dots, 1, \dots, 0)$

If $a^i \in C^\infty(U)$ for $i = 1, \dots, n$

Then $X := \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} \in \Gamma(U)$

In fact, all smooth sections on $U$ are of that form

Let $X \in \Gamma(U)$

Then $\tilde{\Phi} \circ X \circ \phi^{-1} : (x^1, \dots, x^n) \longmapsto (x^1, \dots, x^n, c^1, \dots, c^n)$ $\rightarrow C^\infty$ since $X$ is smooth

where $X_q = \sum_{i=1}^n c^i|_{\phi(q)} \left.\frac{\partial}{\partial x^i}\right|_q \quad \forall q \in U$

In particular, $c^i : \phi(U) \to \mathbb{R}$ is $C^\infty$

Then $a^i := c^i \circ \phi : U \to \mathbb{R}$ is $C^\infty$

And $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$

<u>Proposition</u>: for any coordinate open set $U$, $\Gamma(U)$ is an $n$-dim module over $C^\infty(U)$ with basis
$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$$

<u>Remark</u>: as a vector space over $\mathbb{R}$, $\Gamma(U)$ is infinite dimensional

<u>Smoothness Criterion of vector fields #1</u>:

Let $X : M \to TM$ be a section over $TM$ (a vector field)

$X$ is smooth iff on any chart $(U, \phi = (x^1, \dots, x^n))$, the coefficients $a^i : U \to \mathbb{R}$ are smooth

(where $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ )

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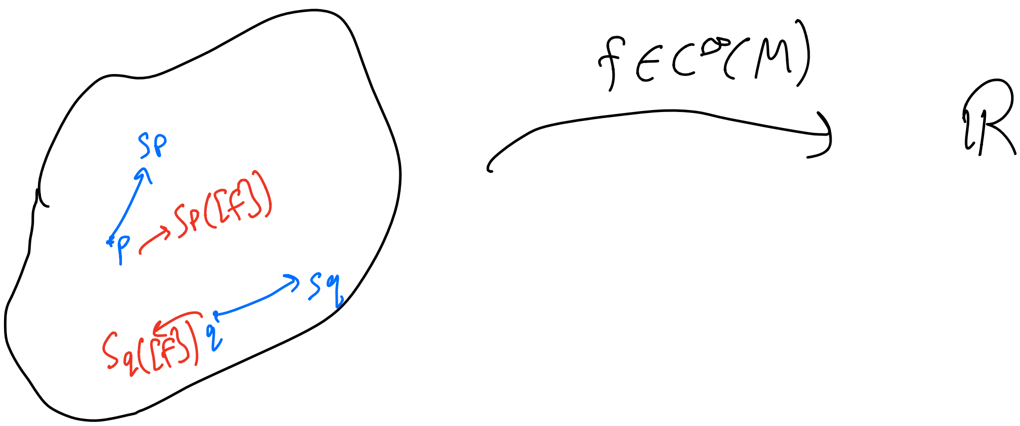
**An equivalent definition of vector fields**

Let $S$ be a section over $TM$

$$S : M \to TM$$
$$p \mapsto (p, S_p)$$




At every point $p$, $S_p$ is a derivation at $p$ $\left( S_p : C_p^\infty(\mathbb{R}) \to \mathbb{R} \right)$

so $S_p[f] \in \mathbb{R}$

This defines a function $D_S(f)$ on $M$

$$D_S(f) : M \to \mathbb{R}$$
$$: p \mapsto S_p([f])$$

So we have the map $D_S$, which takes $C^\infty$ functions to functions on $M$.

Notice:

1) $D_S$ is linear: $D_S(f+g)\big|_p = S_p([f+g])$
$$= S_p([f]) + S_p([g])$$
$$= \big[D_S(f) + D_S(g)\big]\big|_p$$

2) $D_S$ satisfies the Leibniz rule:

$$D_S(fg)\big|_p = S_p([fg])$$
$$= S_p([f])\,g(p) + f(p)\,S_p([g])$$
$$= \big[D_S(f)\,g + f\,D_S(g)\big]\big|_p$$

Suppose $S$ is smooth. Let $(U, \phi = (x^1, \dots, x^n))$ be a chart.

Then on $U$, $S = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ where $a^i \in C^\infty(U)$

(by smoothness criterion)

Let $f \in C^\infty(M)$. Then on $U$,

$$D_S(f)\big|_p = S_p([f])$$
$$= \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x^i}\bigg|_p ([f])$$

$$= \sum_{i=1}^{n} a_i(p) \left.\frac{\partial f}{\partial x^i}\right|_p$$

$$\Rightarrow \quad D_s(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x^i} \in C^\infty(U) \quad \text{on } U$$

$$\Rightarrow \quad D_s(f) \in C^\infty(M)$$

So if $s \in \Gamma(M)$ is a smooth section,
then $D_s : C^\infty(M) \to C^\infty(M)$ is a derivation on the algebra $C^\infty(M)$.

__Remark:__ $v \in T_pM$ is not a derivation on the algebra $C^\infty_p(M)$.
($v$ is a derivation at $p$)
($v$ is a point derivation)

$v : C^\infty_p(\mathbb{R}) \to \mathbb{R}$
liner
Leibniz.

Let $\mathrm{Der}(C^\infty(M)) := \{ D : C^\infty(M) \to C^\infty(M) \mid D \text{ is a derivation} \}$

vectorspace over $\mathbb{R}$
module over $C^\infty(M)$

__Define:__ $\Phi : \Gamma(M) \to \mathrm{Der}(C^\infty(M))$

$$s \mapsto D_s$$

**Thm**: $\Phi$ is an isomorphism (wrt the module and v.s structure)

**Proof**: $\Phi$ is linear and injective ✓ (Assignment 3, 4c)

We show surjectivity

Let $D \in Der(C^\infty(M))$.

We want to find $S \in \Gamma(M)$ s.t. $D = \Phi(S) = D_S$

Roughwork

The section $S$ that we are looking for will satisfy:

for any $p \in M$ and $f \in C^\infty(M)$, $\underline{D(f)|_p} = D_S(f)|_p = \underline{S_p([f])}$

**Attempt**: Define $\cancel{S_p : C^\infty_p(M) \to \mathbb{R}}$

$\cancel{: [f] \mapsto D(f)|_p}$

? $f$ needs to be defined on all $M$.

**Lemma**: for any $[f] \in C^\infty_p(M)$, $\exists \tilde{f} \in C^\infty(M)$ s.t. $\tilde{f} \in [f]$

etc

For $p \in M$, define $S_p : C^\infty_p(M) \to \mathbb{R}$

$: [f] \mapsto D(\tilde{f})|_p$

where $\tilde{f}$ is an extension of $f$

WTS: ① $S_p$ is well defined
② $S_p \in T_pM$

① Let $f_1, f_2 \in [f]$ be functions defined near $p$.
Let $\tilde{f}_1, \tilde{f}_2 \in C^\infty(M)$ be extensions of $f_1$ and $f_2$
$(\tilde{f}_1, \tilde{f}_2 \in [f])$

We want to show: $D(\tilde{f}_1)\big|_p = D(\tilde{f}_2)\big|_p$

i.e. $D(\tilde{f}_1 - \tilde{f}_2)\big|_p = 0$

Note that $\tilde{f}_1 - \tilde{f}_2 = 0$ on a neighbhd $V$ of $p$.
Let $\rho \in C^\infty(M)$ be a bump function at $p$ supported in $V$.

Then $\rho(\tilde{f}_1 - \tilde{f}_2) \equiv 0$ so $D(\rho(\tilde{f}_1 - \tilde{f}_2)) = 0$

So $0 = D(\rho(\tilde{f}_1 - \tilde{f}_2)) = D(\rho)\cancelto{0}{(\tilde{f}_1 - \tilde{f}_2)} + \cancelto{1}{\rho}\, D(\tilde{f}_1 - \tilde{f}_2)$

Evaluate at $p$, we get: $0 = D(\tilde{f}_1 - \tilde{f}_2)\big|_p$

② $S_p : [f] \mapsto D\tilde{f}\big|_p$ is linear and satisfies the Leibniz rule

$\Rightarrow S_p \in T_pM$.

We then define the section $S : M \to TM$
$: p \mapsto (p, S_p)$

Then for $f \in C^\infty(M)$, $D(f)\big|_p = S_p([f]) = D_S(f)\big|_p$

$$\Rightarrow D = D_S = \cancel{\Phi(S)}$$ not yet — we don't know if S is smooth.

We need to show S is smooth as a section.

Let $(U, \phi = (x^1, \cdots, x^n))$ be a chart near $p \in M$.

Then on $U$, $\displaystyle S = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}$

Let $\tilde{x}^j \in C^\infty(M)$ be an extension of $x^j$. $(\tilde{x}^j \in [x^j])$

Then $D(\cancel{x^j}\,\tilde{x}^j)\big|_p = S_p([x^j])$

$$= \sum_{i=1}^n a^i(p) \frac{\partial}{\partial x^i}\Big|_p [x^j]$$

$$= \sum_{i=1}^n a^i(p) \frac{\partial x^j}{\partial x^i}\Big|_p$$

$$= a^j(p)$$

$\Rightarrow D(\cancel{x^j}\,\tilde{x}^j) = a^j$ on $U$ $\Rightarrow a^j \in C^\infty(U)$

as $D(\cancel{x^j}\,\tilde{x}^j) \in C^\infty(U)$

And so $\boxed{D = \Phi(s)}$ $\Rightarrow$ $\Phi$ is surjectivity

$\square$

From now on, denote the space of smooth vector fields by $\mathfrak{X}(M)$. ($S$ and $D_S$ are describing the same element in $\mathfrak{X}(M)$)

$X \in \mathfrak{X}(M)$ is a smooth vector field : $X : M \to TM$ (smooth section)

$X : C^\infty(M) \to C^\infty(M)$ (derivation)

Smoothness criterion of vector fields #2 :

A section $X$ is $C^\infty$ iff $X(f)$ is $C^\infty$ whenever $f \in C^\infty(M)$.

Midterm doesn't cover

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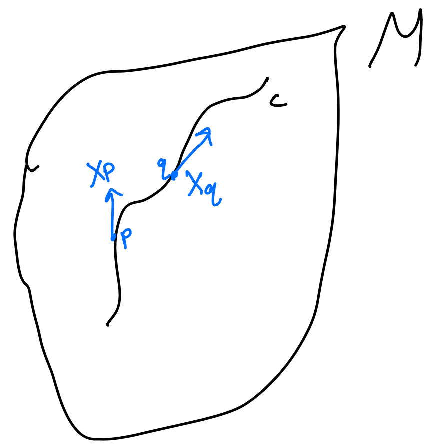
## Integral curves & flow of vector fields

Let $X \in \mathfrak{X}(M)$

Def: An integral curve $c:(a,b)\to M$ of $X$ is a $C^\infty$ curve s.t. $c'(t) = X_{c(t)}$

An integral curve starting at $P$ is an integral curve s.t. $c(0)=P$ and $P$ is called the initial point.

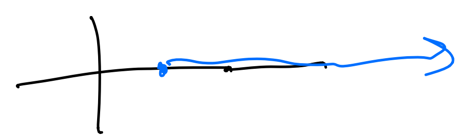



An integral curve is maximal if its domain cannot be extended

example $x^2\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$

integral curve starting at $(1,0)$

is $\gamma(t) = \left(\frac{1}{1-t}, 0\right)$



Let $(U, \phi=(x^1,\dots,x^n))$ be a chart near $P$

Suppose $c:(a,b)\to M$ is an integral curve starting at $P$

Then $$X_{c(t)} = \sum_{i=1}^{n} a^i(c(t)) \frac{\partial}{\partial x^i}\Big|_{c(t)}$$

and $$c'(t) = c_*\left(\frac{d}{dt}\right) = \sum_{i=1}^{n} (x^i\circ c)'(t) \frac{\partial}{\partial x^i}\Big|_{c(t)}$$

Since $c'(t) = X_{c(t)}$,

$$(x^i \circ c)'(t) = a^i(c(t))$$
$$c(0) = p$$

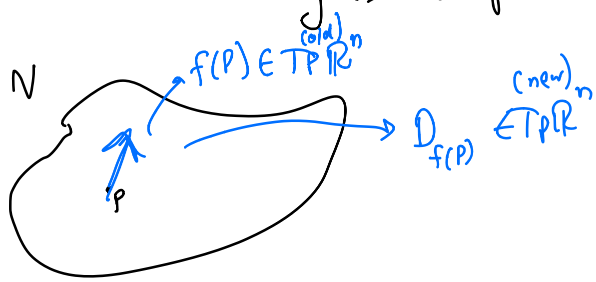
$$(\overbrace{\phi \circ c}^{y})'(t) = \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} = f : \phi(U) \to \mathbb{R}^n$$
$$(\phi \circ c)(0) = (p^1, \dots, p^n)$$

which is a system of ODEs.

<u>Thm</u>: Existence and uniqueness of ODEs:

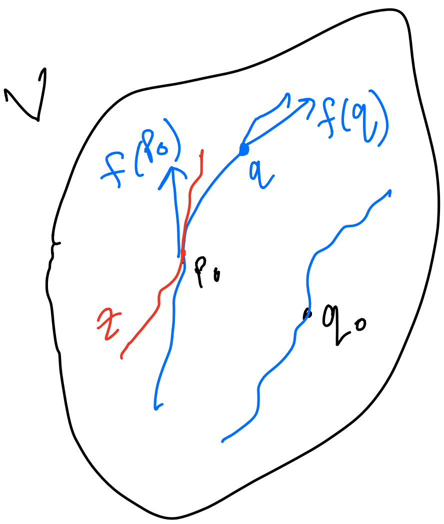
Let $V \subseteq \mathbb{R}^n$ and let $f : V \to \mathbb{R}^n$ be a $C^\infty$ function. Let $P_0 \in V$.

Then the ODE
$$\frac{dy}{dt} = f(y(t))$$
$$y(0) = P_0$$
has a unique $C^\infty$ solution $y : (a(P_0), b(P_0)) \to V$ where $(a(P_0), b(P_0))$ is the maximal open interval containing $0$ on which $y$ is defined.



Notice that a function $f: V \to \mathbb{R}^n$ defines a vectorfield on $V$. The corresponding section

$$X: V \to T\mathbb{R}^n$$
$$: p \mapsto (p, D_{f(p)})$$




If $z: (-\varepsilon_1, \varepsilon_2) \to V$

Satisfies $\dfrac{dz}{dt} = f(z(t))$

and $z(0) = p_0$

Then $(-\varepsilon_1, \varepsilon_2) \subseteq (a(p_0), b(p_0))$

and $z = y\big|_{(-\varepsilon_1, \varepsilon_2)}$

<u>Corollary</u>: Let $U \subseteq M$ be a coordinate open set and let $X \in \mathfrak{X}(U)$.

Then for any $p \in U$, $\exists!$ maximal integral curve of $X$ starting at $p$.

We want to vary the initial point and see if the solution changes smoothly.

Think of $y$ as a function of 2 variables: $y(t,q)$ satisfying

$$\frac{\partial}{\partial t} y(t,q) = f(y(t,q))$$
$$y(0,q) = q$$

<u>Thm</u>: Smooth dependence of solutions on the initial point.

Let $f: V \to \mathbb{R}^n$ be a $C^\infty$ map on an open set $V \subseteq \mathbb{R}^n$. For each point $p_0$, $\exists$ neighbhd of $p_0$ $W \subseteq V$ and $\exists \varepsilon > 0$ and a $C^\infty$ map

$$y: (-\varepsilon, \varepsilon) \times W \to V \quad \text{s.t.}$$

$$\frac{\partial}{\partial t} y(t,q) = f(y(t,q)) \qquad \forall (t,q) \in (-\varepsilon,\varepsilon) \times W$$
$$y(0,q) = q$$

# Post-lecture Practice Problems

1) do the above excercises.

2) Show $\Gamma(U)$ is an infinite dim vectorspace over $\mathbb{R}$ for any coordinate open set

3) Make sure you see that for any section $X : M \to TM$ and a coordinate chart $(U, \phi)$, $\exists\, a^i : U \to \mathbb{R}$ s.t.

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \quad \text{on } U.$$

($X$ might not be $C^\infty$ or even cont, and so $a^i$ is not necessarily $C^\infty$ or even continuous).

Use the fact that at each point, $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^{n}$ is a basis for $T_pM$.

4) Show that any derivation $D : C^\infty(M) \to C^\infty(M)$ is a local operator. That is for any $f \in C^\infty(M)$, $D(f)|_p$ only depends on $f$ near $p$.

5) Problem 14.1, 14.2, 14.3

6) Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^3$ (smooth)

Let $f(x,y,z) := x^2+y^2+z^2$.

Compute $\mathbb{X}(f)$.