MAT 367: Differential Geometry
Exam
Monday, August 24

Instructions:

1. Submit by 11:30 AM through Crowdmark. No late submissions will be accepted.
2. There are 7 problems in this exam.
3. Choose only 4 out of the first 5 problems. **Do not submit the problem that you choose to skip.**
4. Problem 6 is mandatory.
5. Problem 7 is a bonus.
6. You can get up to 56/50 in this exam.

Problem 1 [10]

(a) Consider the vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^2)$ defined by:

$$X = x\frac{\partial}{\partial x}, \qquad Y = y\frac{\partial}{\partial y}$$

Show that they commute and verify by direct computation that their flows commute.

(b) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by $F(x, y) = (xy, e^x)$ and let $\omega \in \Omega^1(\mathbb{R}^2)$ be the 1-form defined by $\omega = xdy$. Verify by direct computation that $F^*d\omega = dF^*\omega$.

(c) Let $\eta \in \Omega^2(\mathbb{R}^3)$ be the closed 2-form defined by $\eta = 2xy^2 dx \wedge dy + zdy \wedge dz$. Verify Poincaré lemma by showing that η is exact. (Find a 1-form $\omega \in \Omega^1(\mathbb{R}^3)$ such that $\eta = d\omega$.)

Hint: Assume $\omega = gdy$ for some $g \in C^\infty(\mathbb{R}^3)$.

Remark: Keep in mind that you do not have to show all your rough work. So you can say directly "the flow of X and Y are ..."

Problem 2 [10]

Let M be a 3 dimensional manifold. Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$ be a vector field and a 1-form satisfying:

$$\iota_X \omega = 1, \qquad \mathcal{L}_X \omega = 0$$

Suppose also that $d\omega$ is nowhere vanishing.

(a) Show that $\omega \wedge d\omega$ is a nowhere vanishing 3-form.

(b) Show that the smooth rank 2 distribution Δ associated to ω is not involutive. Conclude that $\omega \wedge d\omega(Y, Z, [Y, Z])$ is a nowhere vanishing smooth function for any local basis Y, Z of Δ.

(c) Show that there cannot exist a nowhere vanishing function $f \in C^\infty(M)$ such that $d(f\omega) = 0$.
Hint: Compute $\omega \wedge d\omega$.

(d) Suppose $M = S^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$. Define $\alpha \in \Omega^1(\mathbb{R}^4)$ by:

$$\alpha = xdy - ydx + zdw - wdz$$

For $\omega = i^*\alpha$ where i is the inclusion map, show that $\omega \wedge d\omega$ is a nowhere vanishing 3-form on S^3.
Hint: Define the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}$ and use 2a.

Problem 3 [10]

(a) Let $f^1, ..., f^k \in C^\infty(M)$ such that $df^1 \wedge ... \wedge df^k$ is a nowhere vanishing k-form. Show that for each $p \in M$, there exists a coordinate chart such that $f^i = x^i$ for $i = 1, ..., k$. *Hint:* Use the submersion theorem.

(b) Let $\omega^1, ..., \omega^k \in \Omega^1(M)$ be closed 1-forms such that $\omega^1 \wedge ... \wedge \omega^k$ is a nowhere vanishing k-form. Show that near each point, there exists a coordinate chart such that $\omega^i = dx^i$. Conclude that the 1-forms define a smooth involutive distribution of codimension k.

Problem 4 [10]

Let M be a compact manifold with boundary. Let $\omega \in \Omega^n(M)$ be a nowhere vanishing top form on M. Define the map $div_\omega : \mathfrak{X}(X) \to C^\infty(M)$ in the following way: if $X \in \mathfrak{X}(M)$, then $div_\omega(X)$ is the function on M satisfying $\mathcal{L}_X\omega = div_\omega(X)\omega$. This map is well defined thanks to question 3c in assignment 7.

Let N be a smooth outward pointing vector field along ∂M. Define another map $< N, \cdot >_\omega: \mathfrak{X}(M) \to C^\infty(\partial M)$ in the following way: If $X \in \mathfrak{X}(M)$, then $< N, X >_\omega$ is the function on ∂M satisfying $i^*\iota_X\omega =< N, X >_\omega i^*\iota_N\omega$ where $i : \partial M \hookrightarrow M$ is the inclusion map.

(a) Show that the map $< N, \cdot >_\omega$ is well defined. That is show that for every $X \in \mathfrak{X}(M)$, there exists a unique function $< N, X >_\omega$ such that $i^*\iota_X\omega =< N, X >_\omega i^*\iota_N\omega$.

(b) Prove the "divergence theorem": For any $X \in \mathfrak{X}(M)$

$$\int_M div_\omega(X)\omega = \int_{\partial M} < N, X >_\omega i^*\iota_N\omega$$

(c) Show that $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df(X)\omega$ for any $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

(d) Prove the "integration by parts" formula: for any $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$,

$$\int_M f div_\omega(X)\omega = \int_{\partial M} f < N, X >_\omega i^*\iota_N\omega - \int_M df(X)\omega$$

(e) ***(bonus [2])*** Let $M = \overline{B}_1$ be the closed unit ball in $\mathbb{R}^n$. Let $\omega = dx^1 \wedge ... \wedge dx^n$ and $N = x^i \frac{\partial}{\partial x^i}$. Show that the "divergence theorem" implies the divergence theorem:

$$\int_{\overline{B}_1} \nabla \cdot X dx^1 ... dx^n = \int_{S^{n-1}} X \cdot N dA$$

and the "integration by parts" formula implies the integration by parts formula:

$$\int_{\overline{B}_1} f\nabla \cdot X dx^1 ... dx^n = \int_{S^{n-1}} f X \cdot N dA - \int_{\overline{B}_1} \nabla f \cdot X dx^1 ... dx^n$$

Problem 5 [10]

Let $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ be the $n-1$-form defined by

$$\omega = \frac{1}{||x||^n} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

(a) Show that $i^*\omega$ is a nowhere vanishing closed $n-1$-form on the sphere S_r^{n-1} of radius $r > 0$, where $i : S_r^{n-1} \hookrightarrow \mathbb{R}^n$ is the inclusion map.

(b) Show that

$$\int_{S_r^{n-1}} i^*\omega = \frac{n}{r}\mathrm{Vol}(B_r) = \mathrm{SA}(S_r^{n-1})$$

where $\mathrm{Vol}(B_r)$ is the volume of the ball of radius r and $\mathrm{SA}(S_r^{n-1})$ is the surface area of S_r^{n-1}.
Hint: Consider the $n-1$-form $\eta = ||x||^n\omega$. Use stokes theorem.

(c) Show that ω cannot be exact on $\mathbb{R}^n \setminus \{0\}$.

(d) ***(bonus)*** **[2]** Find what is wrong with the following argument: Consider $M = \overline{B}_r \setminus \{0\}$. Then $\partial M = S_r^{n-1}$. Then by Stokes theorem,

$$\int_{S_r^{n-1}} i^*\omega = \int_M d\omega = 0$$

Problem 6 [10]

Are the following true or false? Justify your answer briefly.
2 marks each. 10 is the maximum mark

(a) Any nowhere vanishing vector field is a coordinate vector field near each point. Similarly, any nowhere vanishing 1-form is a coordinate 1-form near each point.

(b) Let $v \in T_pM$ and let $X \in \mathfrak{X}(M)$. Let γ be the integral curve of X starting at p. Then there exists a unique vector field Y on γ such that $Y|_p = v$ and $\mathcal{L}_X Y = 0$.

(c) Let F be the flow of a vector field X and let Y be another vector field. Then for $p \in M$ and an appropriate number t, $Y|_{F_t(p)} = (F_t)_{*,p}\, Y|_p + t(F_t)_{*,p}\, \mathcal{L}_X Y|_p + \frac{t^2}{2}(F_t)_{*,p}\, \mathcal{L}_X \mathcal{L}_X Y|_p + o(t^2)$.

(d) Suppose M is an oriented manifold with boundary. Let $\omega \in \Omega^{n-1}(M)$ with compact support inside a chart $(U, \phi = (x^1, ..., x^n))$. Let $(U \cap \partial M, \phi_{\partial M} = (x^1, ..., x^{n-1}))$ be the corresponding chart on ∂M. Then by Stokes theorem,

$$\int_{\phi(U)} d\omega \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n} \right) (x^1, ..., x^n) dx^1 ... dx^n$$

$$= \int_{\phi_{\partial M}(U \cap \partial M)} \omega \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^{n-1}} \right) (x^1, ..., x^{n-1}, 0) dx^1 ... dx^{n-1}$$

(e) Let $\alpha : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a C^∞-linear map. Then for every $p \in M$, there exists a linear map $\alpha_p : T_pM \to T_pM$ such that $\alpha(X)_p = \alpha_p(X_p)$.

(f) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x$. Define a vector field on $\mathbb{R}$ by $X = x^2 \frac{d}{dx}$. There exists a unique vector field $Y \in \mathfrak{X}(\mathbb{R}^2)$ that is f-related to X.

Problem 7 *(bonus)* [2]

Let $X, Y_1, ..., Y_k \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$ and $f \in C^\infty(M)$. Show that

$$\mathcal{L}_{fX}\omega(Y_1, ..., Y_k) = f\mathcal{L}_X\omega(Y_1, ..., Y_k) + \sum_{i=1}^{k} (\mathcal{L}_{Y_i} f)\, \omega(Y_1, ..., Y_{i-1}, X, Y_{i+1}, .., Y_k)$$

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