Def #1 of a submanifold: $S \subseteq M$ is an embedded submanifold if it's a manifold s.t. $i: S \hookrightarrow M$ is an embedding.

① $i: S \to i(S)$ is a homeomorphism ($\Rightarrow$ S is equipped with the subspace topology)
② i is C^∞ ($\Rightarrow f \in C^\infty(M) \Rightarrow f|_S = f \circ i \in C^\infty(S)$)
③ i_* is injective ($\Rightarrow$ allows for a way to see T_pS as a subset of T_pM)

Tangent space of a submanifold

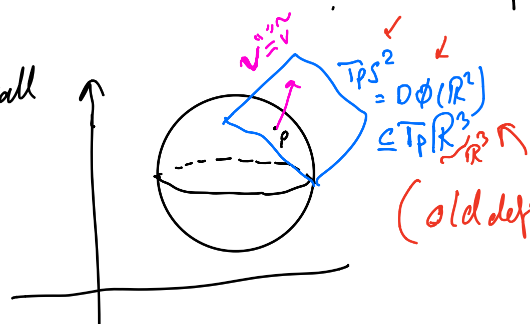
Let $S \subseteq M$ be a k-dim embedded submanifold in M

Let $p \in S$

What is T_pS? is $T_pS \subseteq T_pM$

doesn't make sense

Recall



$v(f|_S)$ ...
$\tilde{v}(f)$

Recall $T_pM = \{ D: C_p^\infty(M) \to \mathbb{R} \mid D \text{ is a derivation at } p \}$

S is a smooth manifold.

$T_pS = \{ D: C_p^\infty(S) \to \mathbb{R} \mid D \text{ is a derivation at } p \}$

Let $v \in T_pS$, can you think of $\tilde{v} \in T_pM$ that is "the same as" v?

Proposition: for every $v \in T_pS$, $\exists! \, \tilde{v} \in T_pM$ with the property

$$\forall f \in C^\infty(M), \quad \tilde{v}(f) = v(f|_S) = v(f \circ i) = i_{*,p}(v)(f)$$

Proof:

$$\iff \tilde{v} = i_{*,p}(v)$$

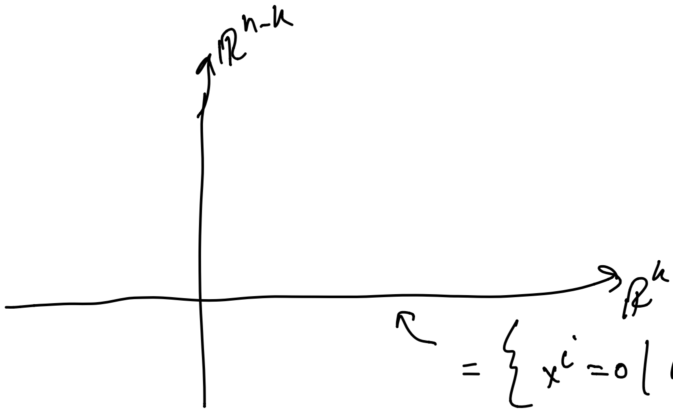
Since $i_{*,p} : T_pS \longrightarrow i_{*,p}(T_pS) \subseteq T_pM$ is an isomorphism (i is an immersion)

$i_{*,p}(T_pS)$ is thought of as T_pS but as seen as a k-dim subspace of T_pM.

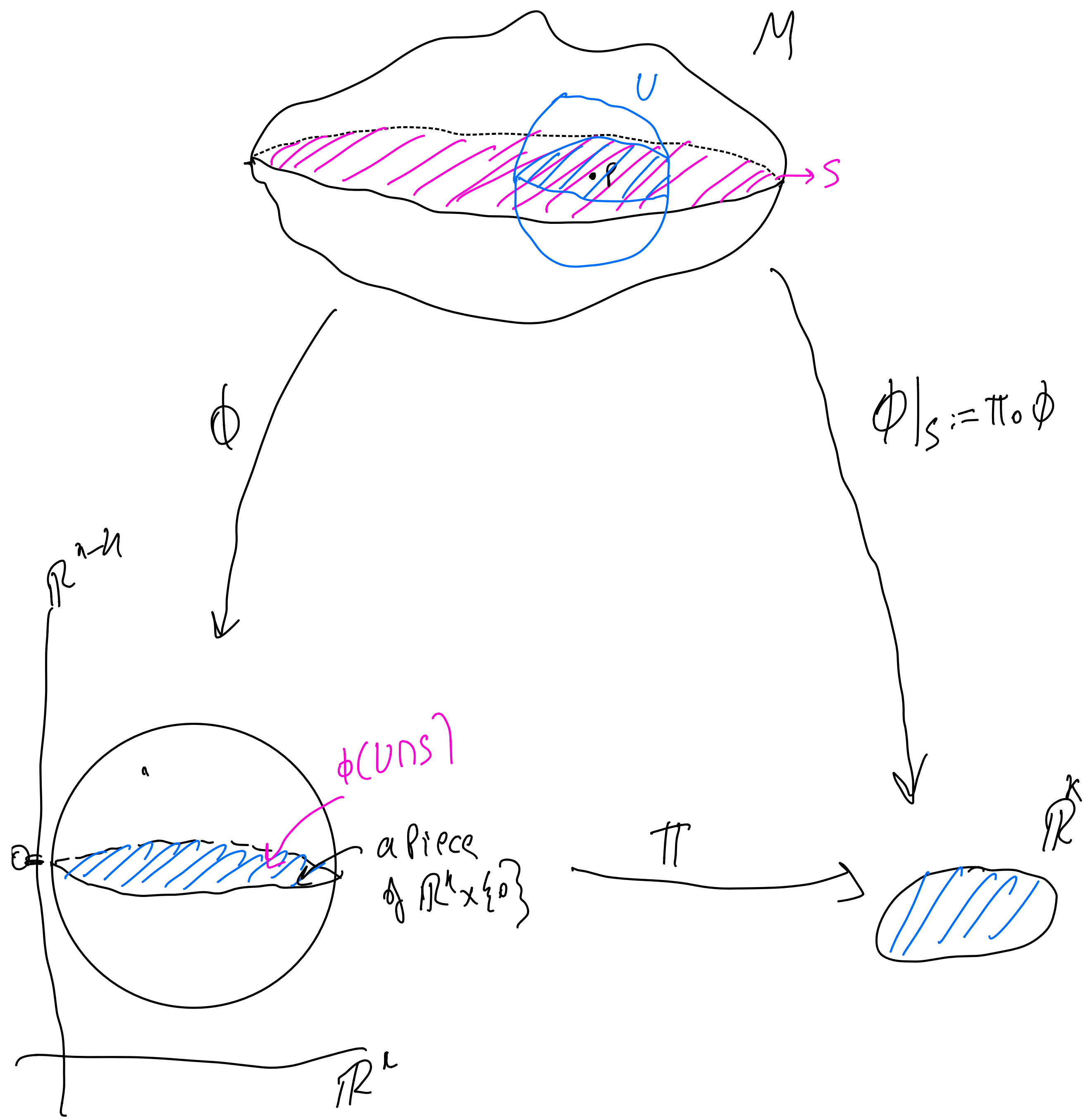
$i_{*,p}(T_pS)$ and T_pS will be used interchangeably.

Regular Submanifolds

Another approach (modelled after $\mathbb{R}^k \times \underbrace{\{0\}}_{\in \mathbb{R}^{n-k}} \subseteq \mathbb{R}^n$)



$$= \{ x^i = 0 \mid i = k+1, \dots, n \}$$



Def #2: $S \subseteq M$ is a regular submanifold of dim k if $\forall p \in S$ $\exists\, (U, \phi = (x^1, \dots, x^n))$ near p s.t. $U \cap S$ is defined by the vanishing of the last $n-k$ coordinates

$$(x^{k+1}, \dots, x^n)\Big|_q = 0 \quad \text{iff } q \in U \cap S$$

A chart (U, ϕ) like this is called an adapted chart relative to S.

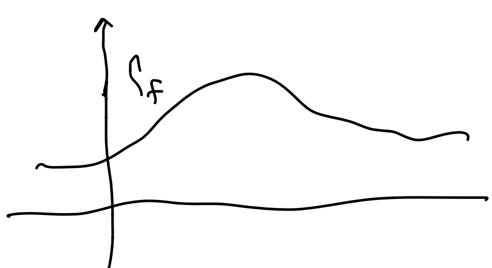
Note that $(U \cap S, \phi_S)$ is a chart on S

where $\phi_S = \pi \circ \phi : U \cap S \to \mathbb{R}^k$
$$p \longmapsto (x^1(p), \dots, x^k(p))$$

If $\{(U_\alpha, \phi_\alpha)\}$ is a collection of adapted charts relative to S covering S,

Then $\{(U_\alpha \cap S, \phi_{\alpha_S})\}$ make a C^∞ atlas on S making a smooth manifold of dim k.

Ex #1: Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function.



Find an adapted of $\mathbb{R}^2$ relative to Γ_f

$$\psi : (x, y) \longmapsto (x, y - f(x))$$

ψ is a diffeomorphism & so $(\mathbb{R}^2, \psi)$ is a chart on $\mathbb{R}^2$.

Also Γ_f is precisely defined by the vanishing of the last coordinate

so $\Gamma_f = \{ p \in \mathbb{R}^2 \mid r^2 \circ \psi(p) = 0 \}$

So $(\mathbb{R}^2, \psi)$ is an adapted chart relative to Γ_f making Γ_f a regular submanifold.

Thm: $S \subseteq M$ is an embedded submanifold if and only if it's a regular submanifold.

($\Rightarrow$) Use immersion Thm, will give you adapted charts

($\Leftarrow$) Use an adapted chart, then

$\Phi_S \circ i \circ \phi^{-1}$ is the canonical immersion

$\Rightarrow i$ is an embedding

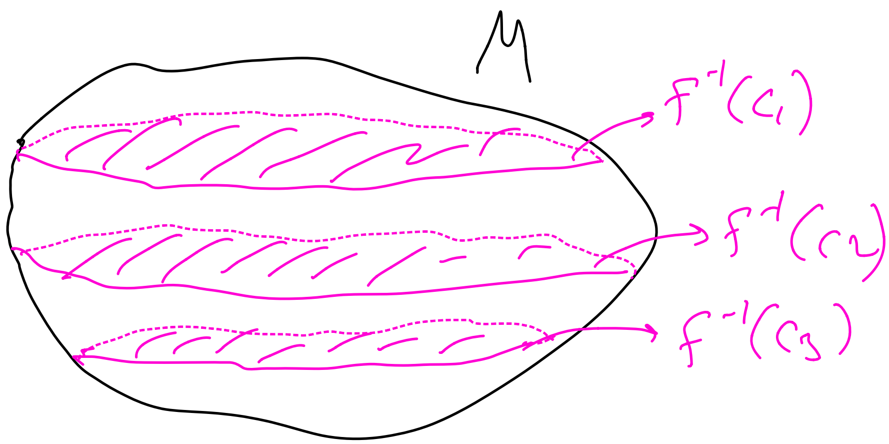
(Thm 11.13 and Thm 11.14)

A Third definition

through level sets.

Def: Let $f: N \to M$ be a C^∞ map. Let $c \in M$.
We say $F^{-1}(c) = F^{-1}(\{c\})$ is the level set of F with level c.

c is a critical value if $\exists p \in F^{-1}(c)$ s.t. $F_{*,p}$ is not surjective.
c is a regular value if it's not a critical value.
If so, $F^{-1}(c)$ is a regular level set.



Thm: Let $f: M \to \mathbb{R}^k$ s.t. 0 is a regular value and $f^{-1}(0) \neq \phi$, Then $F^{-1}(0)$ is a submanifold of M of codimension k. ($\dim = n-k$)

Proof: Let $p \in f^{-1}(0)$, Then $f_{*,p}$ is surjective

Since this is an open condition, Then $F_{*,p}$ is surjective on a neighborhood U of p. (F is a submersion on U)

Then by submersion Thm (Charles), $\exists$ a chart $(\tilde{U}, \phi)$ near p in M and a chart $(\tilde{V}, \psi)$ near 0 in $\mathbb{R}^k$ s.t. $\tilde{U} \subseteq U$

$$\psi \circ F \circ \phi^{-1} : (x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^k)$$

Let $(a^1, \ldots, a^k) = \psi(0)$

Then $f^{-1}(0) \cap \tilde{U} = \left\{ p \in \tilde{U} \mid x^i = a^i \text{ for } i = 1, \ldots, k \right\}$

Define the chart $\tilde{\phi}(p) = (x^{k+1}, \ldots, x^n, x^1 - a^1, x^2 - a^2, \ldots, x^k - a^k)$

Then $f^{-1}(0) \cap \tilde{U}$ is defined by the vanishing k coordinates making $(\tilde{U}, \tilde{\phi})$ an adapted chart near p relative to $f^{-1}(0)$ $\Rightarrow$ $f^{-1}(0)$ is a regular submanifold of cod k

$\boxtimes$

<u>Corollary</u>: <u>Regular Level Set Thm</u>:

Let $f : N \to M$ be a C^∞ map s.t. c is a regular value. Then $f^{-1}(c)$ is a submanifold of N of codim

$\neq \phi$

(what happens $n \leq m$)

Also $\text{Kernel}(f_{*,p}) = i_{*,p}(T_p S)$ "=" $T_p S$
(Proven Assignment 3 Problem 1b)

More general: Constant Rank Level Set Theorem

Let $f: N \to M$ be a C^∞ map and let $c \in N$.
If F is of constant rank K on a neighbhd of $f^{-1}(c)$,
then $f^{-1}(c)$ is a submanifold of Cod K.

Def: $S \subseteq M$ is a "levelset" submanifold of dim k if it's locally a regular level set of a map.

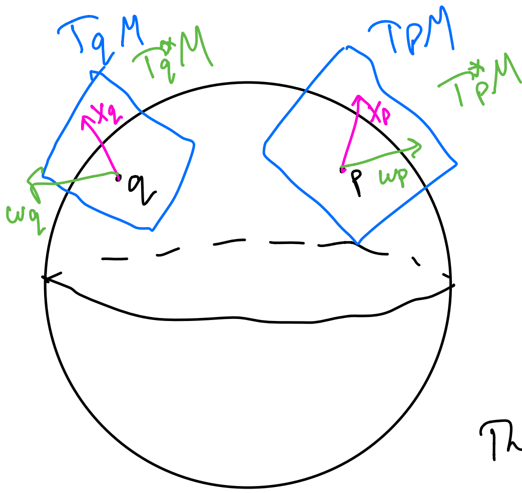
Precisely: if $\forall p \in S$, $\exists f: U \to \mathbb{R}^{n-k}$ a C^∞ map on a neighbhd U of p s.t. 0 is a regular value and $f^{-1}(0) = U \cap S$.

Thm:
$$\underbrace{\text{"levelset" submanifold} \iff \text{embedded submanifold} \iff \text{regular submanifold}}_{\text{submanifold.}}$$

Motivation

define: Vector fields, differential forms, Tensor fields, Riemannian metric

$\to \mathbb{R}^n$ $\quad \to \mathbb{R}^n$ $\quad f: \mathbb{R}^n \to \mathbb{R}^n$



Want to make a "smooth choice" of a vector at each point p.

Want to make a "smooth choice" of a dual vector at each point p.

This will be called a differential 1-form

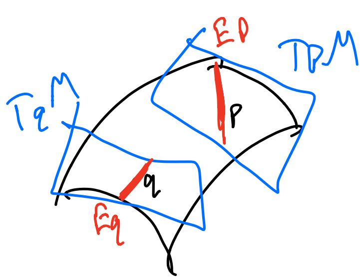
a (k,l) tensor on $\overbrace{T_pM}^{V}$ is

$$T_{p\,(k,l)} : \overbrace{T_p^*\times\cdots\times T_p^*M}^{k} \times \overbrace{T_pM \times\cdots\times T_pM}^{l} \longrightarrow \mathbb{R}$$

and is multilinear

We want to make a "smooth choice" of a tensor $T_{p\,(k,l)}$ at each point. Call it T (smooth tensor field)

A differential k-form is $(0,k)$ smooth tensor field satisfying . . .



Want to make a "smooth choice" of a k-dim subspace E_p of T_pM at each point $p \in M$.

does there exist a submanifold S of M s.t. $i_{*,p}(T_pS) = E_p$?

Answered by Frobenius Thm.

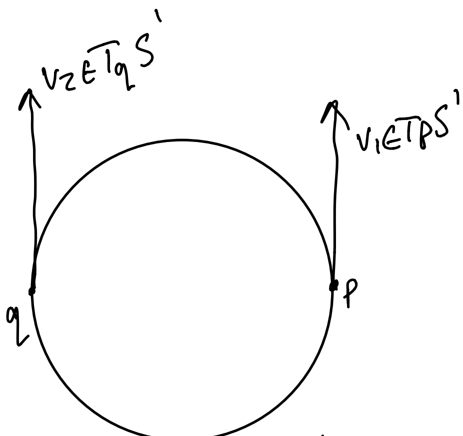
Tangent Bundle

(The key to the 3rd step of generalizing Calculus)

<u>Def</u>: Let M be a C^∞ manifold.
The tangent bundle denoted by TM is defined by

$$TM = \bigsqcup_{p \in M} T_pM := \bigcup_{p\in M} \{p\} \times T_pM$$

$$= \{ (p,v) : p \in M, \ v \in T_pM \}$$



$v_1, v_2 \in TS^1$
$(p,v_1), (q,v_2) \in TS^1$

We have the natural map

$$\pi : TM \to M$$
$$: (p,v) \mapsto p$$

(Not the coarsest topology that makes π continuous)

Let (U,ϕ) be a chart on M. Then $TU = \bigcup_{p\in U} T_pM$

$$\phi : U \to \phi(U) \subseteq \mathbb{R}^n$$
$$p \mapsto (x^1, \ldots, x^n)$$

for $p \in U$, then $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is the coordinate basis of T_pM.

for any $v \in T_pM$, $v = \sum_{i=1}^{n} \underset{v(x^i)}{c^i} \frac{\partial}{\partial x^i}\Big|_p$ where $c^i : TU \to \mathbb{R}$
$: v \mapsto v(x^i)$

Define $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$
$(p, v) \mapsto (x^1, \ldots, x^n, c^1, \ldots, c^n)$

$\tilde{\phi}$ is bijective. $\tilde{\phi}^{-1} : (\phi(p), c^1, \ldots, c^n) \mapsto \sum_{i=1}^{n} c^i \frac{\partial}{\partial x^i}\Big|_p$

Equip TU with the unique topology that makes $\tilde{\phi}$ a homeomorphism.

Define $\mathcal{T} = \left\{ A \subseteq TM \;\middle|\; \begin{array}{l} A \text{ is open in } TU_\alpha \text{ for} \\ \text{every coordinate open set } U_\alpha \end{array} \right\}$

<u>Proposition:</u> $\mathcal{T}$ is a topology on TM

<u>Proposition:</u> 1) $\pi : TM \to M$ is continuous   exc
2) π is an open map.   exc

<u>Proposition:</u> TM is second countable & Hausdorff
exc

$\Rightarrow TM$ is a topological manifold of dim $2n$.

Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M

We want to show $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas on TM.

Let $(TU_\alpha, \tilde{\phi}_\alpha)$ and $(TU_\beta, \tilde{\phi}_\beta)$ be charts

For $v \in T_pM$, $v = \underbrace{\sum c^i \frac{\partial}{\partial x^i}\Big|_p}_{(U_\alpha,\phi_\alpha)} = \underbrace{\sum_{i=1}^n b^i \frac{\partial}{\partial y^i}\Big|_p}_{(U_\beta,\phi_\beta)}$

Then

$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} : (x^1, \dots, x^n, c^1, \dots, c^n) \longmapsto (y^1, \dots, y^n, b^1, \dots, b^n)$

$\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞

independent of c^i

is C^∞ because $D(\phi_\beta \circ \phi_\alpha^{-1})\big|_{(x^1,\dots)}$ is C^∞ wrt $(x^1, \dots, x^n)$

is C^∞ since the relation is linear

$$b^i = v(y^i) = \sum_{k=1}^n c^k \frac{\partial}{\partial x^k}\Big|_p (y^i) = \sum_{k=1}^n c^k \frac{\partial y^i}{\partial x^k}\Big|_p$$

$$= \sum_{k=1}^n c^k D(\phi_\beta \circ \phi_\alpha^{-1})\Big|_{\phi_\alpha(p)}$$

$$\begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} = \underbrace{D(\phi_\beta \circ \phi_\alpha^{-1})\Big|_{(x^1,\dots,x^n)}}_{C^\infty} \underbrace{\begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix}}_{C^\infty}$$

$\Rightarrow \tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ is C^∞

& so $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ is a C^∞ atlas on TM making it a smooth manifold of dim $2n$.

<u>Proposition</u>: $\pi : TM \to M$ is C^∞

$$\tilde{\phi} \circ \pi \circ \phi^{-1} : \underbrace{(x^1, \ldots, x^n, c^1, \ldots, c^n)}_{(p\ ,\ v)} \longmapsto \underbrace{(x^1, \ldots, x^n)}_{p}$$

$c^1, \ldots, c^n$ is C^∞.

<u>Proposition</u> Let $f : N \to M$ be a C^∞ map,

The global differential of f is $f_* : TN \to TM$ is C^∞

$: (p, v) \longmapsto (f(p), f_{*,p}(v))$

Coordinate representation of f_* :

$$\tilde{\psi} \circ f_* \circ \tilde{\phi}^{-1} : (x^1, \ldots, x^n, c^1, \ldots, c^n) \longmapsto \Big(\overbrace{F^1, \ldots, F^n}^{C^\infty}, \overbrace{\underbrace{[f_{*,p}]}_{(x^1, \ldots, x^n)} \begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix}}^{C^\infty} \Big)$$

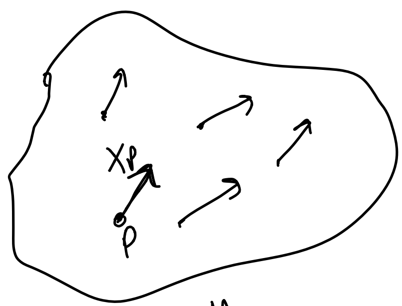
is C^∞

$f_{*,p}(v)$

$(x^1, \ldots, x^n)$ $= \begin{bmatrix} c^1 \\ \vdots \\ c^n \end{bmatrix}$

$F^i = y^i \circ f$

Vector fields



$p \mapsto X_p \in T_pM$

Choice of vector at each point $p \in M$

$X : p \mapsto X_p \in T_pM$

$X : M \to TM$!!
$p \mapsto (p, X_p)$

Def: A section of the tangent bundle is a map $X : M \to TM$ s.t. $\pi \circ X = Id_M$

$$\pi \circ X(p) = \pi(X_p) = p$$

A section is smooth if $X : M \to TM$ is smooth !!

A section is called a vector field on M

A smooth section is called a smooth vector field on M.

Proposition: Let X, Y be C^∞ sections of TM.

Then

1) Define $X+Y : M \to TM$ defined by (C^∞ section)

$$X+Y(p) = X_p + Y_p$$

2) $\forall f \in C^\infty(M)$ define $fX : M \to TM$ (C^∞ section)

defined by $fX(p) = f(p) \cdot X_p$

exc

Denote $\Gamma(M)$ as the space of all C^∞ sections.

The previous proposition implies

$\Gamma(M)$ is
1) vectorspace over $\mathbb{R}$
2) Module over the ring $C^\infty(M)$

Post-lecture Practice Questions

1) Do all the above exercises

2) Let $F: N \to M$ be a submersion. Then $F^{-1}(c)$ is a submanifold of dim $n-m$. what happens when $n \le m$?

3) Application of Regular level set Thm : Problem 9.1, 9.2, 9.3

4) Show that $\{ A \in Mat_{2\times 2}(\mathbb{R}) \mid \|A\|^2 = 1,\ \det(A) = 0 \}$ is a 2-dim submanifold of $Mat_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$

5) If (U, ϕ) is a chart near p and $v \in T_pM$, show that $v = \sum_{i=1}^{n} v(x^i) \frac{\partial}{\partial x^i}\Big|_p$

6) If M is diffeomorphic to $\mathbb{R}^n$, show
TM is diffeomorphic to $\mathbb{R}^{2n}$
In General, show TM is locally diffeomorphic to $M \times \mathbb{R}^n$

7) let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a vector field on $\mathbb{R}^n$.
Describe it as a section of $T\mathbb{R}^n$.

8) for $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = y^2 + x^3 + 5x + 9$
Find $A = \{ (a,b) \mid 0 \text{ is a critical value of } f \}$
sketch typical levelsets $f^{-1}(0)$ for $(a,b) \notin A$.

9) show That if X is a smooth section of S^2, Then it
must vanish somewhere.