MAT 367: Differential Geometry
Exam
Thursday, August 19

Instructions:

1. Submit by 11:30 AM through Crowdmark. No late submissions will be accepted.
2. There are 7 problems in this exam.
3. Choose only 4 out of the first 5 problems. **Do not submit the problem that you choose to skip.**
4. Problem 6 is mandatory.
5. Problem 7 is a bonus.
6. You can get up to 58/50 in this exam.

Problem 1 [10]

Consider the vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ and the 2-form $\omega \in \Omega^2(\mathbb{R}^3)$ defined by

$$X = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad \omega = (x^2 + y^2)dx \wedge dz$$

(a) Compute $\mathcal{L}_X\omega$ by first computing the flow F of X and then taking the derivative of $F_t^*\omega$.

(b) Compute $\mathcal{L}_X\omega$ by using Cartan's magic formula $\mathcal{L}_X = d\iota_X + \iota_X d$.

(c) Let $\alpha \in \Omega^1(\mathbb{R}^2)$ be defined by $\alpha = (x - y^3)dx + x^3dy$. Compute $\int_{S^1} i^*\alpha$ where $i : S^1 \hookrightarrow \mathbb{R}^2$ is the inclusion map.

Hint: Apply Stokes theorem on the unit disk $D := \{(x, y) : x^2 + y^2 \leq 1\}$.

Problem 2 [10]

Let $\omega_1, \omega_2 \in \Omega^1(M)$ be non-vanishing 1-forms that are linearly independent at every point. Define a distribution Δ by $\Delta_p = \mathrm{Ker}\omega_p^1 \cap \mathrm{Ker}\omega_p^2$ for $p \in M$. From problem 4 in Assignment 7, we know that Δ is a smooth rank $n-2$ distribution .

(a) Let $\eta \in \Omega^2(M)$. Show that $\eta \wedge \omega^1 \wedge \omega^2 = 0$ iff η annihilates Δ

i.e. $\eta(X, Y) = 0$ for any local frame X, Y of Δ.

(b) Suppose $\eta \in \Omega^2(M)$ satisfies $\eta \wedge \omega^1 \wedge \omega^2 = 0$. Show that $\eta = \omega^1 \wedge \beta^1 + \omega^2 \wedge \beta^2$ for some 1-forms $\beta^1, \beta^2 \in \Omega^1(M)$.

(c) ***(bonus)*** [**2**] Suppose $d\omega^1 \wedge \omega^1 \wedge \omega^2 = 0$ and $d\omega^2 \wedge \omega^1 \wedge \omega^2 = 0$. Show that ω^1 and ω^2 are locally a linear combination of 2 non-vanishing exact 1-forms df_1 and df_2 for some $f_1, f_2 \in C^\infty(M)$.

i.e. For any p, there exists a neighbourhood U of p and functions $g_1, g_2, h_1, h_2 \in C^\infty(U)$ such that $\omega^1 = g_1 df_1 + g_2 df_2$ and $\omega^2 = h_1 df_1 + h_2 df_2$ on U.

Problem 3 [10]

Let $F : M \to N$ be a surjective submersion. Define a distribution Δ on M by $\Delta_p := \mathrm{Ker} F_{*,p}$ for $p \in M$

(a) Show that Δ is a smooth rank $m - n$ distribution that is involutive.

Hint: Apply the submersion theorem.

(b) Let $\omega \in \Omega^k(N)$ and $\eta := F^*\omega$. Show that $\iota_X \eta = 0$ and $\mathcal{L}_X \eta = 0$ for all sections X of Δ.

(c) Suppose $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, and F is the canonical submersion. Let $\eta \in \Omega^k(M)$ such that $\iota_X \eta = 0$ and $\mathcal{L}_X \eta = 0$ for all sections X of Δ. Show that there exists a k-form $\omega \in \Omega^k(N)$ such that $\eta = F^*\omega$.

(d) ***(bonus)*** [2] Under the assumptions of (c), show that if $\omega' \in \Omega^k(N)$ also satisfies $\eta = F^*\omega'$, then $\omega' = \omega$.

Problem 4 [10]

Let $M := \overline{B_1} \subset \mathbb{R}^{n+1}$ be the closed unit ball with the standard orientation. Let $\omega \in \Omega^n(\partial M)$ be defined by $\omega := i^* \left(\sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \right)$ where $i : \partial M \hookrightarrow M$ is the inclusion map.

(a) Show that ω is an orientation form for $\partial M = S^n$ with the boundary orientation and that

$$\int_M \mathcal{L}_X(dx^0 \wedge ... \wedge dx^n) = \int_{S^n} \omega = (n+1)\mathrm{Vol}(M)$$

where $X = \sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}$ is the outward vector field along ∂M and $\mathrm{Vol}(M)$ is the volume of the unit ball in $\mathbb{R}^{n+1}$.

(b) Let $F : M \to \partial M$ be a smooth map. Show that $F|_{\partial M} : \partial M \to \partial M$ cannot be the identity.

Hint: Show first that $\int_M dF^*\omega = 0$, and arrive to to a contradiction.

Problem 5 [10]

Let $M := [0,1] \times S^n$ with the standard orientation. Let $f_0, f_1 : S^n \to S^n$ be smooth maps and let $F : M \to S^n$ be a smooth map satisfying $F(0,\cdot) = f_0$ and $F(1,\cdot) = f_1$.

(a) Describe the boundary orientation of ∂M.

(b) Show that for any $\omega \in \Omega^n(S^n)$, $\displaystyle\int_{S^n} f_0^*\omega = \int_{S^n} f_1^*\omega$.

Hint: Integrate $dF^*\omega$ on M.

(c) ***(bonus)*[2]** Show that if $f_0 = Id$ and $f_1 : x \mapsto -x$ is the antipodal map, then n cannot be even.

Problem 6 [10]

Are the following true or false? Justify your answer briefly.
2 marks each. 10 is the maximum mark

(a) Let $X, X_1, ..., X_k \in \mathfrak{X}(M)$ such that $[X, X_i] = 0$ for $i = 1, ..., k$. Then for any $\omega \in \Omega^k(M)$, $\mathcal{L}_X\omega(X_1, ..., X_k) = X\big(\omega(X_1, ..., X_k)\big)$.

(b) Let M be a compact oriented 3-manifold. Define the map $F : \Omega^1(M) \times \Omega^1(M) \to \mathbb{R}$ defined by $F(\omega, \eta) = \int_M \omega \wedge d\eta$. F is bilinear and symmetric.

(c) Any rank 2 smooth distribution Δ on S^3 admits a global frame.

(d) Let S be a submanifold of M such that $S = f^{-1}(0)$ for some $f \in C^\infty(M)$. Then $X \in \mathfrak{X}(M)$ is tangent to S if and only if $X(f) = 0$.

(e) Let $\omega \in \Omega^1(M)$ be non-vanishing. Then for any $\eta \in \Omega^1(M)$, $\eta \wedge \omega = 0$ if and only if $\eta = f\omega$ for some $f \in C^\infty(M)$.

(f) Let $\pi : S^n \to \mathbb{R}P^n$ be the usual projection map and let $F : S^n \to S^n$ be the antipodal map $F : x \mapsto -x$. If $X \in \mathfrak{X}(S^n)$ is π-related to $Y \in \mathfrak{X}(\mathbb{R}P^n)$, then $F_*X = X$.

Problem 7 *(bonus)* [2]

Let $\omega \in \Omega^k(M)$. For $X_1, ..., X_{k+1}$, show that

$$d\omega(X_1, ..., X_{k+1}) = \frac{1}{2}\sum_{i=1}^{k+1}(-1)^{i+1}\left[X_i\left(\omega(X_1, ..., \widehat{X_i}, ..., X_{k+1})\right) + \mathcal{L}_{X_i}\omega(X_1, ..., \widehat{X_i}, ..., X_{k+1})\right]$$

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