MAT327 TUT #9

Plan:
- Alternative proof of Urysohn metrization (exercise 6 of Wednesday lecture)
- Embedding Thm (Thm 34.2 of Munkres)
- A characterization of complete regularity (Thm 34.3 of Munkres)
- Tietze extension (to $\mathbb{R}$), ex 7 of Wednesday

Recap: In the proof of Urysohn metrization (normal + second countable $\Rightarrow$ metrizable)

$\{U_n\}$ countable basis

$A := \{(n,m) \mid \overline{U}_n \subset U_m\}$

By Urysohn lemma, $\exists \{f_{n,m}\}_{(n,m)\in A}$ such that $f_{n,m}: X \to [0,1]$

$f_{n,m}(\overline{U}_n) = \{1\}$, $f_{n,m}(U_m^c) = \{0\}$

$$d(x,y) := \sum_{n,m \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|$$

Alternatively ...

Ex 6. Re-index $\{f_{n,m}\}_{(n,m)\in A}$ as $\{g_n\}_{n\in\mathbb{N}}$

Let $F(x) = (g_n(x))_{n\in\mathbb{N}}$

$F: X \to \mathbb{R}^{\mathbb{N}}$ (product topology)

Show F is open.
Conclude F is an embedding.

Proof:

Property: Given x_0 and U, $\exists g_N$. s.t. $g_N(x_0) > 0$ and vanishes outside U

($f_{n,m}$) : $\exists B_m$, $x_0 \subset B_m \subset U$

B_m basis element

B_n, $x_0 \subset B_n$, $\overline{B}_n \subset B_m$

$g_N(x_0) > 0$, and vanishes $\sim$

☆ "Separates points from closed sets"

for family $\{g_n\}$ and space X

Recall: Embedding: homeo onto image

F

Cont.: Easy. componetwise

Open: (later)

Injective: ✓

$$\boxed{F(x) = (g_1(x), g_2(x), \dots)}$$

$x \neq y \in X$,

By ☆, $g_N(x) > 0$, $g_N(y) = 0$

$F(x) \neq F(y)$ diff in N-th

$F: X \to F(X) \subset \mathbb{R}^{\mathbb{N}}$ metric

Open: Fixed U open in X. $F(U)$ open

For any $z_0 \in F(U)$, $z_0 = F(x_0)$

we want to find W open

$z_0 \in W \subset F(U)$

First, choose g_N s.t.

$g_N(x_0) > 0$, vanishes outside U

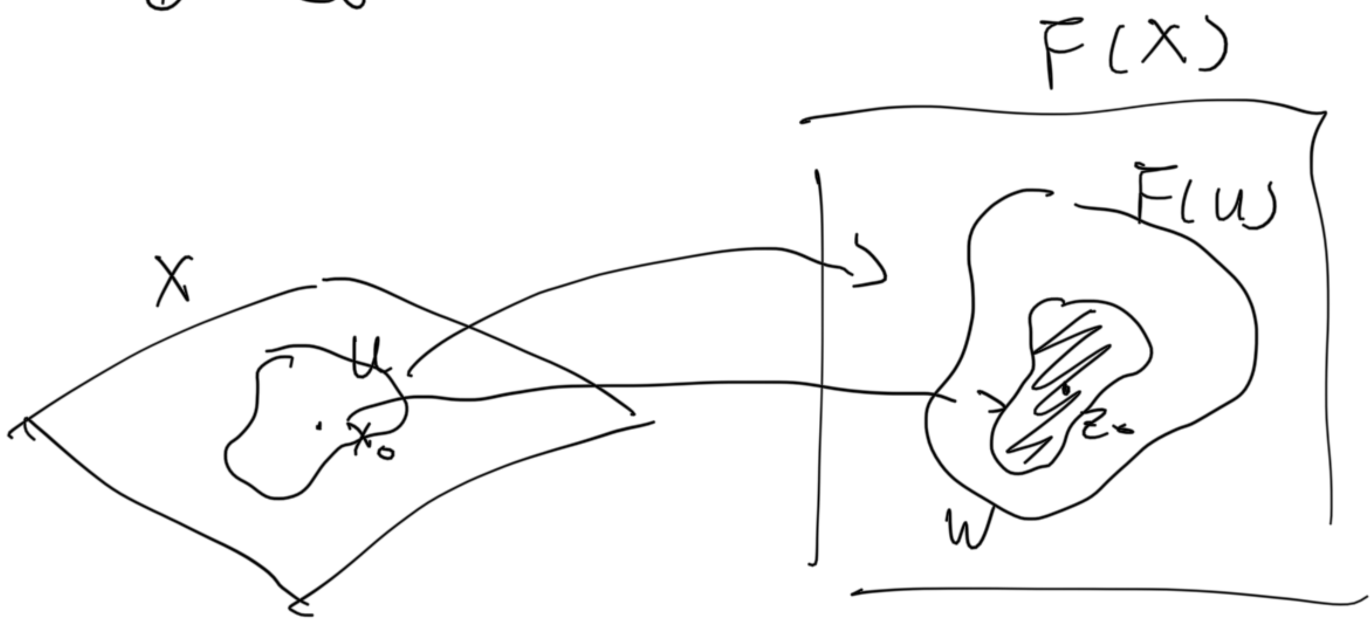
by ☆

$V = \pi_N^{-1}((0, \infty))$

$W = V \cap F(X)$

I claim: $\widetilde{W}$ is what we want.

① $z_0 \in W$



$$\underbrace{\pi_N(z_0) = \pi_N}(F(x_0))$$
$$= g_N(x_0) > 0$$

$z_0 \in V$, $z_0 \in F(X)$

$z_0 \in W$

② $W \subset F(U)$

Any $z \in W$, $\cancel{F(X)}$

$\exists x$ s.t. $z = F(x)$

$$\underbrace{\pi_N(z)}_{>0} = \pi_N(F(x))$$
$$= g_N(x) > 0$$

But

g_N vanishes outside U

$\Rightarrow x \in U$

$\Rightarrow z = F(x) \in F(U)$

$\square$

Observation:

① This proof does not rely on $J = \mathbb{N}$ countabe.

Can use $\{f_\alpha\}_{\alpha \in J}$, $\mathbb{R}^J$

② Assuming $\exists$ family satisfying $\star$,
we don't need much X.
But we do need single points
are closed.

Thm: (Embedding 34.2)

X single points are closed
and $\{f_\alpha\}_{\alpha \in J}$ cont.
and satisfies $\star$

Then, $F: X \to \mathbb{R}^J$
$F(x) = (f_\alpha(x))_{\alpha \in J}$
is an embedding of X in $\mathbb{R}^J$.

If $f_\alpha : X \to [0,1] \ \forall \alpha$,
then F embeds X into $[0,1]^J$

Proof: Same as before.

Thm: (34.3)

X is completely regular
$\Leftrightarrow$ it is homeomorphic to
a subspace of $[0,1]^J$ (for some J)

"$\Leftarrow$": Easy

"$\Rightarrow$": Single points are closed ✓

$\boxed{\text{Have } \{f_\alpha\}_{\alpha \in J} \text{ that separates points from closed sets}}$

Completely regular:

Def. For x and closed A, $x \notin A$

we have f

s.t. $f(x) = 1$, $f(A) = \{0\}$

Equivalent to ☆

$x \in A^c$ open

We have $F : X \to [0,1]^J$

$\square$

Ex 7. X normal, A closed

$\underbrace{f : A \to \mathbb{R}}$

$\Downarrow$ extend

$\tilde{f} : X \to \mathbb{R}$

In lecture, we saw

$f : A \to [a,b]$

$\nearrow$ cont. $\tilde{f} : X \to [a,b]$

a) Use Tietze thm,

$\tilde{h} : X \to [-1,1]$, $\boxed{h = \dfrac{f}{1+|f|}}$

h : ① cont. $1+|f| > 0$

② $h : A \to [-1,1]$

$\tilde{h} : X \to [-1,1]$

b) Use Urysohn's lemma,

$\phi : X \to [0,1]$.

s.t. $\phi(A) = \{1\}$

$\boxed{\phi(\tilde{h}^{-1}(1)) = \{0\}}$

$\boxed{\phi = 1 \text{ on } A}$

Note h does not attain ± 1

$\longrightarrow \tilde{h} : X \to \pm 1$ on A

(3) $\tilde{f} : X \to \mathbb{R}$

$$\tilde{f} = \frac{\phi \tilde{h}}{1 - \phi \tilde{h}}$$

$\phi, \tilde{h}$ cannot multiply to 1

$\tilde{f}$ ✓

Check $\tilde{f}$ extends f

Inside A,

$$\tilde{f} = \frac{\phi h}{1 - \phi h} = \frac{h}{1-h} = \frac{\frac{f}{1+|f|}}{1 - \frac{f}{1+|f|}} = \frac{f}{1+|f| - f}$$

... need to handle the case $f<0$ with more care