MAT 327 Tutorial #6

## The Axiom of Choice and Zorn's Lemma

**Def:** Let $\mathcal{A}$ be non-empty collection of non-empty sets. Then a $f : \mathcal{A} \to \bigcup \mathcal{A}$ is <u>choice function</u> if $f(B) \in \mathcal{A}$.

Examples:

1. All pairs of gloves in the world
   Picking the left glove is a choice function

2. $\mathcal{A} = P(\mathbb{N}) \setminus \{\emptyset\}$, $B \in \mathcal{A}$
   $f(B) = \min(B)$

3. $W$ : well-ordered
   $\mathcal{A} = P(W) \setminus \{\emptyset\}$, $B \in \mathcal{A}$)
   $f(B) = \min(B)$

Q: Do we always have a choice function?

$\phi(B) = \underset{n}{\exists x}\,(x \in B)$

Finite collection $B_1, \ldots B_n$

$$\phi(B_1) \wedge \phi(B_2) \wedge \ldots \wedge \phi(B_n)$$

Conjoin finitely many formulae ✓

$A = \{$ infinitely many sets $\}$ ?

Axiom of Choice: For a non-empty collection $A$ of non-empty sets, there exists a choice function.

Consider a product

$$\prod_{\alpha \in J} U_\alpha \neq \emptyset \quad \text{if} \quad U_\alpha \neq \emptyset$$

Equivalents of AC:

Thm: The following statements are equivalent

1. AC
2. The Well-Ordering Theorem
3. Zorn's Lemma

Every set can be well-ordered

$$\mathbb{N} \longrightarrow \{ \boxed{2}, 4, 8, \ldots \}$$

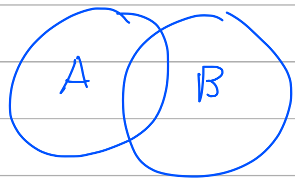
$2 \Rightarrow 1$: $f(B) = \min(B)$

Def: $(P, \leq)$ be a partial order. $A \subseteq P$. Then, we say $p \in P$ is an upper bound for $A$ if $a \leq p \ \forall a \in A$

Partial order : $(P, \leq)$

1. Reflexive : $a \leq a$
2. Anti-symmetric : $a \leq b$, $b \leq a \Rightarrow a = b$
3. Transitive : $a \leq b$, $b \leq c \Rightarrow a \leq c$

$\subseteq$ :




not comparable

total order : every pair of elements is comparable.

Def $(P, \leq)$ partial order

$m \in P$ is maximal if there is no $p \in P$ s.t. $m \leq p$

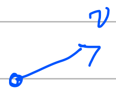
Def : A chain is a totally ordered subset.

Zorn's Lemma :

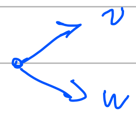
Let $(P, \leq)$ be a partial order s.t. every chain has an upper bound. Then $P$ has a maximal element.

Cor : Every vector space has a basis.

Finite-dim : $V$




$\text{span}\{v\} \neq V$

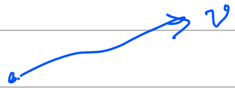



where $w \in V \setminus \text{span}\{v\}$

$\text{span}\{v, w\} \neq V$

$\downarrow$ go on inductively to get a basis

Infinite dim:




$V \setminus \text{span}\{v\}$:
still infinite

Proof: $(P, \leq)$

$P = \{ B : B \text{ is linearly independent} \}$

$\subseteq$: Set inclusion

Claim: If $B \in P$ is maximal, then $B$ is a basis of $V$.

$B$ lin. ind. ✓

$\text{span } B = V$ ?

Suppose $V \setminus \text{span}\{B\} \neq \emptyset$,

$v \in V \setminus \text{span}\{B\}$,

$B' = B \cup \{v\}$

We see that $B' \supseteq B$

$B'$ is lin. ind.

Hence, $B$ is not maximal $\rightarrow\leftarrow$

$\square$

Claim: Every chain has an upper bound.

$C = \{ B_\alpha : \alpha \in J \}$

$B_\alpha \subseteq B_\beta$ or $B_\beta \subseteq B_\alpha \quad \forall\, \alpha, \beta \in J$

$X = \bigcup C$ is an upper bound of $C$

① $B_\alpha \subseteq X$: Obvious

② $X \in P$ : $X$ is lin. ind.

$$\underbrace{\{v_1, v_2, \dots, v_n\}}_{\text{lin. dependent}} \subseteq X$$

Suppose ↳ exists.

Then $v_1 \in B_1 \subseteq X$

$\vdots$

$v_n \in B_n \subseteq X$

$\max\{B_1, \dots, B_n\} = B_k \subseteq X$

$\{v_1, \dots, v_k\} \subseteq B_k$

$B_k \in P$. $B_k$ lin. ind.

Hence $\{v_1, \dots, v_k\}$ is lin. ind. $\rightarrow\leftarrow$

So, $X$ is an upper bound. $\square$

There exists a basis of $V$. $\square$

Summary: Zorn's lemma is useful when you try to construct an object by repeating a procedure, but not done even after infinitely many times.