MAT327 Tutorial #7

Problem 4 of HW4

X locally compact, Hausdorff

$$X^* = X \cup \{\infty\}$$

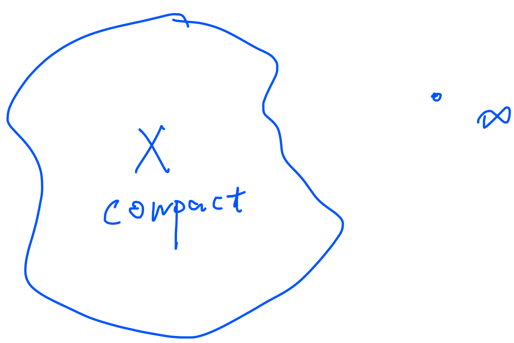
$$\mathcal{T} = \overset{(1)}{\{U : \text{open in } X\}} \cup \underset{(2)}{\{X^* - C : C \text{ compact in } X\}}$$

Observations:

① X compact

$$\mathcal{T} = \{ \sim \} \cup \quad \underset{\sim}{X^* - X} \text{ is open}$$

$$= \{\infty\}$$



② X subspace topology

$\underset{\sim}{X^* - C}$ restricting to X

This is $\underset{\sim}{X - C}$

C compact by def.
X Hausdorff $\Rightarrow$ C closed

$\Rightarrow$ $X - C$ is open

③ $\mathcal{T}$ is a topology

1° Intersections:

(1) ∩ (1) = (1) ✓

$X \quad \{\infty\}$

(2) ∩ (1)

$$(X^* - C) \cap \underset{\cap X}{U} = \underbrace{(X - C)} \cap U \quad ✓$$

(2) ∩ (2)

$$(X^* - C_1) \cap (X^* - C_2)$$

$$= \underbrace{X^* - \underbrace{C_1 \cup C_2}}$$

✓

2° Union

(1) $\bigcup U_\alpha = U$ open in X

(2) $\bigcup (X^* - C_\beta) = X^* - \underbrace{\bigcap C_\beta}$ ✓

Arbitrary intersection of compact sets is compact

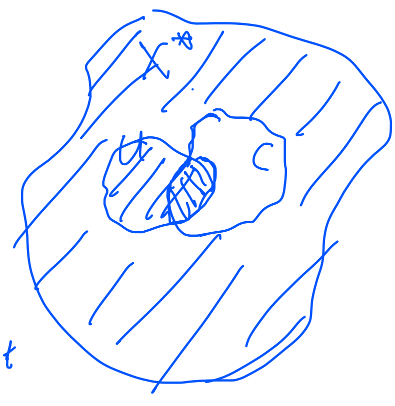
(2) and (1)

$$\left(\bigcup U_\alpha\right) \cup \left(\bigcup (X^* - C_\beta)\right)$$

$$= U \cup \underbrace{(X^* - C)}$$

$$= \underbrace{X^* - \underbrace{(C - U)}}$$

closed and hence compact



(2) ✓

a) X^* is compact Hausdorff

Proof:

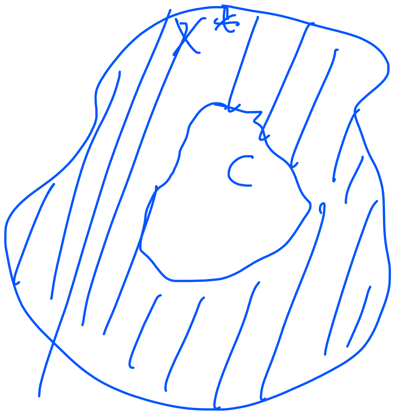
Compact: Consider $\mathcal{O}$ open cover of $X^* = X \cup \{\infty\}$

$X^* - C \in \mathcal{O}$

subcover $\mathcal{O}'$ that covers C

$\mathcal{O}' \cup \{X^* - C\}$

$(X^*, \mathcal{T})$ is compact.



Hausdorff: $x \in X$, $y = \infty$

$x \in U \subset X$

Recall: In locally compact, Hausdorff space,

$\forall x \in X \ \exists \overline{U} \ni x$ compact

$(X^* - \overline{U}) \cap U = \emptyset$

$\infty \in$

This verifies that X^* is indeed Hausdorff

b) $\{x_n\}_{n \in \mathbb{N}}$ in X has converging subsequence

$\Leftrightarrow$ $\{x_n\}$ converges to ∞ in X^*

Proof: What does this mean?

For $X^* - C$, $\exists N \in \mathbb{N}$, s.t. $x_n \in X^* - C$ for every $n \geq N$

In other words, this means that the sequence "escapes" any $C \subset X$

Suppose we have convergent subsequence,

$\{x_{n_k}\}$, s.t. $x_{n_k} \to x$

$\Rightarrow \exists\ U$ of x s.t.

U contains a tail of $\{x_{n_k}\}$

$\Rightarrow \bar{U}$ contains a tail

$\Rightarrow X - \bar{U}$ open does not contain a tail

$\Rightarrow \{x_n\}$ does not converge to ∞

Conversely. No converging subseq.

$\Rightarrow$ Converging to ∞

Recall that: Compactness $\not\Rightarrow$ sequentially compact

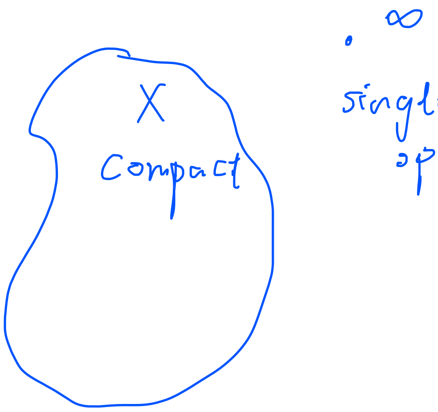
X compact (locally compact) Hausdorff

but not sequentially compact

Then $\exists \{x_n\}$ with no converging subsequence

However, $\{\infty\}$ is open as

$$\{\infty\} = X^* - \underbrace{X}_{\text{compact}}$$



singleton open

So, $\{x_n\}$ does not converge ∞

c) Show that X is open in X^* and has the subspace topology. ✓

X is dense X^* iff X noncompact.

Suppose X is compact. $\Rightarrow$ X is closed

X^* Hausdorff

X is non-compact

$\bar{X} \neq X$ and therefore $\bar{X} \supsetneq X$

$$\bar{X} = X^*$$

$$\infty \subset \underbrace{X^* - C}_{\not\subset X}$$

$$C \subsetneq X$$

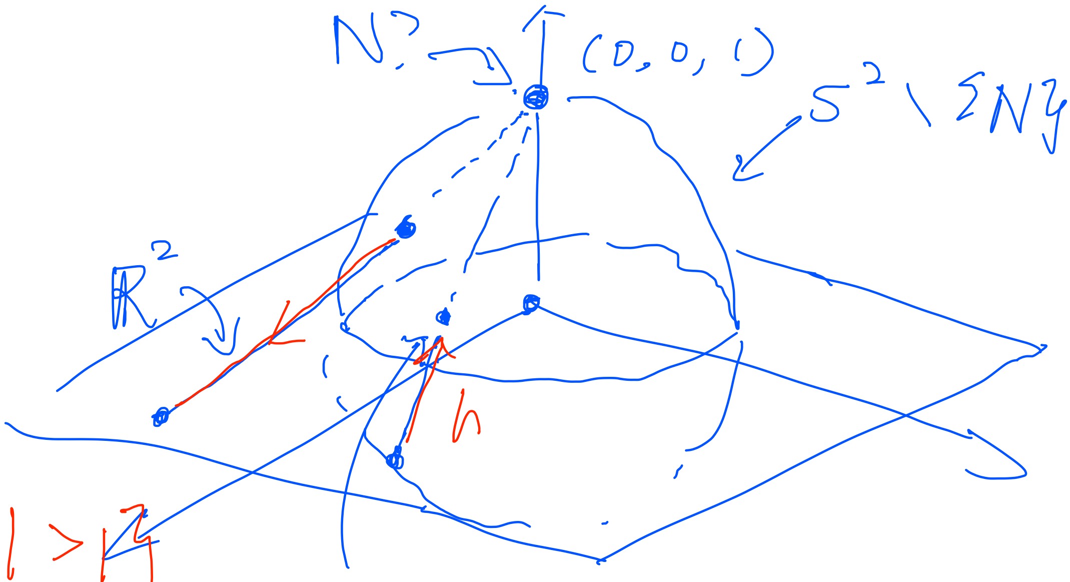
$X^* - C \cap \bar{X}$ Hence, $\infty \in \bar{X}$

d)

$$S^2_{upper} \xrightarrow{h} \{|x| > 1\}$$

$$S^2_{lower} \to \{|x| < 1\}$$

$$N \to \infty$$



$x \leq B_1$

$X - C$

$\mathbb{R}^2$

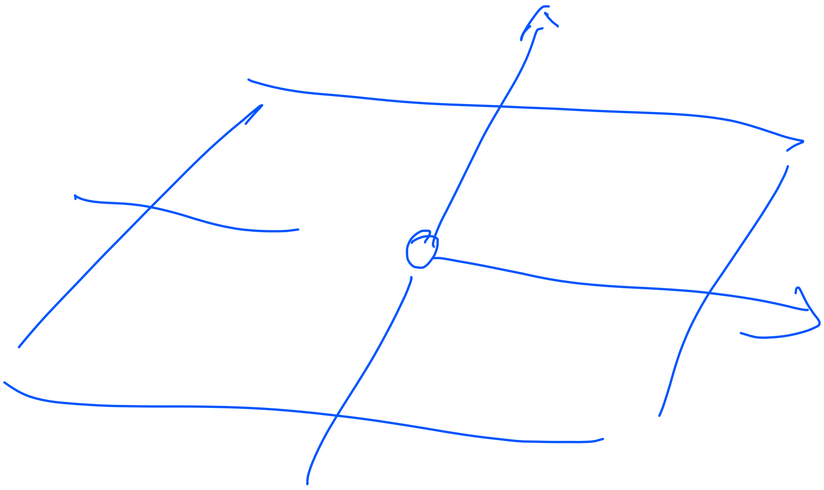
$C \sim S^2$

e) ① $\mathbb{R}^2 \setminus \{(0,0)\}$

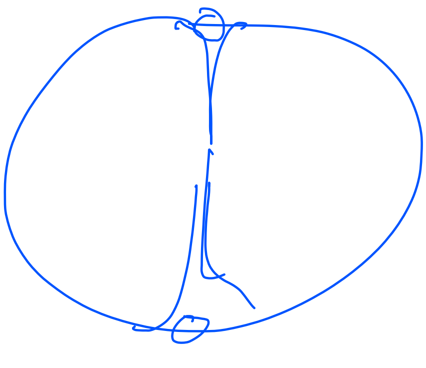
② $\mathbb{R}^2 \setminus \{\text{finitely many points}\}$

③ $\bigcup_{n,m\in\mathbb{Z}} B_{1/2}(n,m)$ in $\mathbb{R}^2$

④ $\mathbb{R}$ discrete



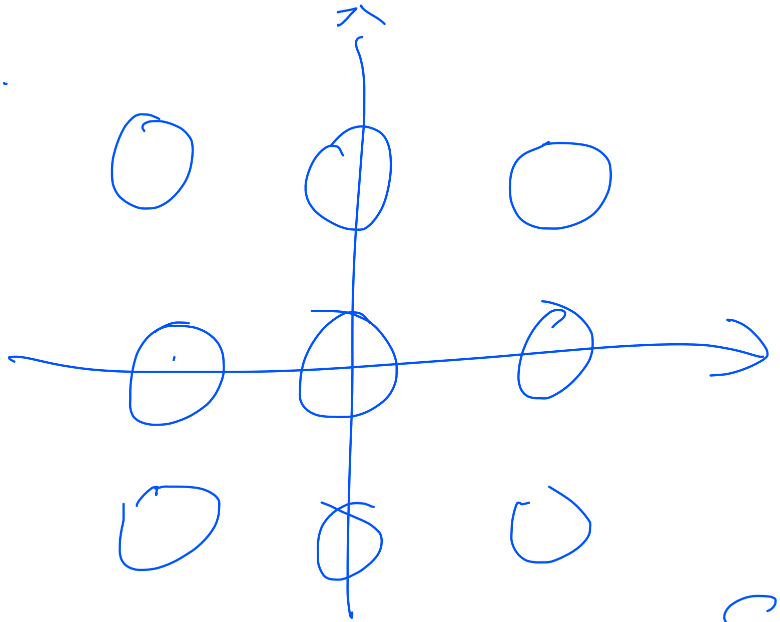
①

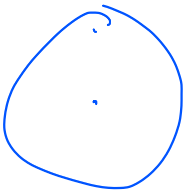


② stitching

to $\circ$ ∞

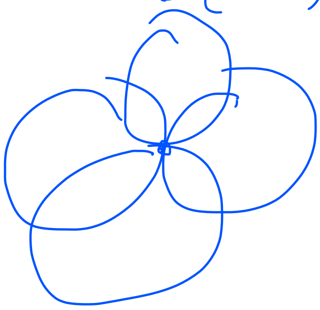
③





$$B \sim \mathbb{R}^2 \to S^2$$

$$(\ ) \sim \mathbb{R}$$



④ $\mathbb{R}$ discrete

Fact: A discrete set is compact iff finite

Open neighborhoods of ∞ are cofinite

$$(X^* - C)$$