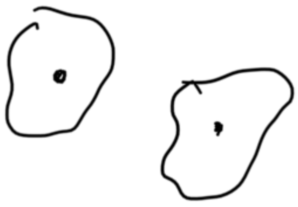
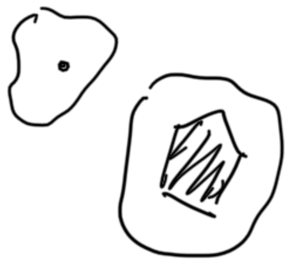
MAT327 TUT #8

Today's plan: Q6, Q7, Q9 on p205 Munkres

Recap:



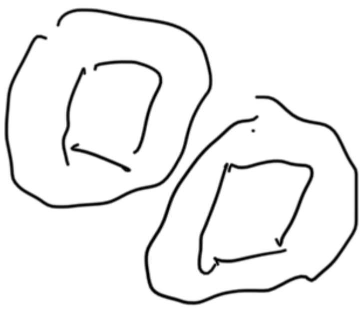
Hausdorff
(T2)



Regular
(T3)

Normal
(T4)

$\Leftarrow$



Completely normal
(T5)

Thm: 1) Subspaces of T2 are T2, products of T2 are T2

2) ~ T3
T3 ~ T3
~ T3

However, no analogous results for T4.

Q6

Def: (Completely normal, T5)
A space X is T5 if every subspace of X is normal

X is T5 $\iff$
$\forall$ A, B separated

$\star$ ($\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$)

" $\Leftarrow$ " Obvious that this implies normal

Furthermore, subspace $Y \subset X$, X satisfies

C and D closed Y

WTS: $\exists\, U, V$ that separate C, D

Then it follows directly

" $\Rightarrow$ " X is T5

A, B that are $\star$

Consider $Y = X - \underbrace{(\bar{A} \cap \bar{B})}_{\text{closed set}}$

Y is open

Y contains A, B

By assumption Y is normal,

and $\exists\, U, V$ separates $\bar{A} \cap Y$, $\bar{B} \cap Y$

Y open $\Rightarrow$

U, V are open in X

7. as A subspace of T5 ✓

Which of the following spaces is T5

Easy: $Y \subset X$, X is T5

Z Subspace of Y is ~ of X

and hence Z is normal

d) A metrizable space ✓

Recall: Metrizable $\Rightarrow$ normal

Recall: Subspaces of metrizable ~

b) The product of " are T5

False

Counterexample: $\mathbb{R}_\ell^2$

$\mathbb{R}_\ell$: lower limit topology

$\boxed{\text{Example 3 on p198} \quad \mathbb{R}_\ell \text{ is normal, but } \mathbb{R}_\ell^2 \text{ is not}}$

$\mathbb{R}_\ell$ is T5

$\mathbb{R}$ is metrizable

$\mathbb{R}_\ell$ is finer :

closures are "smaller"

open set are still open

$\mathbb{R}_\ell$ is T5 as $\mathbb{R}$ is T5.

9. Example 1 on p 203

$\mathbb{R}^J$ not normal,

where J uncountable

① $\mathbb{R}^J$ is regular but not normal

product of $\mathbb{R}$

② $\mathbb{R} \cong (0,1)$

$\mathbb{R}^J \cong (0,1)^J \subseteq [0,1]^J$

Product of compact / Hausdorff

$\Rightarrow$ compact / Hausdorff

$\Rightarrow$ normal

Subspaces of normal space need not be normal

③ Uncountable product of normal spaces need not be normal

Proof: $\mathbb{R}^J$

$X = (\mathbb{Z}_+)^J$

WTS: X is not normal

On p205: A Closed subspace of normal is normal

X is closed

$x \in X$, $B \subset J$ finite

a) $U(x, B) := \{ y \in X : y(\alpha) = x(\alpha) \ \forall \alpha \in B \}$

WTS: $U(x, B)$ is a basis of X

U open in X, $\exists\ U(x, B) \subset U$

$$U(x, B) = \prod_{\alpha \in B} \{x(\alpha)\} \times \prod_{\alpha \notin B} X$$