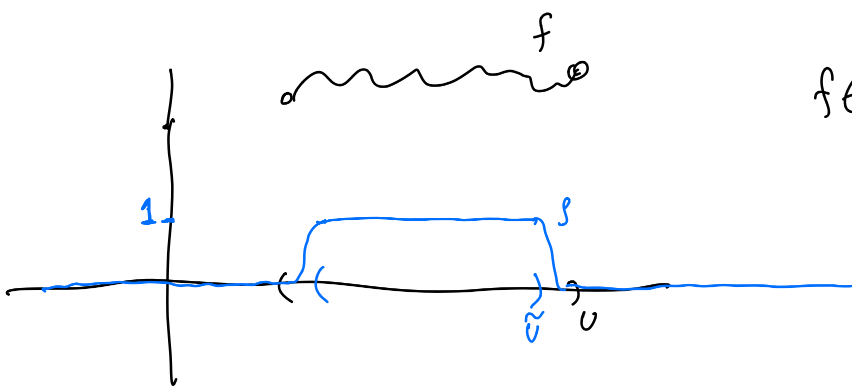
1) optional / mandatory readings

2) Assignment 3 is due by the end of friday but with a 0% penalty per day of lateness until the end of sunday. Then the usual 20% penalty.

Bump functions & Partition of unity

extend local objects to global.

bridge between global & local analysis



$f \in C^\infty(U)$, $U \subseteq \mathbb{R}$

The extension of f will be $\tilde{f}(x) := \begin{cases} \rho(x) f(x), & x \in U \\ 0, & \text{otherwise} \end{cases}$

Then $\tilde{f} \in C^\infty(\mathbb{R})$

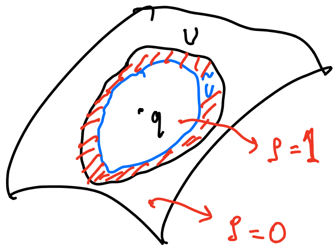
and $\tilde{f}|_{\tilde{U}} = f|_{\tilde{U}}$

Existence of bump function:

Let M be a smooth manifold. Let $q \in M$ and U be a neighbhd of q in M.

Then $\exists\, \rho \in C^\infty(M)$ s.t. $\operatorname{supp}(\rho) := \overline{\{x \in M : \rho(x) \neq 0\}} \subseteq U$

and $\rho|_{\tilde{U}} \equiv 1$ for some neighbhd $\tilde{U} \subseteq U$ of q



Read section 13.1

C^∞ extension lemma:

Let $f \in C^\infty(U)$ where $U \subseteq M$ is a neighbhd of $p \in M$.
Then $\exists\, \tilde{f} \in C^\infty(M)$ s.t. $\tilde{f}|_{\tilde{U}} = f|_{\tilde{U}}$ on some neighbhd $\tilde{U} \subseteq U$ of p.

Proof: Let $\rho \in C^\infty(M)$ be a bump function s.t. $\mathrm{supp}(\rho) \subseteq U$ and $\rho|_{\tilde{U}} \equiv 1$ on some neighbhd $\tilde{U} \subseteq U$ of p.

Define $\tilde{f} : M \to \mathbb{R}$, $\tilde{f}(x) = \begin{cases} \rho(x) f(x), & x \in U \\ 0, & \text{otherwise} \end{cases}$

On U, $\tilde{f} = \rho f$ is C^∞ since ρ and f is C^∞ on U.
On U^c: Let $x \in U^c \subseteq \mathrm{supp}(\rho)^c$ so $\exists$ neighd $V \subseteq \mathrm{supp}(\rho)^c$ of x.
And so $\tilde{f}|_V \equiv 0 \Rightarrow \tilde{f}$ is C^∞ at $x \Rightarrow \tilde{f}$ is C^∞ on U^c

So $\tilde{f} \in C^\infty(M)$.

∎

Let $(U, \phi = (x^1, \dots, x^n))$ be a chart near p.

Then $\frac{\partial}{\partial x^i} : U \to TU$ is a section (vector field) over TU
$q \mapsto \left(q, \frac{\partial}{\partial x^i}\Big|_q\right)$ since $\pi \circ \frac{\partial}{\partial x^i} = Id_U$

Is $\frac{\partial}{\partial x^i}$ a smooth vector field on U?

Recall that $(TU, \tilde{\phi})$ is a chart on TU near $\left(p, \frac{\partial}{\partial x^i}\Big|_p\right)$

$\tilde{\phi} : TU \longrightarrow \phi(U) \times \mathbb{R}^n$
$(q, v) \longmapsto (x^1, \dots, x^n, c^1, \dots, c^n)$

components of v with respect to the basis $\left\{\frac{\partial}{\partial x^1}\Big|_q, \dots, \frac{\partial}{\partial x^n}\Big|_q\right\}$

$$\tilde{\phi} \circ \frac{\partial}{\partial x^i} \circ \phi^{-1} : (x^1, \dots, x^n) \longmapsto \left(x^1, \dots, x^n, 0, \dots, 1, \dots, 0\right)$$
is C^∞

$\Rightarrow \frac{\partial}{\partial x^i}$ is a smooth vector field on U
$\left(\frac{\partial}{\partial x^i} \in \Gamma(U)\right)$

Can we extend $\frac{\partial}{\partial x^i}$, near p to a smooth vector field on M?

Let $\rho \in C^\infty(M)$ be a bump function s.t. $\mathrm{supp}(\rho) \subseteq U$
and $\rho|_{\tilde{U}} \equiv 1$ on a neighbhd $\tilde{U} \subseteq U$ of P.

Define $X : M \to TM$ by

$$X : q \longmapsto \begin{cases} \left(q, \rho(q) \frac{\partial}{\partial x^i}\Big|_q\right), & q \in U \\ (q, 0) & , \text{otherwise} \end{cases}$$

$$\left(X_q = \rho(q) \frac{\partial}{\partial x^i}\Big|_q \right)$$

Then $X \in \Gamma(M)$ is a C^∞ vector field on M s.t.

$$X\big|_{\tilde{U}} = \frac{\partial}{\partial x^i}\Big|_{\tilde{U}} .$$

Partition of Unity

<u>Def</u>: A C^∞ partition of unity is a collection of smooth nonnegative functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ s.t.

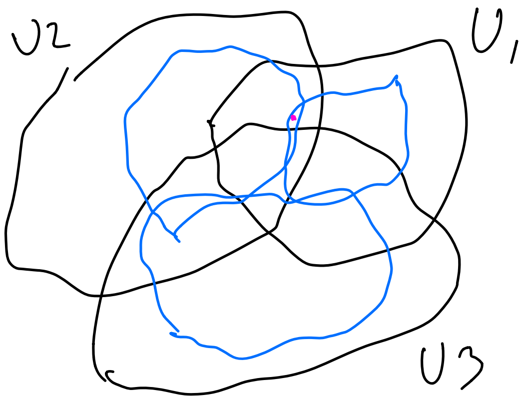
1) $\{\operatorname{supp}\{\rho_\alpha\}\}_{\alpha \in A}$ is locally finite

(means $\forall q \in M$, $\exists$ U neighbhd of q that intersects finitely many of the sets in $\{\operatorname{supp}\{\rho_\alpha\}\}_{\alpha \in A}$)

2) $\displaystyle\sum_{\alpha \in A} \rho_\alpha \equiv 1$

If $\{U_\alpha\}_{\alpha \in A}$ is an open cover, we say $\{\rho_\alpha\}_{\alpha \in A}$

is subordinate to $\{U_\alpha\}_{\alpha\in A}$ if $\operatorname{supp}\{\rho_\alpha\} \subseteq U_\alpha$ for every $\alpha \in A$.



Thm: Existence of a Partition of unity (section B.3 Appendix)

Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover for M

i) There is a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$

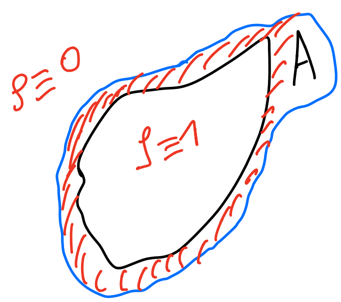
ii) There is a C^∞ Partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ s.t. φ_k has compact support & $\forall k \in \mathbb{N}$, $\operatorname{supp}(\varphi_k) \subseteq U_\alpha$ for $\alpha \in A$.

Consequences:

Existence of bump functions (strong version):

Let $A \subseteq M$ be any closed set and U be any open set containing A.

Then $\exists\, \rho \in C^\infty(M)$ s.t. $\rho|_A \equiv 1$ and $\operatorname{supp}(\rho) \subseteq U$



Outline of Proof: $\{U, A^c\}$ is an open cover and so it admits a partition of unity $\{\rho_1, \rho_2\}$.

Then ρ_1 is the desired function ☺

Show that

Let $A \subseteq M$ be any subset. Recall the definition of "smooth" from Assignment 1.

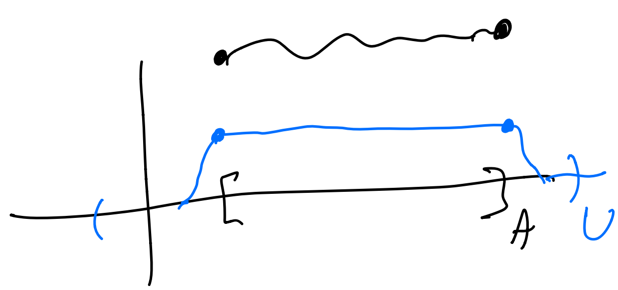
Def: $f: A \subseteq M \to \mathbb{R}$ is "smooth" if $\forall p \in A$ $\exists$ neighbd W_p of p in M and a smooth function $\tilde{f}: W_p \to \mathbb{R}$ s.t. $\tilde{f}|_{W_p \cap A} = f|_{W_p \cap A}$.

Proposition: Let S be a submanifold. $f \in C^\infty(S)$ iff it's "smooth" as defined above.

C^∞ extension lemma (stronger):

Let $f : A \to \mathbb{R}$ be a "smooth" function defined on a closed subset $A \subseteq M$.

Then $\exists\, \tilde{f} : M \to \mathbb{R}$ s.t. $\tilde{f} \in C^\infty(M)$ and $\tilde{f}|_A = f$



If U is an open set containing A, then we can choose $\tilde{f}$ s.t. $\mathrm{supp}(\tilde{f}) \subseteq U$

etc (not straightforward)

Post-Lecture Practice Questions

1) do all the exercises above.
2) If M is compact, then $\rho_\alpha \equiv 0$ for all α except finitely many
3) If $S \subseteq M$ is a closed submanifold, then $\forall f \in C^\infty(S)$, $\exists \tilde{f} \in C^\infty(M)$ s.t. $\tilde{f}|_S = f$. If S is not closed, the statement is false. Find a counterexample.
4) Let $f : \mathbb{R} \to \mathbb{R}$.
   1) If f is a polynomial, $f^{-1}(0)$ is finite set.
   2) If f is analytic, $f^{-1}(0)$ is a discrete set
   3) If f is C^∞, $f^{-1}(0)$ can be $\boxed{\text{any}}$ closed set.

5) Let $S \subseteq M$ is a subset of a manifold M s.t. $f \in C^\infty(M) \Rightarrow f$ is "smooth" on S. Does that imply S is a submanifold.

6) Let M be a compact smooth n-dim manifold. for $q \in M$, Let (U_q, ϕ_q) be a chart near q s.t. $\phi_q(U_q) = B_2(0)$
Let $B_q := \phi_q^{-1}(B_1(0))$.

Then $\{B_q : q \in M\}$ is an open cover, and so admits a finite subcover $\{B_{q_1}, \cdots, B_{q_m}\}$. Let $\rho_i : M \to \mathbb{R}$ be a smooth bump function s.t. $\rho_i|_{\overline{B_{q_i}}} \equiv 1$ and $\text{supp}(\rho_i) \subseteq U_{q_i}$.

Extend ϕ_{q_i} to all of M by defining $\tilde{\psi}_i(x) = \begin{cases} \rho_i(x)\,\phi_i(x), & x \in U_i \\ 0, & x \notin U_i \end{cases}$

Let $F : M \to \mathbb{R}^{nm+m}$

$: p \longmapsto \left(\psi_1(p), \ldots, \psi_m(p), \rho_1(p), \ldots, \rho_m(p) \right)$

Show that F is an embedding.

You have just proven the weak version of Whitney's embedding Theorem: Any manifold can be embedded in some Euclidean space.